Selected Solutions: Problem Set 9.

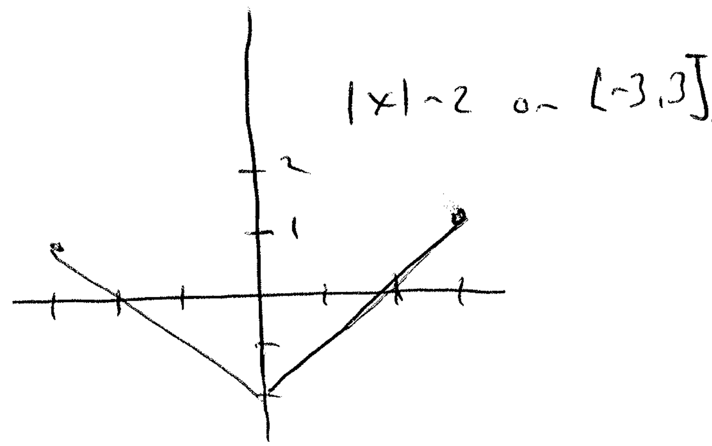
5.1.32 Graph $f(x) = ||x|-2|$ on $[-3,3]$ and determine all local and global extrema on $[-3,3]$.

Strategy: Construct the graph of f from the $|x|$ function one step at a time. Then use the visual graph to determine local and global extrema.

Solution:

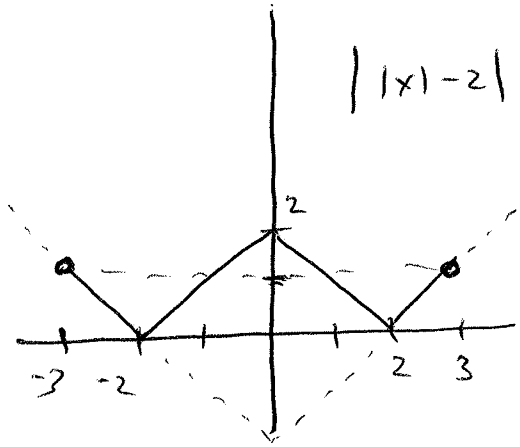
(I) Construct graph:

(a) Here, $|x|-2$ looks like $|x|$ shifted down by 2:

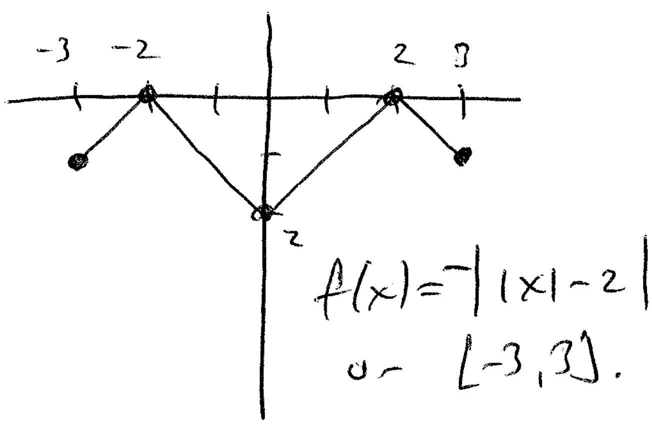


(b) $||x|-2|$ takes the graph of $|x|-2$ and reflects everything negative to its positive counterpart:

$||x|-2|$ on $[-3,3]$.



(c) lastly, $-||x|-2| = f(x)$ reflects the entire graph of part b over the x-axis.



(II) By Inspection, the critical pts are at $(-2,0)$, $(0,2)$, and $(2,0)$. The end pts are at $(-3,1)$ and $(3,1)$. Thus there are local extrema at each of these five: $(2,0)$ and $(-2,0)$ are local maxima and the other three are local minima. The 2 maxima are also global on $[-3,3]$ while the only global minimum is at $(0,-2)$. ∎

5.1.49 A car moves in a straight line and is at distance $s(t) = \frac{1}{100}t^3$, $t \in [0,5]$. in meters.

(a) Find the average velocity between $t=0$ and $t=5$.

