Class 27: 4/11/14 Section 5.5 I

Back to the first optimization problem from last time: $R(p) = pe^{-p}$, or $A(x) = xe^{-x}$.

The "limiting behavior" has $R(0) = 0$ and

$$\lim_{p\to\infty} R(p) = \lim_{p\to\infty} \frac{p}{e^p}.$$

limit is not easy to "see". Cannot evaluate using Quotient Rule for limits since

$$\frac{p \xrightarrow{p\to\infty} \infty}{e^p \xrightarrow[p\to\infty]{} \infty}$$ doesn't determine anything.

Why not? $\frac{\infty}{\infty}$ shows up in each of the following cases, with different results:

$$\frac{x}{2x} \xrightarrow{x\to\infty} \frac{1}{2}$$

$$\frac{x^2}{2x} \xrightarrow{x\to\infty} \infty$$

$$\frac{x}{2x^2} \xrightarrow{x\to\infty} 0$$

all look like $\frac{\infty}{\infty}$ but ultimately have different limiting behavior.

This is why we call $\frac{\infty}{\infty}$ an <u>indeterminate form</u>.

To help better understand this indeterminate form, we study instead of where the numerator and denominator "go", but how "fast" they are going to ∞.

<u>Def</u> Let f, g be differentiable and either

Ⓐ $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

or

Ⓑ $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

for $a \in \mathbb{R}$ or $a = \infty$, or $a = -\infty$.

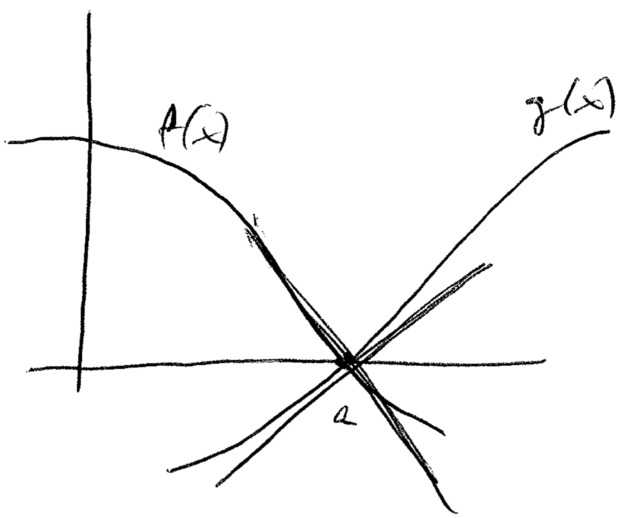
If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Notes ① Called L'Hospital's Rule (L'Hôpital).

② In situation Ⓐ, a direct attempt at evaluation would yield the indeterminate form $\frac{0}{0}$. In Ⓑ, $\frac{\infty}{\infty}$

Caution: These "fractions" are only symbolic. They are not real fractions!!

③ Suppose



Here both $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

If f, g are differentiable, then we can approximate f, g by tangent linear functions, and

$$f(x) \cong \cancel{f(a)}^{0} + f'(a)(x-a) = f'(a)(x-a)$$
$$g(x) \cong \cancel{g(a)}^{0} + g'(a)(x-a) = g'(a)(x-a)$$

Notes cont'd.

(3) cont'd.

Then $\dfrac{f(x)}{g(x)} \cong \dfrac{\cancel{f(a)} + f'(a)\cancel{(x-a)}}{\cancel{g(a)} + g'(a)\cancel{(x-a)}} = \dfrac{f'(a)}{g'(a)}$

for x very close to a.

Hence for the situation of (A) and $\frac{0}{0}$,

$\lim_{x \to c} \dfrac{f(x)}{g(x)}$ looks like $\dfrac{f'(a)}{g'(a)}$ 16 the latter

makes sense.

(4) Works with one-sided limits also.

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ex. $\lim_{x \to \infty} \dfrac{2x-1}{5x+3}$ Here both numerator and denominator go to ∞ as $x \to \infty$

Hence the indeterminate form $\frac{\infty}{\infty}$ as in (B).

By the Theorem,

$$\lim_{x \to \infty} \frac{2x-1}{5x+3} \overset{L'H}{=\!=} \lim_{x \to \infty} \frac{\frac{d}{dx}[2x-1]}{\frac{d}{dx}[5x+3]} = \lim_{x \to \infty} \frac{2}{5} = \frac{2}{5}.$$

$\boxed{\equiv}$

ex. $\lim\limits_{x \to \infty} \dfrac{2x-1}{x^2+3}$. Here direct evaluation yields another $\frac{\infty}{\infty}$ indeterminate form. Since both functions $f(x) = 2x-1$ and $g(x) = x^2+3$ are diff. L'Hospital's Rule applies, so

$$\lim_{x \to \infty} \frac{2x+1}{x^2+3} \overset{L'H}{=} \lim_{x \to \infty} \frac{2}{2x} = \lim_{x \to \infty} \frac{1}{x}$$

and this latter limit is 0.

