Homework 2 Solutions

February 18, 2014

1 Problems from 2.1

8. Give a formula for $N(t)$ based on the information provided: $N_0 = 6$; population doubles every 40 minutes; one unit of time is 60 minutes.

   Answer: Note that if one unit of t is 60 minutes, then 40 minutes is $2t/3$. So, when t reaches $2/3$, we want our population to have doubled. How can model this sort of behavior with a function? We want that $120 = 60c^{2/3}$. In other words, after $2/3$ of a unit of time has passed, our original population of 60 has doubled. Solving this equation, we see that $c = 2^{3/2}$. Thus our formula for $N(t)$ should be

   $$N(t) = 60 \cdot 2^{3t/2}.$$

56. Write N_t as a function of t for the recursion: $N_{t+1} = N_t/3$ with $N_0 = 3500$.

   Answer: If we consider the first few values of this recursion, it should become clear how to write it as a function of t. One point of advice - it might be helpful to avoid simplifying any of the values of N_t at this stage, since that will make the pattern clearer. So,

   $$N_0 = 3500 \quad (1)$$
   $$N_1 = \frac{3500}{3} \quad (2)$$
   $$N_2 = \frac{3500}{3^2} \quad (3)$$
   $$N_3 = \frac{3500}{3^3} \quad (4)$$

   So it should be clear that the recursion can be written as

   $$N(t) = \frac{3500}{3^t}.$$

2 Problems from 2.2

20. Find the next four values of the sequence:

   $$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \ldots$$

Answer: By examining the given values of the sequence, we'll try to find a pattern. The most obvious pattern is that the sequence is alternating. So we know that there's probably a $(-1)^n$ in there somewhere. The next most obvious thing is that the denominators are all perfect squares. What's more, the denominator of the n^{th} term is n^2. This tells us that the denominator is given by n^2. Putting these two facts together, and noticing that there's nothing else going on in this sequence, we can guess that a general form for the sequence is

$$a_n = \frac{-1^n}{n^2}.$$

So, we can determine that the next four terms should be:

$$\frac{1}{36}, -\frac{1}{49}, \frac{1}{64}, -\frac{1}{81}.$$

58. Find the limit a of the sequence $a_n = \frac{1}{\sqrt{n}}$, and determine N such that whenever $n > N$, $|a_n - a| < 0.05$.

Answer: First, a few observations: it might help to notice that $0.05 = 1/20$; also, note that $\sqrt{n}$ is increasing, so $1/\sqrt{n}$ is decreasing; and lastly, $\sqrt{n}$ grows without bound (i.e. "goes to infinity") as n goes to infinity. Since $\sqrt{n}$ just keeps getting bigger and bigger, we can probably guess right off the bat that $a = 0$, and that guess would be correct. Now, let's try to find an N such that whenever $n > N$,

$$|a_n - a| = |a_n - 0| = a_n < 1/20.$$

Now we want to use our observation that $1/\sqrt{n}$ is decreasing. What this means is that as soon as we have a single n such that $a_n < 1/20$, we're home free, since a_n can only get *more* less than $1/20$ after that. So, let's start with what we want and work backwards. What we want is

$$\frac{1}{\sqrt{n}} < \frac{1}{20}.$$

Then we can cross-multiply to see that this requires

$$20 < \sqrt{n}.$$

So, let's choose $N = 20^2 = 400$. Then we can check that if $n > N = 20^2$, we know that

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{20^2}} = \frac{1}{20}.$$

So, indeed, $N = 400$ works.

103. Find all fixed points of the recursion

$$a_{n+1} = \frac{1}{2}(a_n + 5)$$

with $a_0 = 1$.

Answer: For a number c to be a fixed point of the recursion, it means that $c = a_n = a_{n+1}$, since the recursion, once it gets to the fixed point, can't leave it (that's why it's called a fixed point!). So, we solve $a_n = a_{n+1}$. Or, more concretely, we do the following:

$$\begin{aligned}
&\Rightarrow a_n = \frac{1}{2}(a_n + 5) \\
&\Rightarrow 2a_n = (a_n + 5) \\
&\Rightarrow 2a_n - a_n = 5 \\
&\Rightarrow a_n = 5.
\end{aligned}$$

So the only fixed point is 5. Now, let's see if it looks like our recursion tends to 5 as n increases (it may help to use a calculator for the higher values):

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 3 \\
a_2 &= 4 \\
a_3 &= 4.5 \\
a_4 &= 4.75 \\
a_5 &= 4.875 \\
a_6 &= 4.9375 \\
a_7 &= 4.96875 \\
a_8 &= 4.984375 \\
a_9 &= 4.9921875.
\end{aligned}$$

So it does indeed seem that our recursion tends to 5.