<u>Solution</u>: Distance traveled at $t=0$ is $s(0)=0$

at $t=5$ is $s(5) = \frac{125}{100}$

Average velocity is ~~average~~ change in distance over change in time:

$$\text{Avg vel} = \frac{s(5)-s(0)}{5-0} = \frac{1.25-0}{5-0} = .25 \ ^{m}/_{sec}$$

(b) Find the instantaneous velocity for $t \in (0,5)$.

Solution: Here instantaneous velocity is $s'(t)$,

or $$s'(t) = \frac{d}{dt}\left[\frac{1}{100}t^3\right] = \frac{1}{100}(3t^2) = \frac{3}{100}t^2.$$

(c) At what time is the ~~instantaneous~~ instantaneous velocity equal to the average velocity?

Solution: We solve the equation $s'(t) = .25$ for t:

$$s'(t) = \frac{3}{100}t^2 = \frac{1}{4}$$

and get $t = \sqrt{\frac{100}{12}} = \frac{10}{2\sqrt{3}} = \frac{5}{\sqrt{3}} \cong 2.87$ secs

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5.2.14 For $f(x) = \frac{x^2}{x^2+1}$, determine where f is (a) increasing/decreasing, (b) concave up/down. Then graph and label these intervals.

Strategy: Use derivative and 2nd derivative information to determine (a) and (b). Then use a calculator to graph. Then label.

(a) To find increasing and decreasing intervals, calculate $f'(x)$ and look for where $f'(x) > 0$ and $f'(x) < 0$ respectively:

$$f'(x) \overset{\text{Quot}}{\underset{\text{Rule}}{=}} \frac{(2x)(x^2+1) - (x^2)(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}. \quad \text{Here}$$

$f'(x) > 0$ on $(-\infty, 0)$ and $f'(x) < 0$ on $(0, \infty)$.

Hence $f(x)$ increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) Concavity is determined by sign of $f''(x)$ (up when $f''(x) > 0$, down when $f''(x) < 0$).

$$f''(x) \overset{\text{Quot}}{\underset{\text{Rule}}{=}} \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}\right] = \frac{2(x^2+1)^2 - 2x(2(x^2+1))2x}{(x^2+1)^4}$$

