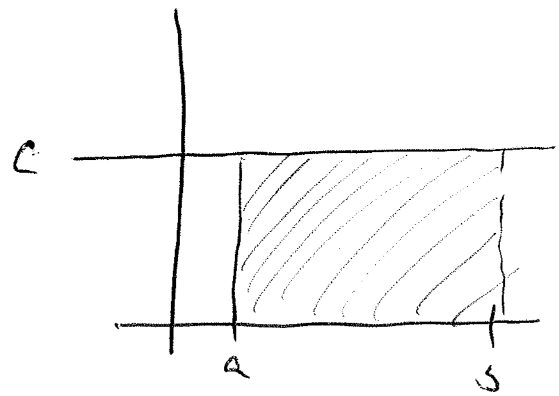
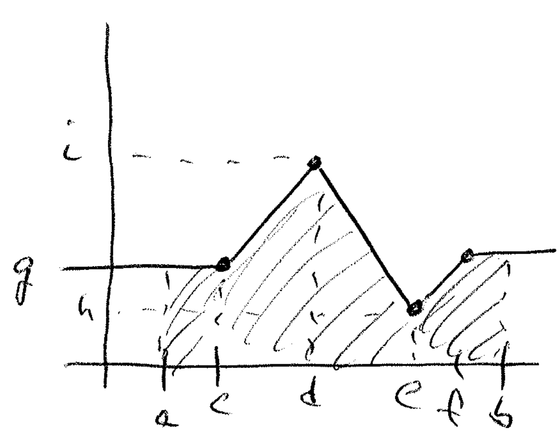
Class 29: 4/16/14 Section 6.1 I

The area between a constant function and the x-axis on the interval $[a,b]$ is easy to find

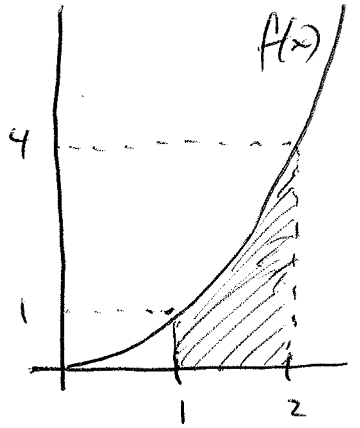


area is (length of $[a,b]$) $\cdot$ height
$= (b-a)c$

It's also straightforward to find the area "under" a piecewise linear function on a closed interval $[a,b]$.



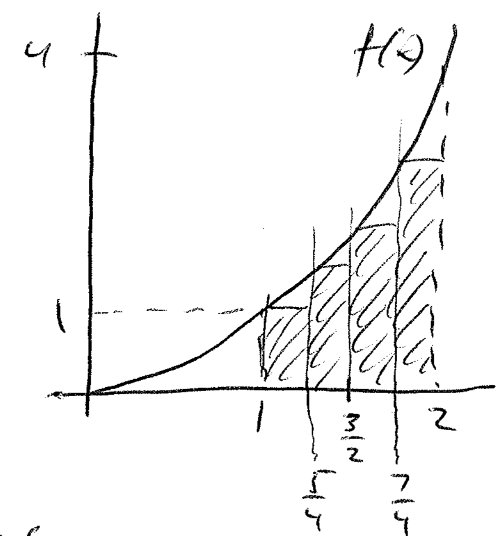
Q: How does one find the area under a curved region?



Q: What is the area of the shaded region?

A: Hard to tell, but easy to estimate....

(I) Break up the interval into pieces. Choose 4 to start, maybe.



(II) Construct rectangles to estimate the area. Use the function to create the tops (in many ways.

For example, choose the left hand endpt on each subinterval.

(III) Add up rectangular areas to estimate area under the curve:

$$S_4 = f(1)\cdot\left(\tfrac{5}{4}-1\right) + f\left(\tfrac{5}{4}\right)\left(\tfrac{3}{2}-\tfrac{5}{4}\right) + f\left(\tfrac{3}{2}\right)\left(\tfrac{7}{4}-\tfrac{3}{2}\right) + f\left(\tfrac{7}{4}\right)\left(2-\tfrac{7}{4}\right)$$

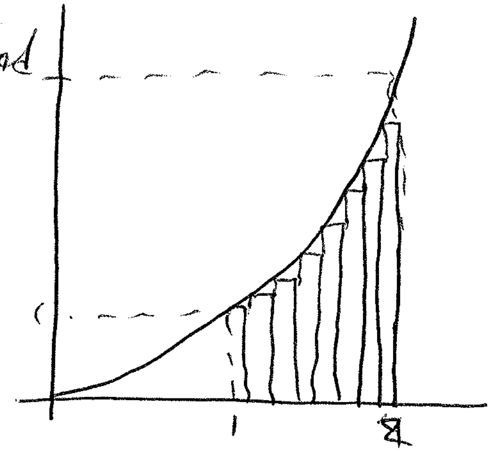
$$= 1\cdot\left(\tfrac{1}{4}\right) + \tfrac{25}{16}\left(\tfrac{1}{4}\right) + \tfrac{9}{4}\left(\tfrac{1}{4}\right) + \tfrac{49}{16}\left(\tfrac{1}{4}\right)$$

$$= \tfrac{16}{64} + \tfrac{25}{64} + \tfrac{36}{64} + \tfrac{49}{64} = \tfrac{124}{64} \doteq 1.54.$$

