Class 4: 2/3/14 Section 2.1 I

Let the function $N(t) = N_0 a^t$, $t \in \mathbb{N}$ model some biological process. Here

- "a" is the growth constant (growth factor) where $a > 0$, $a \neq 1$ as before.
- N_0 is the initial population, since $N(0) = N_0 a^0 = N_0$

This is a model of <u>simple</u> <u>population</u> <u>dynamics</u>

① If $N_0 = 0$, then population never changes. Called a <u>fixed point population</u>, or <u>equilibrium population</u>.

② For all $N_0 > 0$, exponential growth pops. always change. How? Call $N_t = N(t)$

Then
$$N_1 = N(1) = N_0 \cdot a$$
$$N_2 = N(2) = N_0 \cdot a^2 = \underbrace{N_0 \cdot a}_{} \cdot a = N_1 \cdot a$$
$$N_3 = N(3) = N_0 \cdot a^3 = \underbrace{N_0 \cdot a^2}_{} \cdot a = N_2 \cdot a$$

In general, for $i \in \mathbb{N}$, we have $N_{i+1} = N_i \cdot a$, $N_0 = \underline{\text{something}}$

This last equation $N_{i+1} = a N_i$, $N_0 =$ something is called a (first-order) <u>recursion</u>: describes the population at some stage in terms of what it was at the previous stage.

<u>Useful for analysis</u>

ⓐ $N_{i+1} = a \cdot N_i$, $N_0 =$ something is exactly the same as $N(t) = N_0 a^t$, $t \in \mathbb{N}$.

<u>Note:</u> Exponential growth <u>ALWAYS</u> has this feature: Population size measured at regular intervals is always a constant times the previous population size.

ⓑ $\frac{N_{i+1}}{N_i} = a$ (ratio of successive pop sizes is always a constant).

ⓒ Only 2 population sizes needed to construct the equation of the model!

ex A pop. of bacteria was 4500 at some point in time. 1.5 hours later, it was 10,500. Assuming exponential growth, what is the population after 3 hours? after 5 hours.

Strategy ① ~~the~~ Use the 2 points (0, 4500) and (1.5, 10,500) to construct a function modeling the pop. growth.

② Use the new model to construct the population growth after 3 hours.

③ Assuming populations are still growing exponentially in between each 1.5 hours time measurement, find the time that corresponds to 5 hours and calculate the population.

Solution ① Set $N_0 = 4500$ @ $t = 0$, and form $N(t) = N_0 a^t = 4500 a^t$, $t \in \mathbb{N}$

If $t = 1$ corresponds to 1.5 hours, then $N(1) = 4500\, a^{(1)} = 10500 = N_1$

Since growth is assumed exponential,

$$\frac{N_1}{N_0} = \frac{10500}{4500} = a = \frac{7}{3}.$$

The growth model is

$$N(t) = 4500 \left(\tfrac{7}{3}\right)^t$$

where t is measured in 1.5 hour intervals.

② $t = 2$ corresponds to 3 hours, so

$$N_2 = 4500 \left(\tfrac{7}{3}\right)^2 = 24{,}500 \text{ bacteria.}$$

Note that $\frac{N_2}{N_1} = a = \frac{7}{3}$.

Hence $N_2 = \frac{7}{3} \cdot 10500 = 24500$ also.

Solution (cont'd.)

③ Assume pop grows exponentially also in between 1.5 hour intervals
- works when populations are rather large and more or less procreating all the time.

Then $t=1$ @ 1.5 hours
$t=2$ @ 3 hours
$t=3$ @ 4.5 hours

$\Rightarrow t=\frac{1}{3}$ @ $\frac{1}{2}$ hour

$t=?$ @ 5 hours? Answer is $\frac{10}{3}$

So $N(\frac{10}{3}) = 4500(\frac{7}{3})^{10/3} \cong 75823$ bacteria ∎

Note: We could also calculate growth factor @ $t=1$ hour:

$t=0$ $4500 = N_0$

$t=\frac{2}{3}$ (1 hour) $N_{2/3}(\frac{2}{3}) = 4500(\frac{7}{3})^{2/3}$
$\cong 7916$ bacteria.

Then $\frac{N_{2/3}}{N_0} = \frac{7916}{4500} = a \cong 1.76 = (\frac{7}{3})^{2/3}$ = growth constant

for a model where t is measured in hours.

$N(t) = 4500(\frac{7}{3})^{2/3 t}$.

Here then @ 5 hours: $N(5) = 4500(\frac{7}{3})^{2/3(5)} = 4500(\frac{7}{3})^{10/3} \cong 75823.$