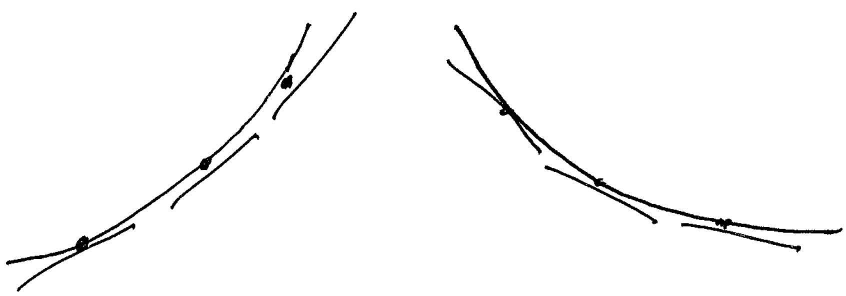
Class 24: 4/4/14 : Section 5.3

So when $f'(x) > 0$, we know $f(x)$ is increasing.

Q: What does $f(x)$ look like when $f'(x)$ is increasing on an interval $[a,b]$?

(This means either $f(x)$ is negative and getting less so, or $f(x)$ is positive and getting bigger).



<u>Facts</u> ① For $f(x)$ diff, $f'(x)$ is just another function.

For $f'(x)$ diff., then $f''(x) = \frac{d}{dx}[f'(x)]$ is again another function.

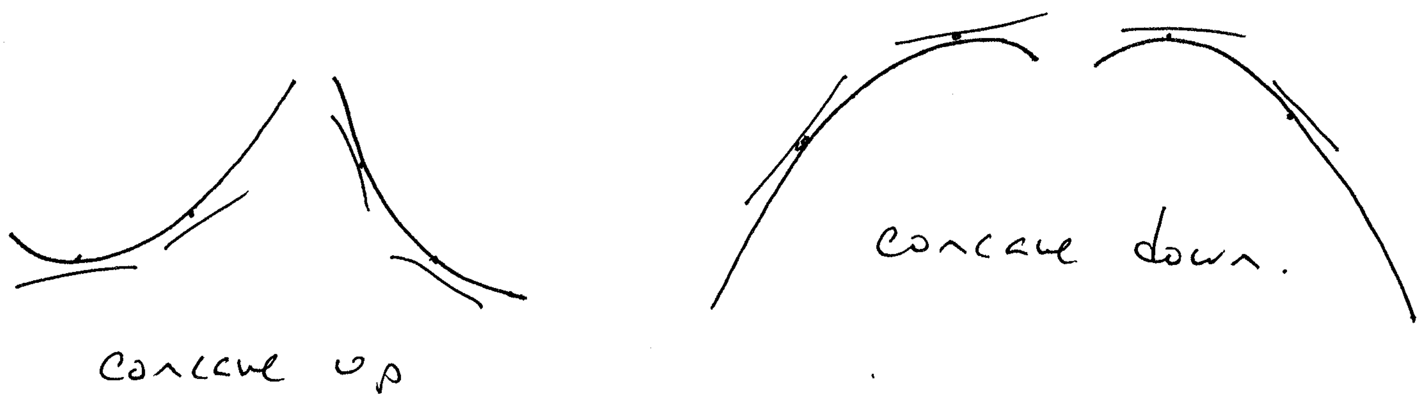
② $f''(x)$ measures how $f'(x)$ is changing. This, in turn, gives an idea on the "bending" of $f(x)$.

Def. For $f(x)$ twice diff. on I,

$f(x)$ is called

<u>concave</u> <u>up</u> if $f'(x) > 0$ on I,

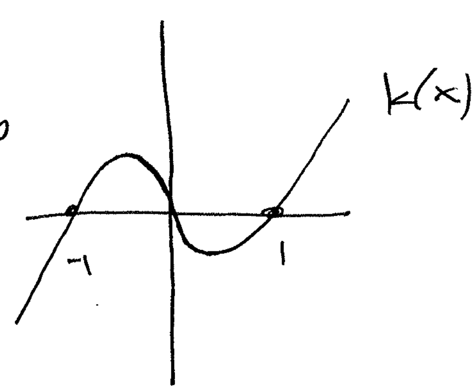
<u>concave</u> <u>down</u> if $f'(x) < 0$ on I.



ex. All quadratic polynomials are either concave up or down everywhere. (why?)

ex. $g(x) = e^x$ and $h(x) = e^{-x}$ are both concave up everywhere.

ex. $k(x) = x^3 - x$ is concave up on $[0,\infty)$ and concave down on $(-\infty, 0]$ since $k''(x) = 6x$.



Note: The monotonicity of $f(x)$ and its concavity are not related. (why?)

