4.6

HW 8
Solutions

14) Differentiate $f(x) = \frac{x-e^{-x}}{1+xe^{-x}}$

$$f'(x) = \frac{(1+e^{-x})(1+xe^{-x}) - (x-e^{-x})(e^{-x}-xe^{-x})}{(1+xe^{-x})^2}$$

$$= \frac{1+e^{-x}+xe^{-x}+xe^{-2x} - xe^{-x} - xe^{-2x} + e^{-2x} + x^2e^{-x}}{(1+xe^{-x})^2}$$

$$= \frac{1+e^{-x}+e^{-2x}+x^2e^{-x}}{(1+xe^{-x})^2}$$

70) Suppose $W(t)$ denotes the amount of a radioactive material at time t. It is modeled by the differential equation

$$\frac{dW}{dt} = -2W(t) \qquad W(0) = 15$$

a) How much material is left at time $t=2$?

b) What's the half life?

First we must solve the equation. The book gives us the solution but its an answer they got from thin air. This is how to solve it, we need integration

$$\frac{dW}{dt} = -2W$$

$$\frac{dw}{w} = -2dt$$ ← integrate both sides

$$\ln w = -2t + C$$

$$W(t) = Ke^{-2t}$$

$$W(0) = 15 = Ke^{-0} \Rightarrow K = 15$$

$$W(t) = 15e^{-2t}$$

Material left at $t=2$ $\qquad W(2) = 15e^{-4}$

Half life: $\frac{15}{2} = 15e^{-2t}$

$\frac{1}{2} = e^{-2t}$

$\ln\frac{1}{2} = e^{-2t}$ $\qquad t_{\frac{1}{2}} = \frac{\ln 2}{2} = \ln\sqrt{2}$

$\frac{-\ln 2}{-2} = t$

4.7

16) Let $f(x) = x^2 + \tan x$ find $\frac{d}{dx} f^{-1}(x)\Big|_{x=\frac{\pi^2}{16}+1}$

given $f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} + 1$

As $f\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} + 1$ $\quad f^{-1}\left(\frac{\pi^2}{16}+1\right) = \frac{\pi}{4}$

We need this to apply

$$\frac{d}{dx} f^{-1}(x)\Big|_{x=a} = \frac{1}{f'(f^{-1}(a))}$$

$f'(x) = 2x + \sec^2(x)$

$f'\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + \sec^2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} + 2$

So $\frac{d}{dx} f^{-1}(x)\Big|_{x=\frac{\pi^2}{16}+1} = \frac{1}{\frac{\pi}{2}+2}$

76) differentiate

$f(x) = \underbrace{e^{x-1}}_{g}\ \underbrace{\sin^2 x}_{h}\ \underbrace{(x^2+5)}_{j}{}^{2x}$

$g'(x) = e^{x-1}$

$h'(x) = 2\sin x \cos x$

$j'(x) = ?$ lets use log-diff ie

$\ln(j(x)) = 2x \ln(x^2+5)$

~~differentiating~~

next page

$$j(x) = (x^2+5)^{-2x}$$

$$\ln j = -2x \ln(x^2+5)$$

$$\frac{-1}{j} \cdot j' = 2\ln(x^2+5) + 2x \frac{1}{x^2+5} \cdot 2x$$

$$j'(x) = -\left[2\ln(x^2+5) + \frac{4x^2}{x^2+5}\right](x^2+5)^{-2x}$$

putting this all together with the product rule

$$f'(x) = e^{x-1}\sin^2 x\,(x^2+5)^{-2x} + e^{x-1}\,2\sin x\cos x\,(x^2+5)^{-2x} - e^{x-1}\sin^2 x\left[2\ln(x^2+5) + \frac{4x^2}{x^2+5}\right](x^2+5)^{-2x}$$

4.8

2) $\sqrt{35}$ let $f(x) = \sqrt{x}$ $a = 36$ $x = 35$

use $f(x) \approx f(a) + f'(a)(x-a)$

$f'(x) = \frac{1}{2\sqrt{x}}$ so

$$f(35) \approx \sqrt{36} + \frac{1}{2\sqrt{36}}(35-36) = \sqrt{36} - \frac{1}{2\sqrt{36}}$$

$$= 6 - \frac{1}{2 \cdot 6} = 6 - \frac{1}{12} = \frac{72}{12} - \frac{1}{12} = \frac{71}{12}$$