Grothendieck Duality for Derived Stacks

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Abstract

This is an (very) incomplete draft of a sketch of ideas behind duality theory of derived stacks, or ∞-topoi modeled as category of sheaves of spaces/spectra over a simplicial site. We also to draw out details of higher Koszul complex and local duality on ringed ∞-topoi.

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1 Ringed Topoi

A *topos* X is a category equivalent to a category equivalent to a category of sheaves of sets on a Grothendieck site $\mathcal{C}$ (which is called a *defining site* for X). Here are a few prperties of topoi

1. A topos X admits finite projective limits, in particular has a final object 1_X, and it admits fibered products.
2. If $(U_i)_{i\in I}$ is a family of objects in X, the "disjoint union" $\coprod_{i\in I} U_i$ exists in X, and commutes with base change
3. Quotients by equivalence relations exist and have the same good properties as in the category of sets.

A theorem of Giraud states that the converse of this is essentially true, i.e. a category T satisfying a these three conditions and some more smallness conditions is a topos.

A continous map of topological spaces $f : X \to Y$ defines a pair of adjoint functors (f^*, f_*) between categories of sheaves of sets on X and Y. The *inverse image* functor f^*, being a left adjoint, commutes with inductive limits. It also commutes with finite projective limits. Now if X and Y are Grothedieck topoi one defines a it geometric morphism $f : X \to Y$ a pair of adjoint functors $(f^* \dashv f_*) : X \to Y$, so that f^* preserves finite limits. Let $\{pt\}$ denote the *point topos*; this is simply the category of sets. Upto a unique isomorphism of functors there is only one geometric morphism from any topos to $\{pt\}$. The direct image functor for the unique morphism $f : X \to \{pt\}$ associates to every object $E \in X$ its "global sections" $\Gamma(X, E)$. The global sections of E, $f_*(E) \simeq \mathrm{Hom}(1_X, E)$, where 1_X is the final object of X. For every $I \in \mathrm{Ob}(Sets)$, i.e. an object in the point topos, and the (unique) morphism $f : X \to \{pt\}$, $f^*(I) \simeq 1_X \times I$. This is called the *constant sheaf* with value I over the topos X.

A *point* of a topos is a geometric morphism $\{pt\} \to \mathfrak{X}$. If $\mathfrak{X}$ is defined by a topological space X, a usual point x of X gives a point of $\mathfrak{X}$, whose inverse image functor is the stalk functor $E \to E_x$.

Consider a Grothendieck topos X which is equivalent to the category of sheaves of sets over a small site $\mathcal{C}$. By a (discrete) commutative ringed valued sheaf on X we mean a functor $X^{opp} \to$ CRings which is left exact. If X is equivalent to the category of sheaves of sets on a a topological space X equipped with a topology of coverings by opens the functor $Top(X)^{opp} \to$ CRings gives a presheaf of commutative rings on X. This can be seen by making use of the Yoneda embedding $\mathrm{Cov}(X)^{opp} \to PSh(X)^{opp} \to Shv(X)^{opp} \to$ CRings, the second functor being sheafification. The condition making any functor $X^{opp} \to$ CRings a sheaf is that it takes limits to limits, i.e. the functor is left exact. Since the the Yoneda embedding is right adjoint it preserves limits. Now if we assume that the toplogy on X is canonical, the the only condition making $X^{opp} \to$ CRings a sheaf on X is that this functor is left exact. This justifies the equivalence between sheaves of commutative rings on a topos X and left exact functors from X^{opp} to CRing. Thus a ringed topos is a topos X equipped with a sheaf of commutative rings A in the above sense. Another way to think of a ringed topos is that it is a topos X with a commutative ring object A in it. This is only a generalization of a *ringed space* in the sense that it allows the use of things more general than spaces. If $(f^* \dashv f_*) : X \to Y$ is a geometric morphism of topoi and A a ring object in Y, since f preserves finite limits, $f^*(A)$ is a ring object in X. Ringed topoi form a category where a morphism $(X, A) \to (Y, B)$ is a geometric morphism $(f^*, f_*) : X \to Y$ of topoi together with maps of sheaves of rings $B \to f_*A$. The final object in the category of ringed topos must be the point topos with sheaf of commutative rings over it. Since the a commutative ring object in the category of sets is simply a commutative ring in the ordinary sense, a ringed point is simply the point topos $\{pt\}$ together with a commutative ring. It can be shown that upto a unique isomorphism of funtors there is only one morphism from any ringed topos to $(\{pt\}, \mathbb{Z})$.

Let (X, A) be a ringed topos. A *A-module* is an object F on X so that for every $U \in X$, $F(U)$ is a $A(U)$-module, and for every $V \to U$ in the topos, the map $F(U) \to F(V)$ is compatible with the module structures via the ring homomorphism $A(U) \to A(V)$. In other words, a *A-module* is a module object in X over the commutative ring object A. The category of A-modules is an abelian category. The commutative group of morphism between A-modules M and N is denoted $\mathrm{Hom}_A(M, N)$.

For two A-modules F and G define the *tensor product* $F \otimes_A G$ to be simply the fiber product object in the topos X (Recall that any topos is closed under finite projective limits). The module homomorphims $Hom_A(F, G)$ form a A-module. On the point topos $\{pt\}$ these constructions are the usual tensor product and Hom of modules over a ring. Given a morphism of ringed topoi $(f^{-1}, f_*) : (X, A) \to (Y, B)$, then $f_*(F)$ is a B-module via the morphisms $B \to f_*A \to f_*F$. For a B-module G, the inverse image functor f^{-1} gives a A-module $f^*G = f^{-1}G \otimes_{f^{-1}B} A$. This gives an adjoint pair of functors $(f^* \dashv f_*) : (X, A)$-modules $\to (Y, B)$-modules. The category of modules over A, (X, A)-modules, is an abelian category. For two a pair of A-modules M and N, there is an abelian group of A-module maps $\mathrm{Hom}_A(M, N)$. The direct image functor for the unique morphism $(X, A) \to (\{pt\}, \mathbb{Z})$ associates to to every A-module F its global sections $\Gamma(X, F) = \mathrm{Hom}_A(1_X, F)$.

Examples of ringed topoi arise in algebraic geometry. We demonstrate by means of the following examples.

1.1 Schemes

A scheme is an example of a ringed topos where the topos is defined by a topological space. A *locally ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$, such that stalk of the sheaf at any point in $x \in X$ is a local ring; denoted by $\mathcal{O}_{(x,X)}$. A *map* of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of topological spaces such that for every point $x \in X$ the induced map on stalks $\mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$ is a map of local rings. For a ring R, $\mathrm{Spec}R$ is a locally ringed space. This defines a full and faithful functor from commutative rings to locally ringed spaces. The image of this functor is the category of *affine schemes*.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that for each point $p \in X$, there is an open subset $U \subset X$ containing U such that $(U, \mathcal{O}_U)$ is an affine scheme.

If a ringed topos is defined by a ringed space, the notion of a module defined over a ringed topos gives the usual notion of a module over a ringed space. In case of sheme X, a special class of $\mathcal{O}_X$-modules is quasi-coherent modules. A module M over R gives rise to a sheaf of modules $\tilde{M}$ over the structure sheaf on $\mathrm{Spec}R$, by taking $\Gamma(\mathrm{Spec}R_f, \tilde{M}) = M \otimes_R R_f$. A module over $\mathrm{Spec}R$ is *quasi-coherent* if it is of the form $\tilde{M}$ for some R module M. For a scheme X, a $\mathcal{O}_X$-module F is quasi-coherent if for every point $p \in X$ there is an open $U \in X$, with $p \in u$, and an exact sequence $\mathcal{O}_X^I \to \mathcal{O}_X^J \to F|_U \to O$ of $\mathcal{O}_X$-modules.

The category of schemes has Grothendieck topologies; Zariski, étale or flat. In any of these topologies there is an full and faithful functor $\text{Schemes}^{opp} \to \text{Sheaves}$ which takes a scheme X to $\dot{X} = \text{Hom}(-, X) \to \text{Sets}$.

If X is a scheme and $(U_i \to X)$ is an open covering of X by affine schemes, then X is completely described by giving just th e family $\{U_i\}$ and the "glueing data", the open subsets $V_{ij} = U_i \cap U_j$ of X and the immersions $V_{ij} \to U_i$ and $V_{ij} \to U_j$. We can write U for the disjoint union of the U_i's. Noting that $V_{ij} = U_i \times_X U_j$, we can write $R = U \times_X U$ for the disjoint union of V_{ij}'s. There is a cononical injection $R \to U \times U$ which identifies R as an equivalence relation on U. Let π_1 and π_2 be the two projections $R \rightrightarrows U$ and $\pi : U \to X$ the covering map. Then in the diagram

$$R \rightrightarrows U \to X$$

$U \to X$ is the coequalizer in the category of schemes. Since R and U are affine schemes X is described as an equivalence relation of affine schemes. But one should be careful here, the equivalence relation $R \rightrightarrows U$ may already have a quotient in affine schemes, but it may be the wrong quotient. The quotient of a diagram of schemes $R \rightrightarrows U$ is taken in the category of locally ringed spaces, so that if X is the quotient, then $\dot{X}$ is a sheaf of affine schemes in the Zariski topology and $\dot{X}$ is a the quotient, in the category of sheaves, of the equivalence relation $\dot{R} \rightrightarrows \dot{U}$.