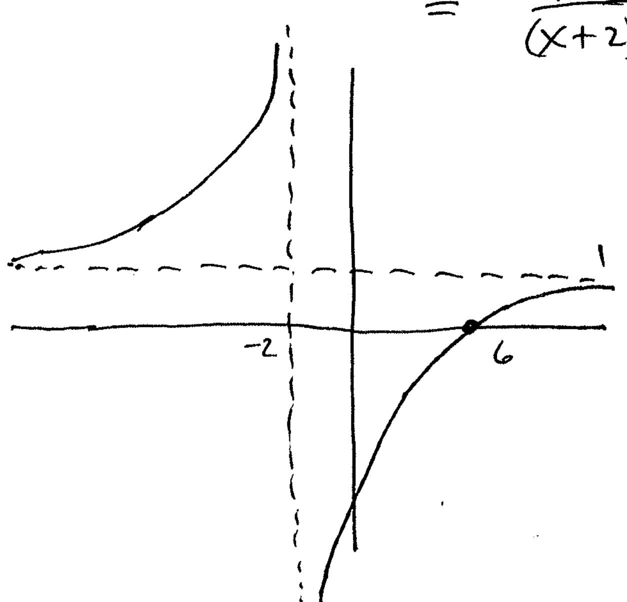
ex. Find the interval(s) where $f(x) = \frac{x-6}{x+2}$ is increasing or decreasing and also its concavity.

Solution: We need both $f'(x)$ and $f''(x)$ here.

- $f'(x) = \frac{1(x+2) - 1(x-6)}{(x+2)^2} = \frac{8}{(x+2)^2}$. The domain is $D = \{x \in \mathbb{R} \mid x \neq -2\}$ and $f'(x) > 0$ on all of D. f(x) is increasing on all of D.

- $f''(x) = \frac{d}{dx}\left[\frac{8}{(x+2)^2}\right] = \frac{0(x+2)^2 - 8 \cdot 2(x+2)}{(x+2)^4} = \frac{-16(x+2)}{(x+2)^4}$

$= \frac{-16}{(x+2)^3}$. Here $f''(x) > 0$ when $x < -2$ hence concave up on $(-\infty, -2)$, and concave down on $(2, \infty)$ since $f''(x) < 0$ here.



So let $f(x)$ be twice differentiable on some interval $I = (a,b)$, and at a pt $c \in (a,b)$ $f'(c) = 0$ and $f''(c) < 0$. What is happening here? And when $f'(c) = 0$ and $f''(c) > 0$?

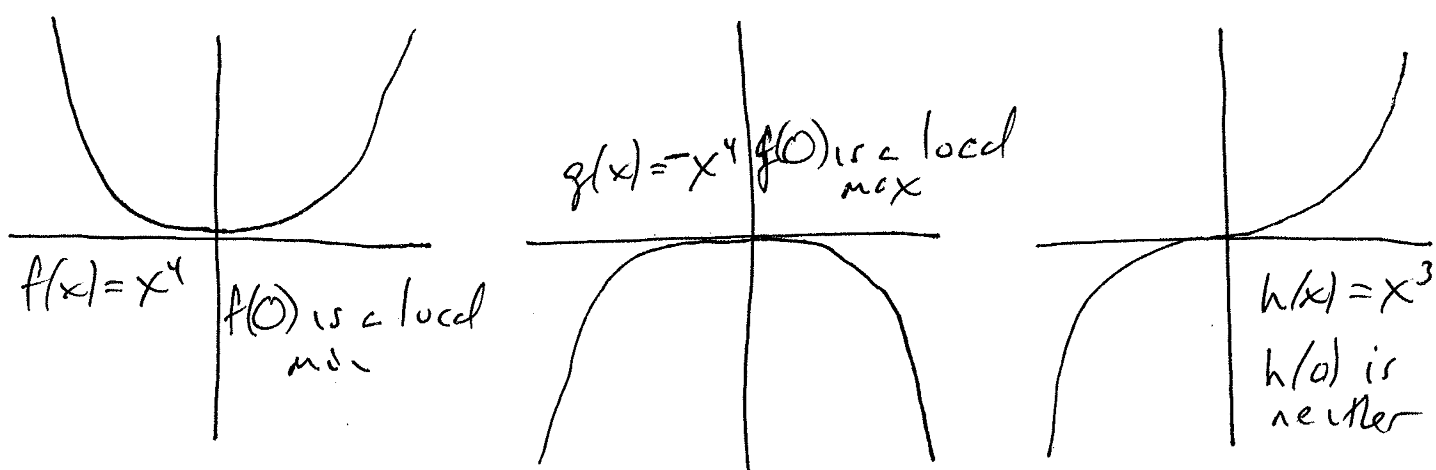
Second Derivative Test for a local extremum

Suppose f is twice diff on an open interval, containing a pt c:

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local max

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local min.

