Solutions HW3

Problems 3.1/22, 50 3.2/10, 41, 46

3.1.22 Q: Use a table or graph to investigate

$$\lim_{x \to 3^+} \frac{1}{x-3}$$

A: We want to choose appropriate values to use in our table. These values should:

- Be greater than 3, as we are approaching from the right.
- Be close enough to 3 that the trend of the function as it approaches the limit is clear.

x	3.1	3.01	3.001	3.0001
$\frac{1}{x-3}$	10	100	1000	10000

In this example, it is clear that as x approaches 3, $\frac{1}{x-3}$ will grow without bound. This should also be clear from the calculation of your second row of the table. We are dividing 1 by the difference between x and 3, and as the distance gets smaller, $\frac{1}{x-3}$ continues to increase.

$$\boxed{\lim_{x \to 3^+} = +\infty, \text{ DNE}}$$

3.1.50 Q: Use the limit laws to evaluate

$$\lim_{x \to 1} \frac{(x-1)^2}{x^2-1}$$

A: Plugging in $x=1$ gives us an indeterminate

$$\frac{(1-1)^2}{1^2-1} = \frac{0^2}{0} = \frac{0}{0}$$

We can factor the denominator which is a difference of perfect squares

$$\lim_{x \to 1} \frac{(x-1)^2}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x-1}{x+1} \cdot \frac{x-1}{x-1}$$

We can use the limit law to separate the product if we know the limit $\lim_{x \to 1} \frac{x-1}{x-1}$ exists.

We know $\lim_{x \to 1} \frac{x-1}{x-1} = 1$ because even though $\frac{x-1}{x-1}$ isn't defined at 1, as we get arbitrarily close we are allowed to cancel the numerator and denominator at any other value of x

So

$$\lim_{x \to 1} \frac{(x-1)^2}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x-1}{x+1} \cdot \lim_{x \to 1} \frac{x-1}{x-1}$$

$$= \frac{0}{2} \cdot 1 = \boxed{0}$$

3.2.10 Q: Determine at which points $f(x) = \frac{1}{x^2-1}$ is discontinuous

A: A rational function is continuous everywhere it is defined, so we must find where f is undefined. This is where the denominator equals 0

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

$$x = -1 \quad\quad x = 1$$

$$\boxed{x = \pm 1}$$

3.2.41 Q: Find the limit

$$\lim_{x \to 0} \frac{e^{2x}-1}{e^x-1}$$

A: Plugging in 0 gives an Indeterminate

$$\frac{e^{2(0)}-1}{e^{(0)}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

So we can rewrite and factor the numerator

$$\frac{e^{2x}-1}{e^x-1} = \frac{(e^x)^2-1}{e^x-1} = \frac{(e^x+1)(e^x-1)}{e^x-1} = (e^x+1)\cdot\left(\frac{e^x-1}{e^x-1}\right)$$

$$\lim_{x \to 0} \frac{e^{2x}-1}{e^x-1} = \lim_{x \to 0} (e^x+1)\frac{e^x-1}{e^x-1} = \lim_{x \to 0} e^x+1 \cdot \lim_{x \to 0} \frac{e^x-1}{e^x-1}$$

$$= 2 \cdot 1 = \boxed{2}$$

3.2.46 Q: Find the limit

$$\lim_{x \to 0} \frac{5-\sqrt{25+x^2}}{2x^2}$$

A: Plugging in 0 gives an indeterminate

$$\frac{5-\sqrt{25}}{2 \cdot 0} = \frac{0}{0}$$

So to resolve this, we need to be able to cancel the x^2. To do that, we must multiply the numerator and denominator by the conjugate

$$\frac{5-\sqrt{25+x^2}}{2x^2} \cdot \frac{5+\sqrt{25+x^2}}{5+\sqrt{25+x^2}} \xrightarrow{\text{diff of squares}} = \frac{25-\left(\sqrt{25+x^2}\right)^2}{2x^2\left(5+\sqrt{25+x^2}\right)}$$

$$= \frac{25-25-x^2}{2x^2\left(5+\sqrt{25+x^2}\right)} = \frac{-x^2}{2x^2\left(5+\sqrt{25+x^2}\right)}$$

$$\lim_{x \to 0} \frac{5-\sqrt{25+x^2}}{2x^2} = \lim_{x \to 0} \frac{-x^2}{2x^2\left(5+\sqrt{25+x^2}\right)} = \lim_{x \to 0} \frac{-1}{2\left(5+\sqrt{25+x^2}\right)} \cdot \lim_{x \to 0} \frac{x^2}{x^2}$$

$$= \frac{-1}{2(5+5)} \cdot 1$$

$$= \boxed{\frac{-1}{20}}$$