The point of all this is that all the constructions for a scheme can be done locally on U modulo gluing data on R. At this point we might replace the space underlying a scheme by a topos. For an affine scheme $\text{Spec}A$ the underlying topos is the category of sheaves on the Zariski site over $\text{Spec}A$; call this the Zariski topos of A. As a topos this is defined by the topological space which is the prime spectrum of A. For an abritrary scheme given by a Zariski equivalence relation $R \rightrightarrows U$ the underlying topos is the category of sheaves of sets on the Zariski site over U with descent data coming from the equivalence relation. More precisely, sheaves $\mathcal{F}$ on U for which $\pi_1^{-1}\mathcal{F} \simeq \pi_2^{-1}\mathcal{F}$ as sheaves of sets on the Zariski site over R. This topos is equivalent to the topos defined by the space X. The structure sheaf $\mathcal{O}_X$ is commuative ring object in this topos. This is an example of a topos with enough points. The structure sheaf $\mathcal{O}_X$ has the special property that for every point $x : \{pt\} \to X$, the inverse image functor (stalk) on the commutative ring object $\mathcal{O}_X$ is local ring. Let us call such a ringed topos (with enough points) a *locally ringed topos*. We can now define a scheme to be a locally ringed topos which is locally isomorphic to $(\text{Spec}A, \mathcal{O}_A)$, where $\text{Spec}A$ is the Zariski topos of A.

1.2 Algebraic Spaces

An equivalence relation $R \rightrightarrows U$ is *étale* if π_1 and π_2 are étale maps of schemes and $\dot{R} \rightrightarrows \dot{U} \times \dot{U}$ is an injection of sheaves. A map of schemes $U \to X$ is an *étale quotient* of $R \rightrightarrows U$ if $\dot{U} \to \dot{X}$ is the sheaf cokernel of $\dot{R} \rightrightarrows \dot{U}$ and $R = U \times_X U$. The sheaf cokernel $\dot{R} \rightrightarrows \dot{U} \to F$ however need not be representable by any scheme. The category of algebraic spaces is in a sense the closure of the category of schemes under the operation of taking quotients under étale equivalence relations.

An *algebraic space* is a functor $A : \text{Schemes}^{opp} \to \text{Sets}$ such that a) A is a sheaf in the étale topology of schemes b) There exists a scheme U and a representable map of sheaves $\dot{U} \to A$ which is étale and surjective, and c) the map of schemes inducing $\dot{U} \times_A \dot{U} \to \dot{U} \times \dot{U}$ is quasi-compact.

Algebraic spaces are the same as étale equivalence relations on schemes. Let A be an algebraic space, U a scheme, and $\dot{U} \to A$ a representable étale covering. Let R be the scheme representing $\dot{U} \times_A \dot{U}$. Then $R \rightrightarrows U$ is an étale equivalence relation of schemes and A is the categorical quotient of $\dot{R} \rightrightarrows \dot{U}$. Conversely, let $R \rightrightarrows U$ be an arbitrary étale equivalence relation in the category of schemes. Suppose $R \to U \times U$ is quasi-compact. Then there is an algebraic space, unique upto unique isomorphism, and a map of sheaves $\dot{U} \to A$ satisfying part (b) of the definition given above, with $\dot{R} = \dot{U} \times_A \dot{U}$.

The local étale site $\text{Et}(A)$ on an algebraic space A. The objects are representable étale morphisms $u : \dot{U} \to A$ where U is a scheme. A map from (u, U) to (v, V) consists of a map of schemes $\phi : U \to V$ and an isomorphism $u \to v \circ \dot{\phi}$. The topology on this category is generated by the pretopology of coverings $\text{Cov}(u, U)$. This is the set of family of maps $((u_i, U_i) \to (u, U))$ such that the morphism of schemes $\coprod_i : U_i \to U$ is étale and surjective.

The category of sheaves of sets on the site $\acute{E}t(A)$ is the topos denoted by $A_{\acute{e}t}$. We can define the site $\underline{\acute{E}t}(A)$ of affine étale coverings of A in a similar way. The topos defined by $\underline{\acute{E}t}(A)$ is equivalent to $A_{\acute{e}t}$.

There is another way to approach the topos underlying an algebraic space, which is more useful for our purposes. Let $p : \dot{X} \to A$ be an étale presentation for the algebraic space A. We can associate a *simplicial scheme* $X_\bullet : \Delta^{opp} \to$ (Schemes). For every object $[n] \in \text{ob}\Delta$, X_n is the scheme representing $(\dot{X}/A)^{[n]}$, the fiber product over A of $(n+1)$ copies of $\dot{X}$. For every $i \in [n]$, the injection $[n-1] \hookrightarrow [n]$ induces the projection $p_{n,i} : (\dot{X}/A)^{[n]} \to (\dot{X}/A)^{[n-1]}$, and the surjection $[n] \twoheadrightarrow [n-1]$, which skips i, induces the diagonal map $d_{n,i} : (\dot{X}/A)^{[n-1]} \to (\dot{X}/A)^{[n]}$. There is a canonical morphism $P_\bullet : X_\bullet \to A$ comprised of the canonical projections $\dot{X}_n \to A$. The *étale topos* $(X_\bullet)_{\acute{e}t}$ is defined as follows. A *simplicial étale sheaf* of sets over $X_\bullet$ is a collection of sheaves of sets $\mathcal{F}_\bullet = (\mathcal{F}_n)_{n\geq 0}$ over the étale site of X_n along with the transition maps $\mathcal{F}_{n'} \to X_\bullet(\delta)_*\mathcal{F}_n$ and $X_\bullet(\delta)^{-1}\mathcal{F}_{n'} \to \mathcal{F}_n$, for every map $\delta : [n'] \to [n]$ in Δ. These transition functions satisfy the evident cocycle conditions. An étale simplicial sheaf over $X_\bullet$ is *cartesian* the second transition function is always an isomorphism. For any algebraic space A with an étale presentation $\dot{X} \to A$, there is an equivalence of topoi between $A_{\acute{e}t}$ and $(X_\bullet)_{\acute{e}t}$.

An *étale simplicial sheaf of rings* $\mathcal{A}_\bullet$ over $X_\bullet$ is a ring object in the topos $(X_\bullet)_{\acute{e}t}$. In other words $\mathcal{A}_\bullet$ is a collection of sheaves of rings $\mathcal{A}_n$ over the local étale site over schemes X_n.The transition map are étale morphism of rings $X_\bullet(\delta)_* : \mathcal{A}_n \to \mathcal{A}_{n'}$ for any map $\delta : [n] \to [n']$ in Δ. $\mathcal{A}_\bullet$ is *flat* if the transition function if for every n and i with $0 \leq i \leq n$, the morphism of sheaves of rings on the local étale site over X_n, $p_{n,i}^{-1} : \mathcal{A}_{n-1} \to \mathcal{A}_n$ is flat. The ring object $\mathcal{O}_A$ in the topos $(X_\bullet)_{\acute{e}t}$ is obtained by setting $\mathcal{A}_n = \mathcal{O}_{X_n}$, the structure sheaf over the local étale site over X_n. The transition maps are evident. $\mathcal{O}_A$ is the structure sheaf of the algebraic space A.

1.3 Deligne-Mumford stacks

Schemes are defined as locally ringed spaces that are locally isomorphic, as locally ringed spaces to affine schemes. Schemes can be also as viewed equivalently as solutions to moduli problems. Therefore is a functor from commutative rings to sets which are sheaves in the Zariski topology of affine schemes and have an open cover, by functors which are represetable by affines, i.e of the form $A \mapsto \text{CRings}(R, A)$. A stack is a generalization of this defintion of a scheme. A *stack* is a category fibered in groupoids over schemes which satisfy effective descent condition. In this sense a stacks are solutions to moduli problems which take values in groupoids rather than sets. The effective descent condition on the category fibered in Groupoids can be encoded by the notion of a *sheaf of groupoids* over commutative rings. The category of presheaves of groupoids over a site has a model structure. The fibrant objects are the sheaves.

