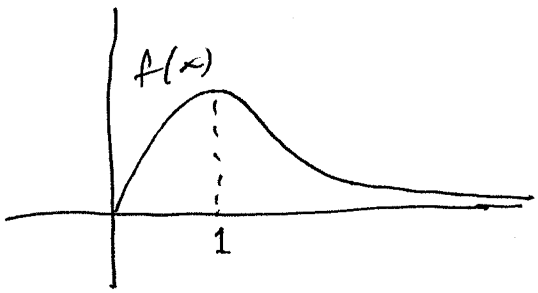
Example from previous class:

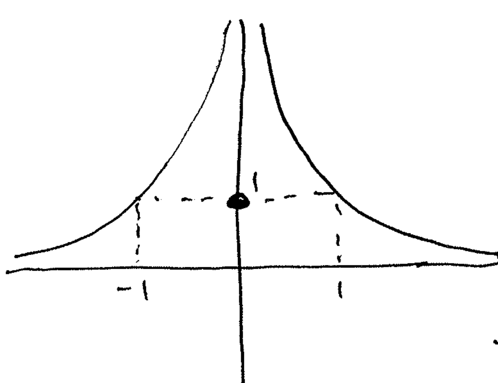
ex. For $f(x) = xe^{-x}$, find all inflection pts.



$f(x)$ is twice differentiable on all of $[0, \infty)$. Hence any inflection pts will occur where $f''(x) = 0$.

Here $f''(x) = \frac{d}{dx}\left[(1-x)e^{-x}\right] = (x-2)e^{-x}$. $f''(x) = 0$ only when $x = 2$. And since $f''(x) < 0$ on $[0, 2]$ and $f''(x) > 0$ on $(2, \infty)$ we conclude $x = 2$ is an inflection pt.

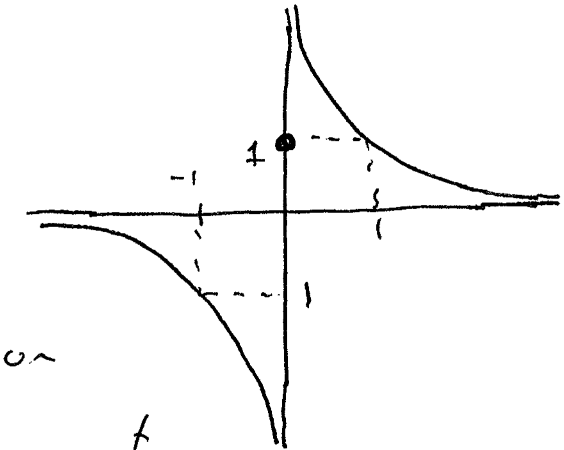
ex. Find all inflection pts for $h(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$



Here $h''(x) = \frac{d}{dx}\left[-\frac{2}{x^3}\right] = \frac{6}{x^4}$.

and since $h''(x) > 0$ on entire domain, except for $x = 0$ there are no inflection pts.

ex. $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$



Here $g(x)$ is defined on the domain $\mathbb{R}$, but is not continuous at $x=0$ (hence $x=0$ cannot be an inflection pt). $g''(x) = \frac{2}{x^3}$ is defined on all $x \neq 0$. On $(-\infty, 0)$ $g''(x) < 0$, and on $(0, \infty)$, $g''(x) > 0$. But $x=0$ is NOT an inflection pt.

Def The horizontal line $y=b$ is called a horizontal asymptote for $f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

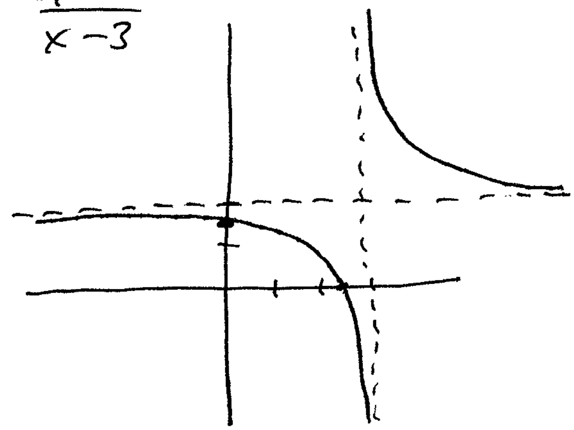
The line $x=c$ is a vertical asymptote if

either $\lim_{x \to c^+} f(x) = \infty$ or $\lim_{x \to c^+} f(x) = -\infty$

or $\lim_{x \to c^-} f(x) = \infty$ or $\lim_{x \to c^-} f(x) = -\infty$.

ex. Let $g(x) = \underbrace{\frac{1}{x-3} + 2}$ $= \frac{1}{x-3} + \frac{2(x-3)}{x-3} = \frac{1+2x-6}{x-3}$

looks like $\frac{1}{x}$ shifted to the right by 3 and up by 2. $= \frac{2x-5}{x-3}$



