Class 33: 4/30/14 Section 7.2

Last class, we learned that given an integral (definite or indefinite) of the form

$$\int f(g(x))\, g'(x)\, dx,$$

one can evaluate or solve the integral via a substitution $u = g(x)$, $du = g'(x)\,dx$, so that

$$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\, dx}_{du} = \int f(u)\, du$$

allowing for possible easier interpretation

This is called the Substitution Method or u-Substitution.

Since it comes from the chain Rule for diff, one can also call it the

Anti Chain Rule.

ex. Find the anti derivative of $6x\cos(3x^2+1)$.

Here we seek to evaluate $\int 6x\cos(3x^2+1)\,dx$, or

$\int \cos(3x^2+1)\,6x\,dx$. Try the substitution $\begin{array}{l} u=3x^2+1 \\ du=6x\,dx \end{array}$

(the inside function of the composition in the integrand).

Then $\int \cos(\underbrace{3x^2+1}_{u})\underbrace{6x\,dx}_{du} = \int \cos(u)\,du = \sin u + C = \sin(3x^2+1)+C$.

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ex. Find $\int_0^{\pi/2} 3\sin^2 x\cos x\,dx$.

Here, let $u=\sin x$, $du=\cos x\,dx$. Then

$$\int 3\sin^2 x\cos x\,dx = \int 3(\underbrace{\sin x}_{u})^2\underbrace{\cos x\,dx}_{du} = \int 3u^2\,du$$

$$= u^3 + C = \sin^3 x + C$$

Thus the antiderivative is $\sin^3 x + C$. Hence

$$\int_0^{\pi/2} 3\sin^2 x\cos x\,dx = \sin^3 x\Big|_0^{\pi/2} = \sin^3\frac{\pi}{2} - \sin^3 0 = 1-0 = 1.$$

Notes ① One could also change the limits of integration according to the u-substitution. Then there is no need to "go back to" x.

ex In the last problem, $\int_0^{\pi/2} 3\sin^2 x\cos x\,dx$, let

$u = \sin x$, $du = \cos x\,dx$, and when $x = 0$, $u = \sin 0 = 0$
$x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$.

$$\int_0^{\frac{\pi}{2}} 3\sin^2 x\cos x\,dx = \int_0^1 3u^2\,du = u^3\Big|_0^1 = 1^3 - 0^3 = 1.$$

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② Constants can be factored in to make the substitution easier:

ex. $$\int x e^{x^2}dx \overset{\substack{u = x^2 \\ du = 2x\,dx}}{\underset{\substack{\text{or} \\ \frac{1}{2}du = x\,dx}}{=\!=}} \int \frac{1}{2}e^u\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$$

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③ The substitution can also help to clean up any leftover parts of the integrand:

ex. Find $\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx$.

Here, the composition $\sqrt{x^2+1}$ indicates the u-substitution is $u = x^2+1$, $du = 2x\,dx$, and when $x=0$, $u = 0^2+1 = 1$, and $x = \sqrt{3}$, $u = (\sqrt{3})^2+1 = 4$.

A straightforward substitution yields the partial result:

$$\int_0^{\sqrt{3}} x^3\sqrt{x^2+1}\,dx = \int_0^{\sqrt{3}} x^2\sqrt{\underbrace{x^2+1}_{u}}\,\underbrace{x\,dx}_{du} = \int_1^4 \frac{x^2\sqrt{u}}{2}\,du$$

We are not done, as we need to render the remaining x^2 part as a function of u:

Here since $u = x^2+1$, $x^2 = u-1$, and

$$\frac{1}{2}\int_1^4 x^2\sqrt{u}\,du = \frac{1}{2}\int_1^4 (u-1)\sqrt{u}\,du = \frac{1}{2}\int_1^4 (u^{3/2} - u^{1/2})\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^4 = \frac{1}{2}\left(\frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2}\right) - \frac{1}{2}\left(\frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{5}(32) - \frac{2}{3}(8)\right) - \frac{1}{2}\left(\frac{2}{5} - \frac{2}{3}\right) = \cdots \quad ∎$$

Yet another Anti-Diff Rule.

Recall the Product Rule for Differentiation.

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x).$$

Rewrite this as

$$(\star) \quad f(x)\,g'(x) = \frac{d}{dx}[f(x)\,g(x)] - f'(x)\,g(x)$$

Note: ⓐ If 2 functions are equal, so are their derivatives.

ⓑ If 2 fncs are equal, so are their antiderivatives, only up to a constant!!!

Hence we can integrate the above equation ($\star$) to get

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv} = \underbrace{\int \frac{d}{dx}[\underbrace{f(x)}_{u}\,\underbrace{g(x)}_{v}]\,dx}_{uv} - \int \underbrace{g(x)}_{v}\,\underbrace{f'(x)\,dx}_{du}.$$

Or, in simpler terms, given the double substitution

$$u = f(x), \quad du = f'(x)\,dx,$$

$$v = g(x), \quad dv = g'(x)\,dx,$$

we get $\boxed{\int u\,dv = uv - \int v\,du,}$

a shorthand way to view this equation.

This is a new possible way to evaluate an integral that is a product of functions; by replacing it with a different, hopefully easier to handle, ~~an~~ integral.

This replacement is called <u>Integration by Parts</u>

It comes from the Product Rule. I call it the <u>Anti-Product Rule</u>.