A morphism of stacks $\mathcal{M} \to \mathcal{N}$ is *representable* if for any scheme Y and a morphism $\dot{Y} \to \mathcal{N}$ the 2-category fiber product $\mathcal{M} \times_Y \mathcal{N}$ in groupoid fibered category is representable by scheme.

An *algebraic stack* is a functor $\mathcal{X} : \text{Schemes}^{opp} \to \text{Groupoid}$ so that a) $\mathcal{X}$ is a sheaf, or equivalently, satisfies effective descent condition as a cateory fibered in groupoids over Schemes, b) every morphism from a scheme $\dot{Y} \to \mathcal{X}$ is representable and c) there is a scheme X and a reprentable morphism $\dot{X} \to \mathcal{X}$ which is smooth and surjective. This map provides a presentation of the stack. An algebraic stack is called *Deligne-Mumford* if the presentation is an étale surjection.

In analogy with algebraic spaces, Deligne-Mumford stacks are equivalent to *étale groupoid relations* on schemes. Given a DM stack with an étale presentation $X \to \mathcal{X}$, let Y be the scheme representing $X \times_{\mathcal{X}} X$. Then $\mathcal{X}$ is equivalent to the quotient stack $[Y \rightrightarrows X]$. Conversely, let $Y \rightrightarrows X$ be an étale groupoid relation on schemes then there is an algebriac stack, unique upto unique isomorphism, with an étale presentation $\dot{X} \to \mathcal{X}$ with $\dot{Y} = \dot{X} \times_{\mathcal{X}} \dot{X}$. This observation enables us to give a more geometric description of a DM stack in terms of a sheaf of commutative rings over a topos.

A Deligne Mumford stack is a ringed topos $(X, \mathcal{O}_X)$ which is locally isomorphic to $(\text{Spec}A, \mathcal{O}_{\text{Spec}A})$ where $\text{Spec}A$ is the étale topos of the ring A and $\mathcal{O}_A$ it's structure sheaf of rings.

Let $A_\bullet$ be an *étale cosimplicial ring.* That is $A_\bullet : \Delta \to \text{Comm Rings}_{\acute{e}t}$ is a functor from the simplicial category to the category of commutative rings with étale maps as the only morphisms. Consider the category $\acute{\text{E}}\text{t}(A_\bullet)$ whose objects are couples $([n], A_n \to B)$, denoted by (n, B), where $[n] \in \text{ob}\Delta$ and $A_n \to B$ is an étale map of commutative rings. A morphism between couples (n, B) and (n', B') is a couples (δ, ϕ) where $\delta : [n] \to [n']$ is an arrow in the category Δ and $\phi : B \to B'$ is an arrow in $\text{Comm Rings}_{\acute{e}t}$ under the étale morphism $A_\bullet(\delta) : A_n \to A_{n'}$ induced by δ.

The category $\acute{\text{E}}\text{t}(A_\bullet)$ can be given a topology generated by the family of covering maps $(\ (n, B) \xrightarrow{(\text{id},\phi_i)} (n, B_i)\ )_{i\in I}$ where $(\phi_i : B \to B_i)_{i\in I}$ is a covering family in the étale topology over A_n.

The topos $(A_\bullet)_{\acute{e}t}$ is the category of sheaves of sets over the site just defined. The objects in this topos are called *simplicial étale sheaves over* $A_\bullet$.

Now we can define a Deligne-Mumford stack as a ringed topos $(X, \mathcal{O}_X)$ which is locally isomorphic to $(\text{Spec}A, \mathcal{O}_A)$, where $\text{Spec}A$ denotes the étale topos of A and $\mathcal{O}_A$ it's canonical sheaf of rings. More precisely, there exists an étale cosimplicial ring $A_\bullet$ so that there is an equivalence of topoi $X \simeq (A_\bullet)_{\acute{e}t}$, with $A = A_0$.

2 Brave new topoi

2.1 Homotopical sheaf theory

We want to consider the problem of defining the notion of a sheaf over a Grothendieck site that takes values in a model category, or a category that has homotopy types associated to it's objects. Let us denote by $\mathcal{S}$ such an category. Some examples are simplicial sets, chain complex of modules over acommutative ring and the category of spectra (S-modules, symmetric spectra). We shall call objects of such categories *spaces*.

There is a model category structure on presheaves of spaces on any small site discovered by Jardine and Joyal. This makes use of the model structure on the category of spectra as well as the topology of the site under consideration. Given a small category $\mathcal{C}$ the *global injective model structure* on the category of diagrams of spaces $\mathcal{S}^{\mathcal{C}^{opp}}$ in which the cofibrations and weak equivalances are a given objectwise. If $\mathcal{C}$ is equipped with a Gothendieck topology, i.e. $\mathcal{C}$ is a small site, this gives a model structure on the category of presheaves of spaces $\mathcal{S}^{\mathcal{C}^{opp}}_{inj,global}$. However this is not very interesting as it does take the topology of the site $\mathcal{C}$ into account. The *local injective model structure* on the S-module$^{\mathcal{C}^{opp}}$ is the Bousfield localization of the global injective model structure along *local weak equivalences* of spectral presheaves. A morphism $\mathcal{F} \to \mathcal{G}$ of spectral presheaves is a local weak equivalence if it induces an isomorphism of all sheaves of homotopy groups. If the site has enough points this is equivalent to saying that the map induces a weak equivalence of stalks $\mathcal{F}_x \simeq \mathcal{G}_x$ in the model category $\mathcal{S}$. The fibrant objects in this local injective model structure on S-modules$^{\mathcal{C}^{opp}}$ are the those presheaves which are

1. fibrant in the global injective model structure;

2. satisfy descent for all *hypercovers*.

The *Čech injective model structure* on presheaves is obtained from the global injective model structure by Bousfield localizing along *Čech covering maps*. Given a covering family $(U_i \to V)_{i \in I}$ in $\mathcal{C}$, let $U_\bullet$ denote the Čech nerve of the covering family. For a presheaf $\mathcal{F}$ we can associate the Čech presheaf $\check{C}\mathcal{F}$ so that $\check{C}\mathcal{F}(V) \simeq \mathrm{holim}_\Delta \mathcal{F}(U_\bullet)$. A Čech cover morphism is the natural morphism of the form $\check{C}\mathcal{F} \to \mathcal{F}$ for any spectral presheaf $\mathcal{F}$. The injective Čech model category obtaoned by Bousfield localizing the the global injective model category along Čech cover morphism is midway towards the full local injective model category in the sense that the local model structure is the Bousfield localization of the global model structure along all *hypercover morphims* and not just Čech cover morphisms. The fibrant objects in the injective Čech model structure are those presheaves which are

1. fibrant in the global injective model structure;

2. satisty descent for all *Čech covers*.

The fibrant objects in the local injective model structure on spectral presheaves give us the right notion of *spectral sheaves*. However working with hyperdescent is cumbersome. This can be avoided by working with sites where hypercovers are equivalent to Čech covers. In such a situation, there is a Quillen equivalence between the local injective model structure and the Čech injective model structure on spectral presheaves over $\mathcal{C}$. All the sites we consider in this paper are of this nature. Let us call such a site a *Verdier site*(needs definition..). Now we can characterize spectral sheaves, which are the fibrant objects in Jardine-Joyal local injective model structure, in terms of Čech descent.

If the topos is defined by a small Verdier site $\mathcal{C}$, a sheaf of E_∞-rings is a presheaf $\mathcal{F} : \mathcal{C}^{opp} \to E_\infty$-rings taking values in fibrant S-modules and satisfying Čech descent: given a covering $(U_i \to U)_{i \in I}$ in site $\mathcal{C}$ there is a weak equivalence (in the model structure on S-modules) $\mathcal{F}(U) \simeq \mathrm{holim}_\Delta \mathcal{F}(U_\bullet)$. Here $U_\bullet$ is the Čech nerve of the covering $(U_i \to U)_{i \in I}$.

2.2 ∞-topoi

Let $\mathcal{C}$ be a small category enriched over simplicial sets, equipped with a Grothendieck topology on the homotopy category $\mathrm{Ho}(\mathcal{C})$. Let us such a category a *simplicial site*. An ∞-topos $\mathfrak{X}$ is category equivalent to the category of (pre)sheaves of *spaces* on $\mathcal{C}$. By "spaces" we mean objects of a $(\infty, 1)$-category Spaces, for example, topological spaces, simplicial sets, S-modules etc. The category of presheaves of spaces on $\mathcal{C}$ is the category of (enriched) functors. We shall denote this category by $[\mathcal{C}^{opp}, \mathrm{SSet}]$. This a full simplicial subcategory of the category of functors $\mathrm{SSet}^{\mathcal{C}^{opp}}$.