Here $\lim_{x\to\infty} g(x) = 2$

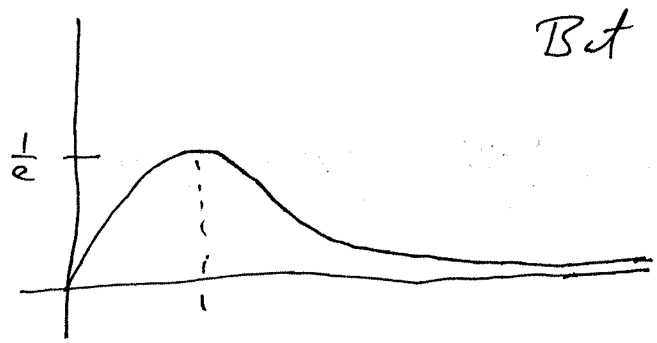
$\lim_{x\to-\infty} g(x) = 2$

$\lim_{x\to3^+} g(x) = +\infty$ and $\lim_{x\to3^-} g(x) = -\infty$.

ex. Back to $f(x) = xe^{-x}$. Does $f(x)$ have any asymptotes. Find them, if any.

Solution: $f(x)$ is continuous on all of $\mathbb{R}$. Hence it cannot have a vertical asymptote. (why not?).

As for horizontal asymptotes, the graph looks to have ~~x~~ $y=0$ as a horizontal asymptote.



But $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$ is hard to evaluate. We will learn a good technique soon.

There are also inclined asymptotes. These are harder to see. Example 4 pg 232 is a good example.

Note: If a function $f(x)$ can be written as a sum of 2 other functions

$$f(x) = g(x) + h(x)$$

where $\lim_{x\to\infty} h(x) = 0$, then as x gets large, $f(x)$ will look more and more like $g(x)$, even if $g(x)$ is not horizontal.

ex (4, pg 232) Let $f(x) = \frac{x^2-3}{x-2}$ on all $x \neq 2$.

We look for asymptotes and find $x=2$ to be a vertical asymptote. But $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ so <u>No</u> horizontal asymptotes.

But there is another one!

ex (cont'd)

Here $f(x)$ is an improper rational function, and via long division, we can rewrite it:

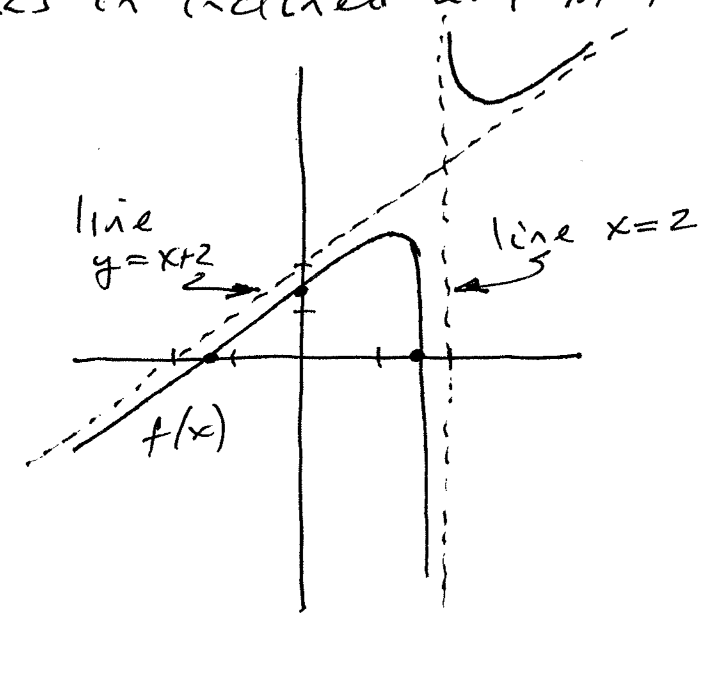
$$\begin{array}{r|l} & x+2 \\ \hline x-2 & x^2-3 \\ & x^2-2x \\ \hline & 2x-3 \\ & 2x-4 \\ \hline & 1 \end{array}$$

$$f(x) = \frac{x^2-3}{x-2} = \underbrace{\underbrace{x+2}_{g(x)} + \underbrace{\frac{1}{x-2}}_{h(x)}}_{\text{from description before example began.}}$$

Hence $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) + \lim\limits_{x \to \infty} h(x)$

$= \lim\limits_{x \to \infty} g(x)$, since $\lim\limits_{x \to \infty} h(x) = 0$.

Hence when x is very large, $f(x) = \frac{x^2-2}{x-2}$ looks more and more like $g(x) = x+2$. We say $f(x)$ has an inclined asymptote at $y = x+2$:



All of the features ~~are~~ of the graph of $f(x)$ given by calculus techniques allow us to "see" $f(x)$:

(a) horizontal and vertical intercepts, if any.

(b) where $f'(x)$, $f''(x)$ are positive, negative or 0.