$$=$$

$$=$$

$$= \frac{(2 - 8x^2)(x^2+1)}{(x^2+1)^4} = \frac{2 - 8x^2}{(x^2+1)^3}$$

(5) cont'd Here $f''(x) = \frac{2-8x^2}{(x^2+1)^3} > 0$ when

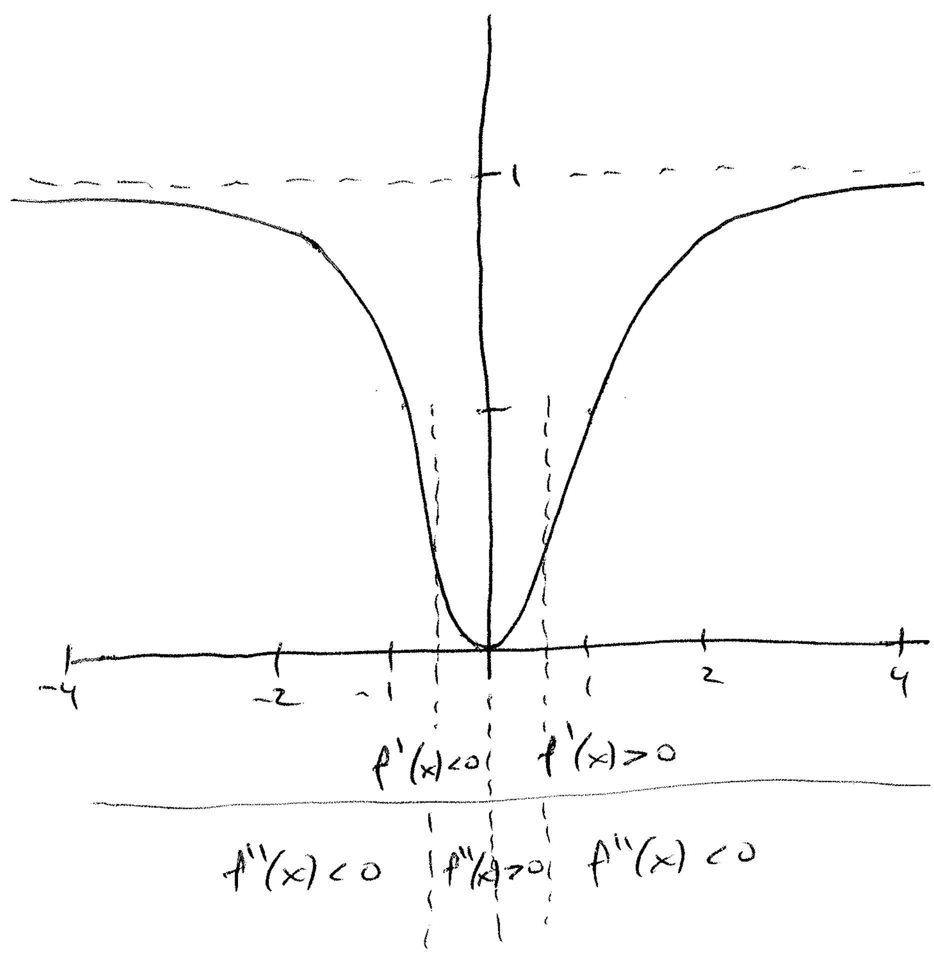
$2-8x^2 > 0$ or when $2 > 8x^2$ or when

$\frac{1}{4} > x^2$ or when $-\frac{1}{2} < x < \frac{1}{2}$.

and $f''(x) < 0$ when $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

Thus $f(x)$ is concave up on $[-\frac{1}{2}, \frac{1}{2}]$ and concave down on $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

Graph:



5.2.23 ⓐ Explain why there is only one solution to the equation $f(x)=0$ if $f(x)$ is cont on $[a,b]$, differentiable on (a,b) and $f'(x)$ is strictly positive or strictly negative on (a,b), and where $f(a)$ and $f(b)$ are of different signs:

<u>Solution</u>: Since $f(a)$ and $f(b)$ have different signs and $f(x)$ is continuous on $[a,b]$, by the Intermediate Value Thm, there must be at least one pt $c\in(a,b)$ where $f(c)=0$.

Since $f'(x)$ is never 0 or changes sign on (a,b), $f(x)$ is monotonic on $[a,b]$. Call $c\in(a,b)$ the first place in (a,b) where $f(c)=0$. But once $f(x)$ crosses at c, it cannot cross again without the derivative being 0 and changing sign.

Since $f'(x)$ doesn't change sign on $[a,b]$, there can be only one place where $f(c)=0$. ∎

(b) Use part (a) to show that $x^3-4x+1=0$ has exactly one solution on $[-1,1]$.

<u>Solution:</u> $f(x)=x^3-4x+1$ is a polynomial, and hence continuous on $[-1,1]$ and differentiable on $(-1,1)$, with $f'(x)=3x^2-4$.

Since $f(-1)=(-1)^2-4(-1)+1=4>0$ and
$f(1)=1^3-4(1)+1=-2<0$, by the IVT we know there is a root in $[-1,1]$, at least one.

But since $f(x)$ is diff on $(-1,1)$ and
$f'(x)=3x^2-4<0$ strictly on $(-1,1)$, it follows by (a) that there is only one place in $[-1,1]$ where $f(x)=0$. ∎

That is, only one solution to $x^3-4x+1=0$ on $[-1,1]$.

5.2.32 Determine whether the function

$$f(P) = \left(1+\frac{aP}{k}\right)^{-k}, \quad P \geq 0 \text{ and } a, k > 0$$

constants, is increasing or decreasing as a function, on $[0,\infty)$.

<u>Solution</u>: Check the sign of the derivative.

$$f'(P) = \frac{d}{dP}\left[\left(1+\frac{aP}{k}\right)^{-k}\right]$$

$$= -k\left(1+\frac{aP}{k}\right)^{-k-1} \cdot \frac{a}{k} = -a\left(1+\frac{aP}{k}\right)^{-(k+1)}$$

Here $a, k > 0$ are positive constants. Hence for $P \geq 0$, $1+\frac{aP}{k} > 0$ on $[0,\infty)$. Hence $\left(1+\frac{aP}{k}\right)^{-(k+1)} > 0$ on $[0,\infty)$, and so $-a\left(1+\frac{aP}{k}\right)^{-(k+1)} = f'(P) < 0$ on $[0,\infty)$. Hence $f(P)$ is decreasing on $[0,\infty)$. $\blacksquare$