

Class 26: 4/9/14 Section 5.4

I

Optimization Problems - Find a global min or a max of a function on some interval.

4 examples

(I) Problem 5.6.15 The Ricker Curve of Ash populations in Fisheries

For what size of a Ash population

$$R(P) = \alpha P e^{-\beta P}, \quad P \geq 0, \quad \alpha, \beta > 0 \text{ constants.}$$

parental stock P is the number of recruits R maximized?

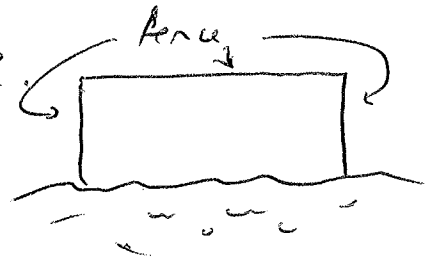
Note: For $\alpha\beta = 1$, this is the function

$$f(x) = x e^{-x}.$$

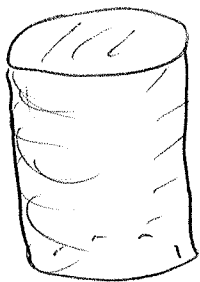
(II) Let Y be the yield of a crop as a function of nitrogen level N in the soil, modelled as $Y(N) = \frac{N}{1+N^2}$, $N \geq 0$

Find the ~~best~~ nitrogen level that optimizes yield.

(III) Using 1200 feet of fencing, find the dimensions of the largest enclosed rectangular area using a river as one side of the enclosure.



(IV) Minimize the amount of material required to enclose 1 liter of liquid in a closed cylindrical can.



(1 liter = 1000 cm³).

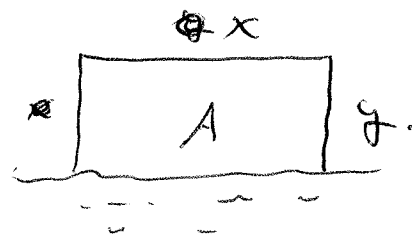
All of these problems are identical in nature:

Find the largest or smallest value of a function on some interval.

(I) Maximize $R(P) = \alpha P e^{-\beta P}$, for $P \in [0, \infty)$
For clarity, choose $\alpha = \beta = 1$.

(II) Maximize $Y(N) = \frac{N}{1+N^2}$ on $[0, \infty)$.

(III) Maximize Area $A = xy$
for fixed perimeter



(IV) Minimize Surface Area subject to fixed

volume $S = \pi r^2 h + 2\pi r^2$

and $V = 1000 \text{ cm}^3$.



Technique is straightforward:

(a) Find critical pts and function values

(b) Check end pts and limiting behavior

if interval is closed

asymptotes or limits if interval is not closed.

Special Note: How to do (III) and (IV) when there are two variables to play with?

Answer: When there are 2 variables, look for another relation between them to write optimized function as a function of only 1 variable.

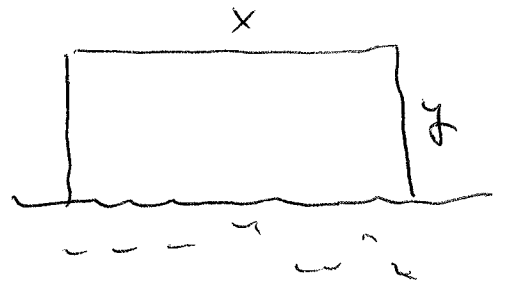
(III)

~~Maximize~~ Maximize $A = xy$

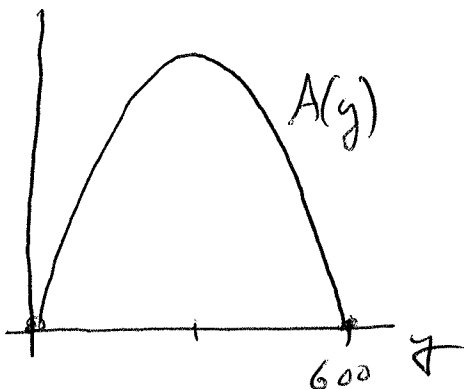
where amount of fencing

is fixed: Perimeter $P = 1200 = x + 2y$, or

$$x = 1200 - 2y.$$

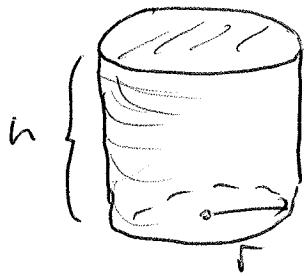


Hence $A = xy = (1200 - 2y)y = 1200y - 2y^2$



This is now only a function of y , and $A(y)$ is a parabola opening downward.