An $(\infty,1)$-category $\mathfrak{C}$ is *presentable* if there is a simplicial model category whose homotopy coherent nerve is the the ∞ category $\mathfrak{C}$ (as a weak Kan complex). All the infinity categories we consider are presentable.

We can use the model structure on the presentation of the infinity category of spaces to define the notion of sheaves of spaces. The model structure on the category of spaces induces a model structure on the underlying ordinary category of the functor category $[\mathcal{C}^{opp}, \mathrm{SSet}]$ where the cofibrations and weak equivalences are defined objectwise. This is the global injective model structure on $[\mathcal{C}^{opp}, \mathrm{SSet}]$. The simplicial nerve of the fibrant-cofibrant objects in $[\mathcal{C}^{opp}, \mathrm{SSet}]_{inj,global}$ is a model for an ∞-topos.

In order to define sheaves of simplicial sets on the simplicial site we need to define a model structure on the enriched functor category that take the topology of the site into account. The *local* injective model structure on the ordinary category underlying $[\mathcal{C}^{opp}, \mathrm{SSet}]$ is defined as the left Bousfield localization of the global injective model structure along *hypercovering maps* so that the cofibratons are exactly the cofibrations in the global injective model structure and the weak equivalences are the *local weak equivalecens.* A morphism of presheaves of spaces is a local weak equivalence if it induces an isomorphism of all presheaves of homotopy groups.

Given $x \in \mathrm{Ob}(\mathcal{C})$, the induced topology on the over category $\mathrm{Ho}(\mathcal{C})/x$ and the canonical functor $\mathrm{Ho}(\mathcal{C}/x) \to \mathrm{Ho}(\mathcal{C})/x$ defines a Grothendieck topology on $\mathrm{Ho}(\mathcal{C}/x)$. For $n \geq 0$, $\pi_n(F)$ is a sheaf on the site $\mathrm{Ho}(\mathcal{C})$. A morphism $f : F \to G \in [\mathcal{C}^{opp}, \mathrm{Sset}]_{inj,global}$ is a local weak equivalence if

1. $\pi_0 F \to \pi_0 G$ is an isomorphism of sheaves over $\mathrm{Ho}(\mathcal{C})$.
2. $\pi_n(F, s) \to \pi_n(G, f(s))$ is n isomorphism of sheaves over $\mathrm{Ho}(\mathcal{C}/x)$ for $s \in \pi_0 F(x)$

Denote this model category by $[\mathcal{C}^{opp}, \mathrm{SSet}]_{inj,local}$. The fibrant objects are precisly the sheaves on the simplicial site $\mathcal{C}$ taking values in simplicial sets. The simplical model category $\mathfrak{X} = [\mathcal{C}^{opp}, \mathrm{SSet}]^{cf}_{inj,local}$ is also a model for a ∞-topos $\mathfrak{X}$. The category $\mathcal{C}$ is called a *defining site.*

A continous functor of simplicial sites $f : \mathcal{C} \to \mathcal{D}$ induces a pair of adjoint functors (f^*, f_*) between the catgory of sheaves of spaces on $\mathcal{C}$ and $\mathcal{D}$. The inverse image functor f^*, being a left adjoint, commutes with homotopy colimits. It also commutes with finite homotopy inverse limits (why?). Now if $\mathfrak{X}$ and $\mathfrak{Y}$ are ∞-topoi ($\mathfrak{X}$ and $\mathfrak{Y}$ are topological model categories), define a *geometric morphism* $f : \mathfrak{X} \to \mathfrak{Y}$ to be a pair of Quillen adjoint functors $(f^* \dashv f_*)$ between the simplicial model categories $\mathfrak{X}$ and $\mathfrak{Y}$ such that f^* commutes with finite homotopy limits.

The *point* ∞-*topos* $\{pt\}$ is simply the category Spaces(for example the category of simplicial sets). Given a ∞-topos $\mathfrak{X}$, there is a unique geometric morphism $\mathfrak{X} \to \{pt\}$ in the sense that the $(\infty, 0)$ category of geometric morphisms from $\mathfrak{X}$ to $\{pt\}$ is contractable (as a Kan complex). Any ∞-topos $\mathfrak{X}$ admits a final object $1_{\mathfrak{X}}$. This is a final object in the sense that that if we use a topologial model category to 'model' the $(\infty, 1)$ category $\mathfrak{X}$, the hom space $\mathrm{Hom}_{\mathfrak{X}}(E, 1_{\mathfrak{X}})$ is contractible. The direct image functor associated to the unique upto homotopy functor $f : \mathfrak{X} \to \{pt\}$, $f_*(E)$ gives the homotopy "global sections" of the object $E \in \mathfrak{X}$. This can also be identified with the derived hom-space $\mathrm{RHom}_{\mathfrak{X}}(1_{\mathfrak{X}}, E)$ in $\mathfrak{X}$. The inverse image functor $f^*(I)$ is the homotopy constant sheaf associated to the space $I \in$ Spaces. $f^*(I) \simeq 1_{\mathfrak{X}} \times_{\mathfrak{X}} I$.

Ringed ∞-topoi. A ringed ∞-topos is a pair $(\mathfrak{X}, \mathcal{A})$, where $\mathfrak{X}$ is ∞-topos and $\mathcal{A}$ is a *homotopy commutative* ring object in $\mathfrak{X}$ (so that $\mathcal{A}$ is a strictly commutative ring object in $\mathrm{ho}(\mathfrak{X})$). Precisely, $\mathcal{A}$ is a fibrant-cofibrant object in $\mathfrak{X}$ with a choice of commutative ring structure given by an element $f \in \mathrm{RHom}_{SbOp}(E^2_\infty, \mathcal{E}^2_{\mathcal{A}})$. Here $SbOp$ denotes the category of simplicial bi-operads, E_∞ denotes the commutative monoid operad and $\mathcal{E}_{\mathcal{A}}$ is the endomorphism operad of $\mathcal{A}$. The little-cubes operad can be used as a model for a cofibrant replacement of E_∞.

Given a geometric morphim of ∞-topoi $f : \mathfrak{X} \to \mathfrak{Y}$, and homotopy commutative ring objects $(\mathcal{A}, i)$ in $\mathfrak{X}$ and $(\mathcal{B}, j)$ in $\mathfrak{Y}$. Since f^* preserves finite homotopy limits there is a canonical homotopy commutative ring structure on $f^*\mathcal{B} \in \mathfrak{X}$ via the maps $j : E_\infty \to \mathcal{E}_{\mathcal{B}} \to f^*\mathcal{E}_{\mathcal{B}} \simeq \mathcal{E}_{f^*\mathcal{B}}$. If we denote by $\mathcal{C}\{\mathcal{A}\}$ the moduli space of $\mathcal{C}$-algebra structures on $\mathcal{A}$, i.e. $\mathcal{C}\{\mathcal{A}\} = \mathrm{RHom}_{SOp}(\mathcal{C}, \mathcal{E}_{\mathcal{A}})$, then this is the image of the element j along the map $E_\infty\{\mathcal{B}\} \to E_\infty\{f^*\mathcal{B}\}$. The direct image functor f_* being a Quillen right adjoint, preserves homotopy limits. Therefore there is a canonical ring structure on $f_*\mathcal{A}$ as well. The ring structure map is the image of i along the map of moduli spaces $E_\infty\{\mathcal{A}\} \to E_\infty\{f_*\mathcal{A}\}$.

A sheaf of $\mathcal{A}$-modules is module object over the E_∞-ring $\mathcal{A}$ in the simplicial model category $\mathfrak{X}$. As an object $\mathcal{M}$ in $[\mathcal{C}^{opp}, \mathrm{SSet}]^{cf}_{inj,local}$, $\mathcal{M}(U)$ is a $\mathcal{A}(U)$-module. If $\mathcal{A}(U)$ is seen as a *connective* commutative S-algebra, $\mathcal{M}(U)$ is a connective $\mathcal{A}(U)$-module. For $U \to V \in \mathcal{C}$, $\mathcal{M}(V) \to \mathcal{M}(U)$ is a map of $\mathcal{A}(U)$-modules. The category of $\mathcal{A}$-modules is a simplicial model category. Denote this by $(\mathfrak{X}, \mathcal{A})$-modules, or simply, $\mathcal{A}$-modules.

Define $K(\mathcal{A})$ to be the category of sheaves of $\mathcal{A}$-modules (not necessarily connective). That is for any $U \in \mathcal{C}$, $\mathcal{M}(U)$ is a $\mathcal{A}(U)$-module spectrum. $K(\mathcal{A})$ is a *stable* simplicial model category. The derived category $\mathrm{ho}K(\mathcal{A})$ has the structure of a triangulated category. We shall denote this by $D(\mathcal{A})$.