And if both $f'(c) = 0$ and $f''(c) = 0$? Nothing.



$f'(0) = f''(0) = 0$ $\qquad$ $g'(0) = g''(0) = 0$ $\qquad$ $h'(0) = h''(0) = 0$.

ex. Find the extrema of $f(x) = xe^{-x}$ on the interval $[0, 10]$.

Solution: ~~Ext~~ Since $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$ exists on all of $(0,10)$, the extrema will occur either at an endpt or at a place where $f'(x) = 0 = (1-x)e^{-x}$. Thus, the extrema will occur at $x=0$, $x=10$, or $x=1$.

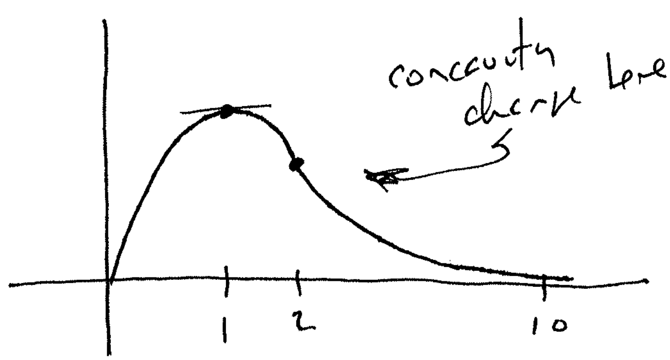
We could simply check the values here.

$$f(0)=0, \quad f(10) = 10e^{-10}, \quad f(1) = \frac{1}{e}.$$

Hence minimum at $x=0$, and max at $x=1$.

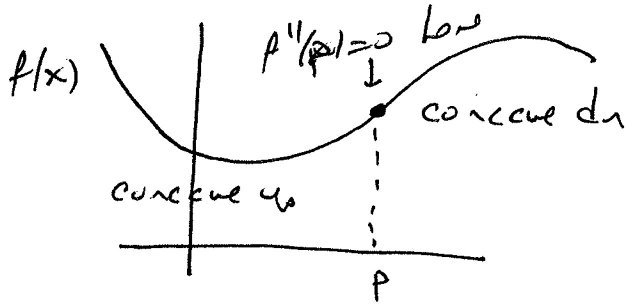
We could also test the max at $x=1$ by the SDT: $f''(x) = -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$

Since $f'(1)=0$ and $f''(1) = (1-2)e^{-1} < 0$ at $x=1$, we know by 2nd Der test that there is a local max at $x=1$.

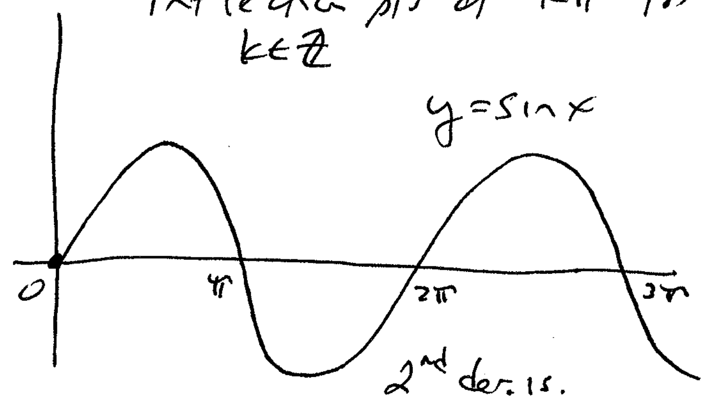


Def Let $f(x)$ be continuous at a pt p.

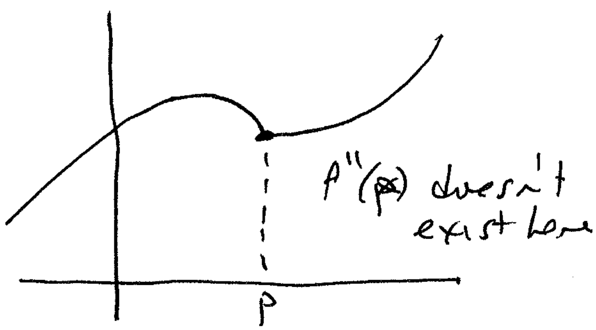
p is called an inflection pt for f if $f''(x)$ exists and changes sign as the graph of f passes through p.



inflection pts at $k\pi$ for $k \in \mathbb{Z}$



2nd der. is. $y'' = -\sin x$.



$y'' = 0$ precisely at $x = k\pi$.

Fact if $f(x)$ is twice diff and has an inflection pt at p, then $f''(p) = 0$.

ex Study the cases $f(x) = x^4$, $g(x) = -x^4$, $h(x) = x^3$.

Notice that as you pass through p. $f'(p)$ stops rising and starts falling or stops falling and starts rising. p is an inflection pt.