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ex. $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$?

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ex $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$.

Other indeterminate expressions:

Hidden inside $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are other forms:

$\boxed{0 \cdot \infty}$ $\quad \lim\limits_{x \to 0^+} x \ln x$

(Rewrite as $\lim\limits_{x \to 0^+} \dfrac{\ln x}{1/x}$ to get indeterminate $\frac{\infty}{\infty}$)

$\boxed{\infty - \infty}$: $\lim\limits_{x \to 0^+} \frac{1}{x^2} - \frac{1}{x^4}$

(Rewrite as $\lim\limits_{x \to 0} \frac{x^2-1}{x^4}$. Then by L'H:

$$\lim_{x \to 0^+} \frac{1}{x^2} - \frac{1}{x^4} = \lim_{x \to 0^+} \frac{x^2-1}{x^4} \overset{L'H}{=\!=} \lim_{x \to 0^+} \frac{2x}{4x^3}.$$

$$= \lim_{x \to 0^+} \frac{2}{4x^2} = 0.$$

Also $\lim\limits_{x \to a} [f(x)]^{g(x)}$ may be problematic:

0^0 : $\lim\limits_{x \to 0^+} x^x$

∞^0 : $\lim\limits_{x \to 0^+} \left(\frac{1}{x}\right)^x$

0^∞ : $\lim\limits_{x \to 0^+} x^{1/x}$

1^∞ : $\lim\limits_{x \to 0^+} (\cos x)^{1/x}$

All involve competing quantities.

ex: $\lim\limits_{x \to 0^+} (\cos x)^{1/x} = \lim\limits_{x \to 0^+} \exp\left[\frac{1}{x} \ln \cos x\right] = \exp\left[\lim\limits_{x \to 0^+} \frac{\ln \cos x}{x}\right]$

where this last limit is $\frac{0}{0}$ indeterminate. By L'H.

$$\overset{L'H}{=\!=} \exp\left[\lim_{x \to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{1}\right] = \exp\left[\lim_{x \to 0^+} -\tan x\right] = \exp[0]$$

$$= 1.$$

Experiment:

$$\lim_{x\to\infty} \frac{x}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$$\lim_{x\to\infty} \frac{x^3}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{6x}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$$

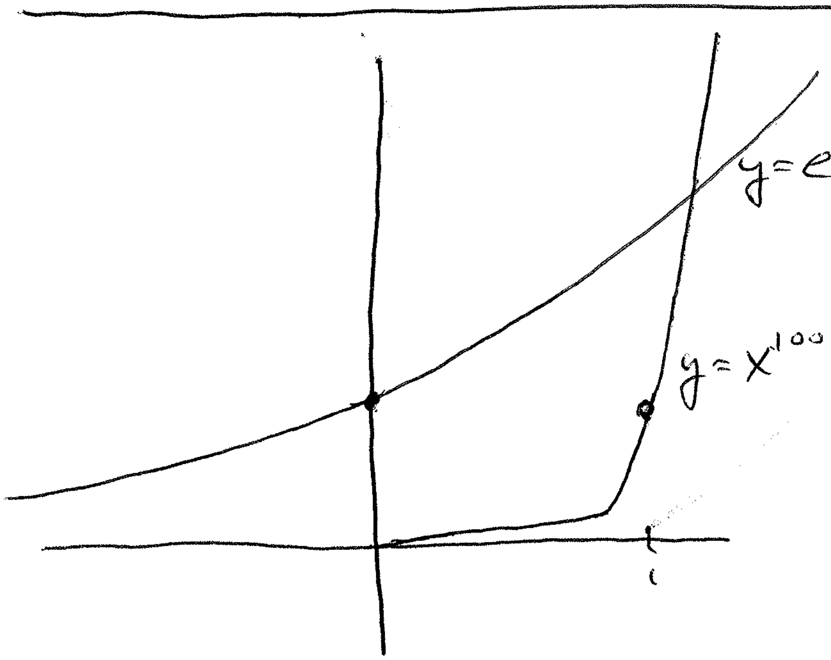
$$\vdots$$

$$\lim_{x\to\infty} \frac{x^{400}}{e^x} = ?$$

$$\vdots$$

$\lim_{x\to\infty} \frac{x^p}{e^x} = ?$ for $p \in \mathbb{N}$.

Show $\lim_{x\to\infty} \frac{x^p}{e^x} = p!\left(\lim_{x\to\infty} \frac{1}{e^x}\right) = (p!)(0) = 0.$



It looks like x^{100} will approach ∞ faster than e^x. But

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} = 0.$$