2.3 Derived Algebraic Stacks

Zariski ∞-topos of a E_∞-ring space: A_{Zar}. A map of E_∞-rings $f : A \to B$ is a *Zariski* if

1. $\pi_0 A \to \pi_0 B$ is a Zariski map of ordinary commutative rings ($\mathrm{Spec}\pi_0 B \to \mathrm{Spec}\pi_0 A$ is an open immersion of affine schemes)

2. $\pi_0 B \otimes_{\pi_0 A} \pi_i A \to \pi_i A$ is an isomorphism for every i.

Let A be an E_∞ ring. Define $\mathrm{Zar}(A)$ the local Zariski site over A as follows. The category underlying $\mathrm{ZAr}(A)$ has as objects Zariski maps of E_∞ ring spaces $A \to B$, and as morphisms triangles

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \nwarrow \quad \nearrow & \\ & A & \end{array}$$

. A family of morphisms $(B \to B_i)_{i \in I}$ over A in $\mathrm{ho}(\mathrm{Zar}(A))$ is a cover in the Zariski topology if the family $(\pi_0 B \to \pi_0 B_i)_{i\in I}$ is a covering in the ordinary Zariski topology over the ring $\pi_0 A$. The opposite category $\mathrm{Zar}(A)^{opp}$ is a simplicial site.

Define the ∞-topos $A_{Zar} = [\mathrm{Zar}(A), \mathrm{SSet}]^{cf}_{inj,local}$. The derived affine scheme $\mathrm{Spec}A$ corresponding to an E_∞ ring space A is the ringed ∞-topos $(A_{Zar}, \dot{A})$, where $\dot{A} = \mathrm{Hom}_{\mathrm{Zar}(A)}(A, -)$. Denote this by $\mathcal{O}_A$.

Given a Zariski map of E_∞-rings $A \to B$, there is a geometric morphism of ∞-topoi $B_{Zar} \to A_{Zar}$ induced by the continous map of simplicial sites $\mathrm{Zar}(B) \to \mathrm{Zar}(A)$.

Ordinary topos associated to the derived affine scheme $\mathrm{Spec}^{Zar} A$; denote by $\pi_0 \mathrm{Spec}^{Zar} A = \mathrm{Shv}_{\mathrm{Sets}}(ho(\mathrm{Zar}(A)^{opp}))$. This is a ringed topos. The ring object is $\dot{A} = \pi_0 \mathrm{Hom}_{ho(Zar(A))}(A, -)$.

Étale ∞-topos of an E_∞ ring A: $A_{ét}$. A map of E_∞-ring spaces $f : A \to B$ is étale if

1. $\pi_0 A \to \pi_0 B$ is an étale morphism

2. $\pi_0 B \otimes_{\pi_0 A} \pi_i A \to \pi_i A$ is an isomorphism for every i.

The local étale site associated to A is defined in the way similar the definition if the local Zariski site. Denote this simplicial site by $\mathrm{Ét}(A)$. The ∞-topos associated to the simplicial site $\mathrm{Ét}(A)^{opp}$ is denoted $A_{ét}$. That is, $A_{ét} = [\mathrm{Ét}(A), \mathrm{SSet}]^{cf}_{inj,local}$. The derived affine scheme $\mathrm{Spec}^{ét} A$ is the ringed ∞-topos $(A_{ét}, \dot{A})$, where $\dot{A} = \mathrm{Hom}_{\mathrm{Ét}(A)}(A, -)$. Given an étale morphism of E_∞-rings $A \to B$, there is a geometric morphism of ∞-topoi $A_{ét} \to B_{ét}$.

A *derived algebraic (Deligne-Mumford) stack* is a ringed ∞-topos $(\mathfrak{X}, \mathcal{A})$ which is locally isomorphic to the $\mathrm{Spec}^{ét} A$. In order to make this precise we need the notion of an ∞-topos assocuited to a cosimplicial E_∞ ring. A *étale cosimplicial E_∞-ring $A_\bullet$* is a cosimplicial object in the category of E_∞-rings, $A_\bullet : \Delta \to E_\infty\mathrm{Rings}$, whose face and degeneracies are étale. Define the local étale site $\mathrm{Ét}(A_\bullet)$ associated to $A\bullet$ as follows. The category underlying $\mathrm{Ét}(A)$ has as objects couples $([n], A_n \to B)$, denoted by (n, B), where $[n] \in \mathrm{ob}\Delta$ and $A_n \to B$ is étale. A morphism between couples (n, B) and (n', B') is a pair (δ, ϕ) where $\delta : [n] \to [n']$ is an arrow in Δ and $\phi : B \to B'$ is an étale map over the map $A_\bullet(\delta) : A_n \to A_{n'}$ induced by δ.

A family of maps $(\ (n, B) \xrightarrow{\mathrm{id}, \phi_i} (n, B_i)\ )_{i \in I}$ is a covering if $\phi_i : B \to B_i$ is an étale covering in $\mathrm{Ét}(A_n)$. This defines a topology on the homotopy category $\mathrm{hoÉt}(A_\bullet)$, making $\mathrm{Ét}(A_\bullet)$ a simplicial site. The ∞-topos $(A_\bullet)_{ét}$ is the category of sheaves of spaces over $\mathrm{Ét}(A_\bullet)$; $(A_\bullet)_{ét} = [\mathrm{Ét}(A_\bullet), \mathrm{SSet}]^{cf}_{inj,local}$. An object E in $(A_\bullet)_{ét}$ consists of a family of sheaves $E_n \in (A_n)_{ét}$ along with transition maps $E_n \to A_\bullet(\delta)_* E_{n'}$ for every map $[n] \to [n'] \in \Delta$ satisfying the cocycle conditions.

A derived Deligne-Mumford stack is a ringed ∞-topos $(\mathfrak{X}, \mathcal{A})$ which is locally isomorphic to the affine derived scheme $(\mathrm{Spec}^{ét} A, \mathcal{O}_A)$. More precisely, there is an étale cosimplicial E_∞ ring space $A_\bullet$, so that there in an equivalence of ∞-topoi $\mathfrak{X} \simeq (A_\bullet)_{ét}$.

2.4 Stable ∞-topoi and E_∞-ringed topoi

A stable ∞-topos is a category equivalent to the category of (pre)sheaves on a simplicial site $\mathcal{C}$ taking values in a simplicial *stable* model category $\mathfrak{C}$. In other words a stable ∞-topos $\mathfrak{X}$, as a simplicial stable model category, can be expressed as $\mathfrak{C}^{\mathcal{C}^{opp}}_{local}$. We shall use $\mathfrak{C} =$ S-modules for the stable model catgeory. An E_∞ ringed topos is a stable ∞-topos with the choice of a commutative ring object in it.

There are a number of categories of *spectra* in which one can do stable homotopy theory. We shall work with the category of S-modules. S-modules is a stable presentable $(\infty,1)$-category. There a closed symmetric monoidal stable model catgeory presenting this infinity category. The monoid objects and commutative monoid objects in this category are A_∞-rings and E_∞-rings respectively. The homotopy category of S-modules is the stable homotopy category. Let X be an ∞-topos which is modeled as $sN(\text{S-modules}_{local}^{\mathcal{C}^{opp}})^\circ$, where $\mathcal{C}$ is a small simplicial site. We propose to define a sheaf of E_∞-rings over X. This is commutative ring object in X. The category of E_∞ rings is higher categorical in nature. There is a model categroy of E_∞ rings representing this infinty category. The notion of a sheaf of E_∞ rings must involve this model structure in some way. Loosely speaking, a sheaf is a functor $\mathcal{F} : X^{opp} \to E_\infty$-rings which is left exact in the sense that it takes limits to homotopy limits. We will have to consider some of the different model structures on the spectral presheaves in order to make the idea of a sheaf of E_∞-rings precise.

If $(f^{-1}, f_*) : X \to Y$ is a geometric morphism of topoi. Let us suppose $X = \text{Top}(\mathcal{C})$, $Y = \text{Top}(\mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are small sites, and the morphism of topoi is induced by a continous morphism of sites $f : \mathcal{C} \to \mathcal{D}$. Given a sheaf of spectra $\mathcal{F}$, $f_*\mathcal{F}$ is a homotopy sheaf of spectra. The Čech descent condition can be easily checked; given a covering $(V_i \to V) \in \mathcal{D}$, one obtains a covering $(f^{-1}V_i \to f^{-1}V) \in \mathcal{C}$. There are weak equivalences of spectra $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V) \simeq \text{holim}_\Delta \mathcal{F}(f^{-1}V_\bullet) = \text{holim}_\Delta f_*\mathcal{F}(V_\bullet)$. For $\mathcal{G}$ a sheaf of spectra on Y, a presheaf of spectra can be definedd using the association $\mathcal{F} \mapsto \mathcal{G} \circ f$. Define $f^{-1}\mathcal{G}$ to be the fibrant replacement of $\mathcal{G} \circ f$ in the injective local model structure on spectral presheaves. Let $f : X \to \{pt\}$ be the unique geometric morphism from X to the point topos. The direct image functor f_* associates to every sheaf of spectra $\mathcal{F}$ over X it's *homotopy global sections* spectrum $\Gamma(X, \mathcal{F})$. For any point $p : \{pt\} \to X$ in the topos, the inverse image functor associates to every sheaf of spectra $\mathcal{F}$ the *homotopy stalk* $\mathcal{F}_p = p^{-1}\mathcal{F}$.