(c) local extrema

(d) concavity and inflection.

(e) Vertical and/or horizontal asymptotes and/or limits at $\pm\infty$.

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2 excellent examples of curve sketching in book.

Here is another

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ex. Sketch $y = 3x^4 - 4x^3$.

Strategy Use list above.

Solution: Intercepts: vertical at $(0,0)$.

horizontal: $y = 0 = 3x^4 - 4x^3 = x^3(3x-4)$

Intercepts at $(0,0)$, and $(\frac{4}{3}, 0)$.

ex. Sketch $y = 3x^4 - 4x^3$ cont'd.

(b) Derivative info: Here

First derivative
$$\begin{cases} y' = 12x^3 - 12x^2 = 12x^2(x-1) \\ y' = 0 \text{ when } x = 0, 1. \\ y' > 0 \text{ on } (1, \infty) \\ y' < 0 \text{ on } (-\infty, 1) \end{cases}$$

schematic depicting sign of $f'(x)$.

y': $- - - - 0 - - - - 0 + + + +$; zeros at 0 and 1

$$y'' = 36x^2 - 24x = 12x(3x-2)$$

$y'' = 0$ when $x = 0, \frac{2}{3}$

$y'' > 0$ on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$

$y'' < 0$ on $(0, \frac{2}{3})$

y'': $+ + + 0 - - - - - - 0 + + + + +$; zeros at 0 and $\frac{2}{3}$

(d) Here concave up on $(-\infty, 0) \cup (\frac{2}{3}, \infty)$ and concave down on $(0, \frac{2}{3})$.

Inflection pts at $(0,0)$ and $(\frac{2}{3}, \frac{-16}{27})$.

(c) $x=1$: Here $y(1) = -1$, $y'(1) = 0$, $y''(1) > 0$

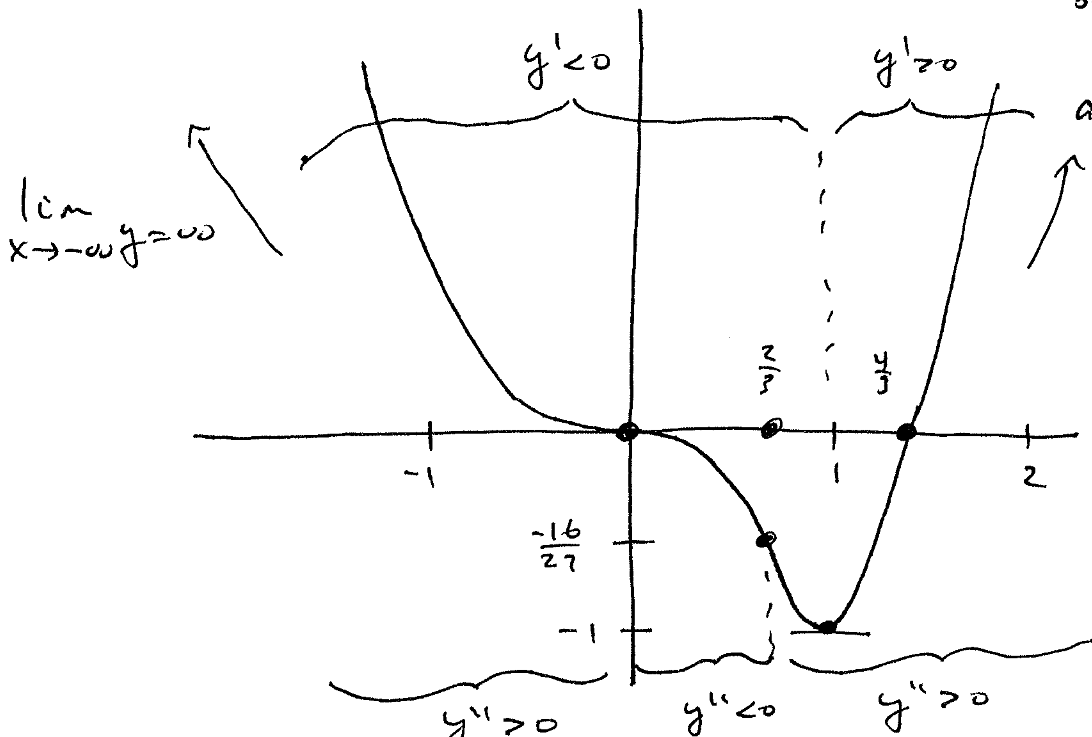
Hence there is a local min at $x=1$ by 2nd derivative test.

$x=0$: Here $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ hence we do not yet know if there is an extremum at $x = 0$.

ex Sketch $y = 3x^4 - 4x^3$ cont'd.

(d) Since $y(x)$ is a polynomial (degree 4), there are no horizontal or vertical asymptotes.

but $\lim_{x \to \infty} y = \infty$ and $\lim_{x \to -\infty} y = \infty$. (why?)



Connecting all of the relevant information in the only way possible allows us to sketch the function.