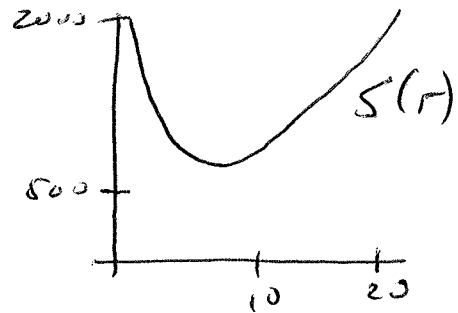
(IV) Minimize $S = 2\pi r h + 2\pi r^2$, where



$V = 1000 = \pi r^2 h$, so $h = \frac{1000}{\pi r^2}$.

Thus $S = S(r) = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$
 $= \frac{2000}{r} + 2\pi r^2$

S is now only a function of r and looks like this graph



Solutions

(I) Choose $\alpha = \beta = 1$. Maximize $R(p) = pe^{-p}$ on $[0, \infty)$.

Only critical pt is where $R'(p) = 0 = (1-p)e^{-p}$ at $p = 1$. And since here $R''(p) = (p-2)e^{-p}$ here $R''(1) < 0$, $p = 1$ is a local max.

And since $R(0) = 0$ and $\lim_{p \rightarrow \infty} R(p) = 0$, we conclude $R(p) = \frac{1}{e}$ is the max value of $R(p)$ on $[0, \infty)$. (we will show why soon)

(II) Max $Y = \frac{N}{1+N^2}$ on $(0, \infty)$. Y here is diff on $(0, \infty)$, and $Y' = \frac{(1+N^2) - N(2N)}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$

Here $Y'(N) = 0$ only when $N = 1$.

And since $Y''(N) \Big|_{N=1} = \frac{-2N(1+N^2)^2 - (1-N^2)2(1+N^2)2N}{(1+N^2)^4} \Big|_{N=1}$

$$= \frac{-2(4)}{16} = -\frac{1}{2} < 0, \quad Y(1) \text{ is a local max.}$$

Again, $Y(0) = 0$ and $\lim_{N \rightarrow \infty} Y(N) = 0$, hence

$$Y(1) = \frac{1}{2} \text{ is a global max.}$$

(III) Maximize $A(y) = 1200y - 2y^2$. This is diff on $[0, 600]$ (where $A \geq 0$), and

$$A'(y) = 1200 - 4y = 0 \text{ when } y = 300.$$

Here $A''(y) = -4 < 0$ so $A(300)$ is a local max.

And since $A(0) = A(600) = 0$, $A(300)$ is a global max.

Hence $y = 300$, and this by

$$\text{Perimeter } P = 1200 = \cancel{2x} + 2y,$$

$$x = 1200 - 2y = 1200 - 2(300) = 600$$

Dimensions of largest enclosure are

$$x = 600 \text{ ft}, \quad y = 300 \text{ ft}.$$

IV

Minimize $S(r) = \frac{2000}{r} + 2\pi r^2$, diff

on $(0, \infty)$. Here $S'(r) = -\frac{2000}{r^2} + 4\pi r$

and $S'(r) = 0$, where $\frac{2000}{r^2} = 4\pi r$, or

$$r = \sqrt[3]{\frac{2000}{4\pi}} \approx 5.419 \text{ cm}.$$

Here the amount of material needed can be as

$$\text{low as } S(5.419) \approx 369.05 + 34.05$$

$$\approx 403.1 \text{ cm}^2.$$

Check $S''(r) = \frac{4000}{r^3} + 4\pi > 0$ on $(0, \infty)$

Hence $r \approx 5.419$ is the global min.