Differentiability implies Continuity

Here is a theorem that we talked about in class but never really explored:

Thm If $f(x)$ is differentiable at $x=c$, then $f(x)$ is continuous at $x=c$.

pf. Since $f(x)$ is diff at $x=c$, we know

$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. So what can we say about $\lim_{x\to c} f(x)-f(c)$? If $f(x)$ is cont at $x=c$, this limit should be 0:

$$\begin{aligned}
\lim_{x\to c} f(x) - f(c) &= \lim_{x\to c} (f(x)-f(c))\left(\frac{x-c}{x-c}\right) \\
&= \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)(x-c) \\
&\overset{\text{prod rule}}{=} \underbrace{\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right)}_{f'(c)} \underbrace{\lim_{x\to c}(x-c)}_{0} = 0
\end{aligned}$$

Hence $\lim_{x\to c} f(x) - f(c) = 0$ so that $\lim_{x\to c} f(x) = f(c)$

and $f(x)$ is cont. at $x=c$. ∎