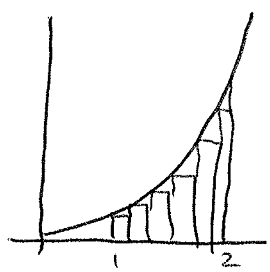
This gives us a low estimate of the area between $f(x)=x^2$ and the x-axis on $[1,2]$.

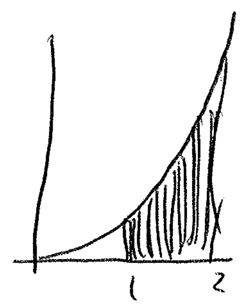
(IV) Repeat with a larger set of boxes:

A larger set of boxes defined the same way will leave out less material: a better approximation.

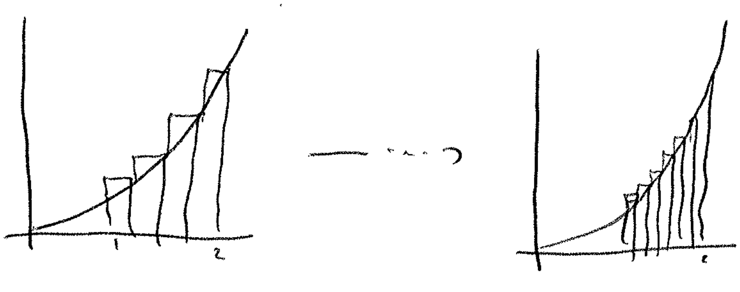


(V) Keep repeating. What happens if we push the number of subintervals between 1 and 2 to infinity?





Note: Would it matter if we chose the right hand end pt or any other function value on each subinterval?



Σ-notation Let $a_1, \dots, a_n$ be real numbers.

Then $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n = S_n$ (the sum)

ex Let $a_k = k$ for $k = 1, \dots, 101$.

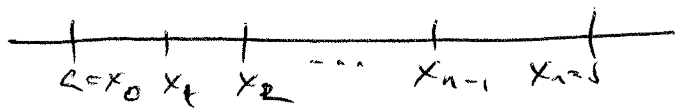
What is $\sum_{k=1}^{5} k$? $S_5 = \sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

$\sum_{k=1}^{10} k$? $S_{10} = 1+2+3+4+5+6+7+8+9+10$

It turns out, there is a pattern:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{try it!}$$

Given an interval $[a,b]$, divide it up into n-subintervals $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$



and call this partition $P = [x_0, x_1, \dots, x_n]$

Call the size of each subinterval $\Delta x_i = x_i - x_{i-1}$

and call the <u>norm</u> of P

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}.$$

so that $\|P\|$ is the length of the largest subinterval.

Back to estimating

- Let P be any partition of $[1,2]$, so
  $$P = [x_0 = 1, x_1, \cdots, x_{n-1}, x_n = 2].$$

- For each subinterval $[x_{i-1}, x_i]$, i/
  - its length is $\Delta x_i = x_i - x_{i-1}$
  - choose any function value $f(c_i)$ for $c_i \in [x_{i-1}, x_i]$.

- Then $S_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$ is the estimate of the area under $f(x) = x^2$ on $[1,2]$.

Need a better estimate?

- Choose a larger partition (more boxes)

Q: How good an estimate can you achieve by letting $\|P\| \to 0$?

A: Ultimately a perfect one. Note: As $\|P\| \to 0$, the number of intervals in the partitions go to ∞. This is true in calculus.

Def. The <u>definite</u> <u>integral</u> of $f(x)$ on the interval $[a,b]$ if it exists is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

If this limit exists, we say $f(x)$ is <u>integrable</u> on $[a,b]$.

Parts of a definite Integral:

upper limit

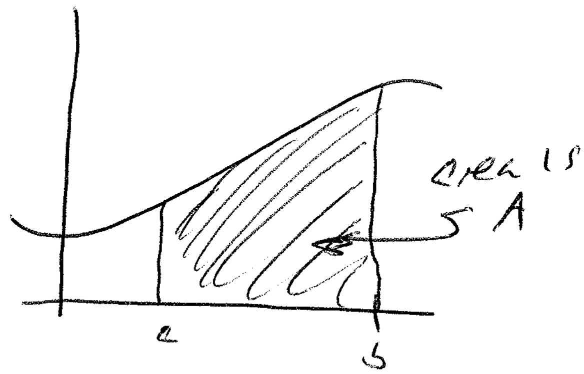
integrand

integration sign

$$\int_a^b f(x)\; dx = A$$

lower limit

variable of integration



Notes: ① What happens when the function dips below the x-axis?