A module M over a sheaf of E_∞ rings $\mathcal{A}$ is a functor $\mathcal{C}^{opp} \to S$-modules satisfying the sheaf condition; $M(U) \simeq \text{holim}_\Delta M(U_\bullet)$, for every covering $(U_i \to U)_{i\in I}$, so that for each object $U \in \mathcal{C}$, $M(U)$ is a $\mathcal{A}(U)$-module. For any $V \to U \in \mathcal{C}$ the map $M(U) \to M(V)$ is a map of $\mathcal{A}(U)$-modules. The $\mathcal{A}$-module M is *cartesian* if the homomorphism $\mathcal{A}(V) \wedge_{\mathcal{A}(U)} M(U) \to M(V)$ is an equivalence of $\mathcal{A}(U)$-modules. Given two $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{G}$ define the *tensor product* $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ as the sheaf associated to (fibrant replacement of) the presheaf $U \mapsto \Gamma(U, \mathcal{F}) \wedge_{\Gamma(U,\mathcal{A})} \Gamma(U, \mathcal{G})$. The presheaf $U \mapsto F_{\Gamma(U,\mathcal{A})}(\Gamma(U, \mathcal{F}), \Gamma(U, \mathcal{G})$ satisfies Čech descent. The associated sheaf is $Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$.

A morphism of ringed topoi $(\phi, f^{-1}, f_*) : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a geometric morphism $(f^{-1}, f_*) : X \to Y$ of topoi and a map between sheaves of E_∞-rings $\phi : \mathcal{B} \to f_*\mathcal{A}$. For a $\mathcal{A}$-module $\mathcal{F}$, $f_*\mathcal{F}$ is a $\mathcal{B}$-module via the maps of E_∞-ringed sheaves $\mathcal{B} \to f_*\mathcal{A} \to f_*\mathcal{F}$. For a $\mathcal{B}$-module $\mathcal{G}$, $f^*\mathcal{G}$ is the $\mathcal{A}$-module $\mathcal{A} \otimes_{f^{-1}\mathcal{B}} f^{-1}\mathcal{G}$. The point topos $\{pt\}$ can be E_∞ringed by means of the constant sheaf S. Then $(\{pt\}, S)$ is the final object in the category of E_∞-ringed topoi in the sense that that there is a contractible space pf maps from any $(X, \mathcal{A})$ to $(\{pt\}, S)$.

Examples of E_∞-ringed topoi arise in derived(spectral) algebraic geometry. We demonstrate by means of the following examples.

2.5 Derived Schemes

Let A be an E_∞ ring. The topological space underlying the Spec$A$ is the ordinary Zariski spectrum of the commutative ring $\pi_0 A$: i.e. the set of prime ideals of $\pi_0 A$. We give the space Spec$A$ with the usual Zariski topology of A, with a basis of opens given by $U_f = \{\mathfrak{p} | f \notin \mathfrak{p}\}$, $f \in \pi_0 A$. The Zariski site over the space Spec$A$ is a Verdier site. Thus a sheaf of E_∞-ring is a presheaf that satisfies Čech descent. Now we have to give a structure sheaf of E_∞ rings on the space Spec$A$. If suffices to define $\mathcal{O}_{\text{Spec}A}$ over each of the opens $U_f \subseteq \text{Spec}A$. If A were a commutative ring one would define $\mathcal{O}_{\text{Spec}A}(U_f) = A[f^{-1}]$. This definition also works for E_∞ rings. Given an E_∞ ring A and a $f \in \pi_0 A$, the assignment $A \to A[f^{-1}]$, obtained by standard constructions over S-modules, is characterized by the the properties: the map $\pi_* A \to \pi_* A[f^{-1}]$ identifies $\pi_* A[f^{-1}]$ as $(\pi_* A)[f^{-1}]$), and secondly, .. The map $A \to A[f^{-1}]$ is well defined upto canonical equivalence and enables us to define $\mathcal{O}_A$.

A *derived scheme* is a E_∞-ringed space $(X, \mathcal{A})$ which is locally isomprphic to the E_∞-ringed space $(\text{Spec}A, \mathcal{O}_A)$. We call the sheaf of E_∞ rings $\mathcal{A}$ the *structure sheaf* of the derived scheme, denoted by $\mathcal{O}_X$. The covering of the scheme X by affines $\{U_i\}_{i\in I}$ gives a description of the underlying space of the scheme X in terms of the Zariski topos $\text{Zar}(U_\bullet)$. Here $U_\bullet$ is the Čech nerve of the covering $(U_i \to X)_{i\in I}$ and $\text{Zar}(U_\bullet)$ is the category of simplicial Zariski sheaf over $U_\bullet$.

A module M over the commutative S-algebra A gives a sheaf of modules $\widetilde{M}$ over the derived affine sheme $(\text{Spec}A, \mathcal{O}_A)$ by taking $\Gamma(\text{Spec}U_f, \widetilde{M}) = A[f^{-1}] \wedge_A M$. A cartesian module over a $\mathcal{O}_X$, where $\mathcal{O}_X$ is the structure sheaf of a derived scheme, is a a *quasi-coherent module* if it is locally over Spec$A$ isomorphic to $\widetilde{M}$, where M is an A-module.

A morphism of derived schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of the associated E_∞-ringed topoi, which is *local.* A map of sheaves of E_∞-rings $f : \mathcal{O}_X \to \mathcal{O}_Y$ is local if it induces a local homomorphism $\pi_0\mathcal{O}_{Y,f(y)} \to \pi_0\mathcal{O}_{X,x}$ of commutative rings for every $x \in X$. Since the category of spectra is higher categorical in nature, we have a space of morphisms between any two given derived schemes. So we have $\mathrm{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) = \coprod_{f:X\to Y} \mathrm{Hom}_0(\mathcal{O}_Y, f_*\mathcal{O}_X)$. Here Hom_0 means local maps of E_∞-ring sheaves.

Let $(X, \mathcal{O}_X)$ be an ordinary scheme. Every ordinary commutative ring can be regarded as an E_∞-ring by means of the Eilenberg-MacLane functor. The functor H : CommRings $\to E_\infty$-rings that associates to every commtative ring R, the Eilenber-Maclane spectrum HR, is full and faithful. There an associated functor on presheaf categories H : CommRings$^{\mathcal{C}^{opp}} \to E_\infty$-rings$^{\mathcal{C}^{opp}}$ whenever $\mathcal{C}$ is a small site. However H doesn't normally take sheaves of commutative rings to sheaves of E_∞-rings. Therefore given a sheaf F of commutative rings over $\mathcal{C}$ we will denote by $\tilde{H}F$ the fibrant replacement (sheafification) of $H(F)$ in the local injective model structure on spectral presheaves. The functor $\tilde{H}$ enables us to regard any ordinary scheme as a derived scheme. In other words there is a full and faithfull functor from ordinary schemes to derived schemes which takes an ordinary sheme $(X, \mathcal{O}_X)$ to the derived scheme $(X, \tilde{H}\mathcal{O}_X)$.

Conversely, for any E_∞ ring A we have a commutative ring π_0A. For any sheaf $\mathcal{F}$ of E_∞-rings, the functor $U \mapsto \pi_0\mathcal{F}(U)$ is a presheaf of comuutative rings. Let $\pi_0\mathcal{F}$ denote the sheafification of this presheaf. If $(X, \mathcal{O}_X)$ is a derived scheme, the pair $(X, \pi_0\mathcal{O}_X)$ an ordinary scheme. We call this the *underlying ordinary scheme* od any derived scheme. If we use only connective E_∞-rings instead of arbitrary E_∞-rings, the functor $(X, \mathcal{O}_X) \to (X, \pi_0\mathcal{O}_X)$ is right adjoint to the inclusion functor from shemes to derived schemes. Without connectivity assumptions however, no such adjunction exists.

Given any derived scheme $(X, \mathcal{O}_X)$, we have a quasi-coherent sheaves of modules over the underlying ordinary scheme $(X, \pi_0\mathcal{O}_X)$ given by $\pi_n\mathcal{O}_X$ for each n.

A map of E_∞ rings (following Lurie) $A \to B$ is étale if the induced map $\pi_0A \to \pi_0B$ on ordinary rings is étale and there are isomophisms $\pi_nA \otimes_{\pi_0A} \pi_0B \to \pi_nB$ of π_0B modules. This gets easily generalized to derived schemes. A morphism of derived schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is étale if

1. the map of the uderlying schemes $\pi_0X \to \pi_0Y$ is étale;
2. $f^*\pi_n\mathcal{O}_X \to \pi_n\mathcal{O}_Y$ is an isomorphism of $\pi_0\mathcal{O}_Y$ modules.

A *derived algebraic space* is an E_∞ ringed topos $(X, \mathcal{A})$ so that X is the étale topos of an ordinary algebraic space and, $\pi_0\mathcal{A}$ is the structure sheaf of rings over X.

A derived algebraic space can be also obtained by an étale equivalence relation of derived schemes. A diagram of derived schemes $(p_1, p_2) : X \rightrightarrows Y$ is an étale equivalence relation if the maps p_1 and p_2 are étale, in the sense described in the previous section, and if the digram of the underlying schemes $\pi_0X \rightrightarrows \pi_0Y$ is an equivalence relation.

3 Local (co)-homology on ∞-topoi

3.1 Local cohomology on ringed topoi

Let X be a topos. E an object in X. Then the category $X_{/E}$ of objects in X over E is a topos. This follows from Giraud's characterization of a topos. If the topos X can be realized as the category of sheaves over the site $\mathcal{C}$, so that that for a choice of object $E \in \mathcal{C}$, the object $E \in X$ is the Yoneda embedding of $E \in \mathcal{C}$ in Sh$\mathcal{C}$; then $X_{/E} = \mathrm{Sh}\mathcal{C}_{/E}$ is equivalent to $\mathrm{Sh}\left(\mathcal{C}_{/E}\right)$, where the $\mathcal{C}_{/E}$ has the topology induced by $\mathcal{C}$. The topos $X_{/E}$ is the *topos induit* over the object E in X. There is a canonical morphism of topoi $j_E : X_{/E} \to X$, the inclusion morphism of the topos $X_{/E}$ in the ambient topos X, or the *localization morphism* of X along E. The gemetric morhism j_E corresponds to a triple of adjoint functors $(j_{E!} \dashv j_E^* \dashv j_{E*})$. The functor $j_{E!} : X_{/E} \to X$ forgets the structural arrow. The functor $j_E^* : X \to X_{/E}$ is defined by $j_E^*(Z) = (E \times Z, \mathrm{pr}_1)$, where $\mathrm{pr}_1 : E \times Z \to E$ is the projection of the first factor. This can also be interpreted as the change of base funtor relative to the morphism $E \to 1_X$, where 1_X is the final object in X. j_E^* is left adjoint to $j_{E*} : X_{/E} \to X$. For an object E' over E, the object $j_{E*}(E')$ is sometimes denoted by $\Pi_{E/1_X}(E'/E)$, or $\mathrm{Hom}_{E/1_X}(E, E')$.

A morphism of topos $f : Y \to X$ is an *extension* if the functor $f_* : Y \to X$ is fully faithful (i.e. if the adjunction map $f^*f_* \to \mathrm{id}_Y$ is an isomorphism). A *sub-topos* of X is a strict full subcategory X' of X such that the inclusion functor $\alpha : X' \to X$ is of the form i_*, where $i : X' \to X$ is a morphism of topoi (i.e. α admits a left adjoint which is left exact. The morphism i is then called the *inclusion morphism* of the the sub-topos X' in X. It is clear that the inclusion map i is an extension map.

Open sub-topos. Let U be an *open* set of a topos X; i.e. a sub object of the final object 1_X in X. Consider the localization morphism $j : X_{/U} \to X$. Since $U \to 1_X$ is a monomorphism, the functor $j_!$ is a fully faithful (??), and therefore the biadjoint j_* is also fully faithful. In these terms, the localization morphism j associated to an open set U of the topos X is an extensiof topos. An extension of topoi $X' \to X$ is an *open extension* if the map is defined by an open set U in X. A sub-topos X' of X is *open sub-topos* if the if the inclusion morphism $X' \to X$ is an open extension. The open set U associated to the open extension $j : X' \to X$ is determined by the subobject $j_!(1_{X'})$ of 1_X, where $1_{X'}$ is the final object in X'. in confitmation with the previous definitions, an open sub-topos of X associated to the open set U in X is a the full strict sub-category of X obtained by the essential image of the functor $j_* : X_{/U} \to X$. This is also canonically equivalent to the essential image of the functor $j_!$, which is formed of the objects E for which the structure morphism $E \to 1_X$ is factors through U.

We can combine these observations to charaterize an open sub-topos of X as an open extension of topoi $j : \mathcal{U} \to X$. Then the functor $j_!$ and it's right adjoint j^* exists, so that we have a suite of three functors,

$$\mathcal{U} \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{j^*}}}}} X$$

where, a) $j_!$ and j_* are full and faithful, b) $j_!$ commutes with filtered colimits and c) for every object E of X, the adjunction counit $j_!j^*E \to E$ is a monomorphism.

The closed complement of an open sub-topos of X. Let U be an open set in a topos X, and consdier the localization map $j : \mathcal{U} = X_{/U} \to X$, which is an open extension morphism of topoi. For every object E of X, let $E_{cU} = U \coprod_{U\times E} E$, where the coproduct is taken over the canonical projections $\mathrm{pr}_1 : U \times E \to U$ and $\mathrm{pr}_2 : U \times E \to E$. If T is topolocical space and U an open set in T, the topos $\mathrm{Top}(U)$ can be identified as an open sub-topos of T. If Y is the closed complement of U in T, the topos $\mathrm{Top}(Y)$ is a sub-topos of $\mathrm{Top}(T)$, and is formed of objects of $\mathrm{Top}(T)$ whose restrcition to U is the final object in $\mathrm{Top}(U)$. If E is an object in $\mathrm{Top}(X)$, the object E_{cU} is canonically isomorphic to $i_*i^*(E)$, where $i : Y \to T$ is the inclusion, and $E \to E_{cU}$ is the natural adjunction unit. It follows from construction that for any $E \in X$ there exists a F and an isomorphism $E \simeq F_{cU}$ if and only if the canonical morphism $E \to E_{cU}$ is an isomorphism, or equivalently, the canonical morphism $\mathrm{pr}_1 : U \times E \to U$ is an isomorphism (i.e. $j^*(E)$ is the final object in $X_{/U}$).

Following the discussion in the previous paragraph, in the setting of a an open extension map of topoi $j : \mathcal{U} \to X$, the full strict sub-category $\mathcal{Y}$ of X consisiting of objects E of X whose pullback $j^*(E)$ is the final object of $\mathcal{U}$, is a sub-topos of X. The inclusion functor $i_* : \mathcal{Y} \to X$ is therefore the direct image morphism of a geometric morphism of topoi $i : \mathcal{Y} \to X$. The unit morphism $E \to i_*i^*E$ of the adjunction $i_* \dashv i^*$ is the canonical morphism $E \to E_{cU}$. The sub-topos $\mathcal{Y}$ is the *closed complement* of the open sub-topos $\mathcal{U}$ in X. Conversely, a sub-topos $\mathcal{Y}$ of X is a closed sub-topos of X if there exists an open subset U of X so that $\mathcal{Y}$ is the closed complement of $X_{/U}$. The open set U is dtermined by $i_*(\emptyset_\mathcal{Y})$, where $\emptyset_\mathcal{Y}$ is the initial object in $\mathcal{Y}$, which in turn is $i^*(\emptyset_X)$.

With the notation as above there are morphism of topoi $j : \mathcal{U} \to X \leftarrow \mathcal{Y} : i$, and a two suites of adjoint functors $(j_! \dashv j^* \dashv j_*)$ and $(i^* \dashv i^*)$,

$$\mathcal{U} \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{j^*}}}}} X \underset{i^*}{\overset{i_*}{\underset{\longrightarrow}{\longleftarrow}}} \mathcal{Y}$$

The object $1_{X\,cU} \in X$ is the *Koszul* object in the the topos X. It represents $\mathcal{Y}$-local sections, i.e sections which have 'support in $\mathcal{Y}$'. Given any $E \in X$, the $\mathcal{Y}$-local sections $\Gamma_\mathcal{Y}(X, E) = \mathrm{Hom}_X(1_{X\,cU}, E)$. Compare with global sections $\Gamma(X, E) = \mathrm{Hom}_X(1_X, E)$.

Let (X, A) be a ringed topos, E be an object in X. $j_E : X_{/E} \to X$ be the localization morphism. Since j_E^* ia a left adjoint and preserves finite limits, the sheaf $j_E^*(A)$ is canonically equipped with the structure of a ring. Let us denote the sheaf of rings $j_E^*(A)$ by $A|E$. The topos $X_{/E}$ is ringed by $A|E$. The sheaf $j_{E*}j_E^*(A)$ is canonically a ring. The adjunction morphism $A \to j_{E*}j_E^*(A)$ is a morphism of rings.

If M is an A-module, the sheaf $j_E^*(M)$ has the structure of a $A|E$-module, and the functor $j_E^* : (X, A)$-modules $\to$ $(X_{/E}, A|E)$-modules is the restriction to $X_{/E}$. j_E^* commutes with inductive and projective limits, i.e. in particular, j_E^* is exact. Given a $A|E$-module N, the sheaf $j_{E*}(N)$ is a $j_{E*}(A|E)$-module, and is a A-module by the unit map of adjunction $A \to j_{E*}(A|E)$. This defines a functor $j_{E*} : (X_{/E}, A|E)$-modules $\to (X, A)$-modules which commutes with projective limits. For a A-module M, the adjoint morphism $M \to j_{E*}j_E^*(M)$ is a morphism of A-modules, and for a $A|E$-module N the natural map $\mathrm{Hom}_{A|E}(j_E^*M, N) \to \mathrm{Hom}_A(M, j_{E*}N)$ is an isomorphism. Therefore they are adjoint functors $j_E^* \dashv j_{E*}$.

The functor j_E^* admits a left adjoint $j_{E!} : (X_{/E}, A|E)$-modules $\to (X, A)$-modules called *extension by zero*. $j_{E!}$ commutes with colimits, since it is a left adjoint. Suppose X is the topos of presheaves of sets over a site $\mathcal{C}$ containg the object E. Then the topos $X_{/E}$ is equivalent to the category of presheaves on $\mathcal{C}_{/E}$, and modulo this equivalence, the functor $j_E^* : X \to X_{/E}$ is composition with the forgetful functor $\mathcal{C}_{/E} \to \mathcal{C}$. This results in a very explicit construction of $j_{E!}$ for any $A|E$-module N and for every object F in $\mathcal{C}$, $(j_{E!}N)(F) = \oplus_{u \in \mathrm{Hom}_{\mathcal{C}}(F,E)} N(u)$. In a more general situation where the the topos $X_{/E}$ is equivalent to sheaves over the site $\mathcal{C}_{/E}$ (with the induced topology) and the functor is composition with the forgetful (continous) functor on the sites $\mathcal{C}_{/E} \to \mathcal{C}$, the extension by zero funtor is obtained by composition with the extension by zero functor on presheaves and the sheafification functor. For a $A|E$-module N, the sheaf of sets associated to $j_{E!}N$ is not in general isomorphic to the sheaf associated to the extension by empty set of the sheaf of sets associated to N, where, the extension by empty set is the functor $j_{E!} : X_{/E} \to X$ discussed earlier.

Given a ringed topos (X, A) and an object E an object in X. The A-module $j_{E!}(A|E)$, denoted by A_E or $A_{E,X}$, is the *A-module freely generated by E*, i.e. for every A-module M, there is a canonical isomorphism, $\mathrm{Hom}_X(E, M) \simeq \mathrm{Hom}_A(A_E, M)$, which is functorial on M. The free module A_E generated by E gives an isomorphism between the global sections of a $A|E$-module of the form j_E^*M, $\Gamma(X_{/E}, j_E^*M) = \mathrm{Hom}_{X_{/E}}(1_{X_{/E}}, j_E^*M)$ and $\mathrm{Hom}_{A|E}(A|E, j_E^*M)$ via the series of adjunction isomorphisms,

$$\mathrm{Hom}_{A|E}(A|E, j_E^*M) \simeq \mathrm{Hom}_A(A_E, M) \simeq \mathrm{Hom}_X(E, M) = \mathrm{Hom}_X(j_{E!}1_{X_{/E}}, M) \simeq \mathrm{Hom}_{X_{/E}}(1_{X_{/E}}, j_E^*M).$$

<u>The Koszul complex</u>. Let (X, A) be a ringed topos, U an open subset in X. Let $\mathcal{U}$ be the open sub-topos of X defined by U. Let $\mathcal{Y}$ be the sub-topos of X that is the complement of $\mathcal{U}$. There are canonical morphism of topoi $j : \mathcal{U} \to X$ and $i : \mathcal{Y} \to X$. The topos $\mathcal{U}$ is ringed by means of the sheaf j^*A and $\mathcal{Y}$ is a ringed by i^*A, denoted by $A|\mathcal{Y}$. The morphism of topoi j and i become morphisms of ringed topoi under these considerations. These two morphisms of ringed topoi $j : (\mathcal{U}, A|U) \to (X, A)$ and $i : (\mathcal{Y}, A|\mathcal{Y}) \to (X, A)$ define five functors between the vcategories of modules

$$(\mathcal{U}, A|U)\text{-modules} \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{j^*}}}}} (X, A)\text{-modules} \underset{i_*}{\overset{i^*}{\rightleftarrows}} (\mathcal{Y}, A|Y)\text{-modules}$$

The ring object $K_\bullet(\mathcal{Y}) := A_{cU} \in X$ is the Koszul complex associated to the closed sub-topos $\mathcal{Y}$. $K_\bullet(\mathcal{Y})$ can be identified with *colim* $\left(\begin{array}{ccc} A_U & \longrightarrow & A_U \\ \downarrow & & \\ A & & \end{array} \right)$, where the horizontal arrow is the identity map, and the verical arrow is the counit of the adjunction $j_{U!} \dashv j_U^*$; $A_U = j_{U!}j_U^*A \to A$.

The category of a modules over any ringed topos (X, A) is an abelian. We can consider it's derived category. This is a triangulated category denoted by $D(X, A)$. For a A-module M the unit of adjunction $N \to j^*j_*N$ gives a morphism of chain complexes of A-modules in $D(X, A)$, $N \to Rj^*j_*N$. Consider the triangle in the triangulated category $D(X, A)$

$$\Gamma_{\mathcal{Y}}N \to N \to Rj^*j_*N \to \Sigma\Gamma_{\mathcal{Y}}N$$

Define the *$\mathcal{Y}$-local cohomology* of the A-module N to be the object $\Gamma_{\mathcal{Y}}N$ in the derived category. Define the *$\mathcal{Y}$-local Čech cohomology* of N to be Rj^*j_*N in $D(X, A)$. The local cohomology functor is representable by the Koszul complex $K_\bullet(\mathcal{Y})$; $\Gamma_{\mathcal{Y}}(N) = \mathrm{Hom}_{(X,A)-\text{modules}}(K_\bullet(\mathcal{Y}), N)$ in $D(X, A)$.

3.2 The ∞-Koszul complex

Let $\mathfrak{X}$ be a ∞-topos an E an object in $\mathfrak{X}$. The over catgeory $\mathfrak{X}_{/E}$, as a simplicial model category, is a ∞-topos. This follows from the fact that there is a Quillen equivalence between simplicial model catgeories $[\mathcal{C}_{/E}^{opp}, \mathrm{SSet}]_{inj,global} \simeq$

$[\mathcal{C}^{opp}, \mathrm{SSet}]_{inj,global}{}_{/y(E)}$, and with the canonically induced model structure on the over category. Here $y(E)$ is the ∞ categorical Yoneda embedding of E in the presheaf category. The ∞-topos $\mathfrak{X}_{/E}$ is the *topos induit*. There is a canonical geometric morphism of ∞-topoi $j_E : \mathfrak{X}_{/E} \to \mathfrak{X}$, called *localization*. This consists of a triple of Quillen adjoint functors $(j_{E!} \dashv j_E^* \dashv j_{E*})$. The functor $j_{E!}$ takes the object $F \to E$ in the the over category to F and $j_E^*(F) = \mathrm{pr}_1 : F \times_{\mathfrak{X}} E \to E$.

A morphism of ∞-topoi $f : \mathfrak{X} \to \mathfrak{Y}$ is an *extension* if f_* is a fully faithful functor of $(\infty, 1)$ categories. A *sub* ∞*-topos* of $\mathfrak{X}$ is a full sub-simplicial model category $\mathfrak{X}'$ such that the inclusion functor $\alpha : \mathfrak{X}' \to \mathfrak{X}$ is of the form i_* for some ∞-geometric extension of topoi $i : \mathfrak{X}' \to \mathfrak{X}$.

A functor of simplicial model categories $F : \mathcal{X} \to \mathcal{Y}$ is full and faithfull if the morphism of mapping spaces $\mathrm{Hom}_{\mathcal{X}}(A, B) \to \mathrm{Hom}_{\mathcal{Y}}(f(A), f(B))$ is (-1)-truncated, i.e. the homotopy fiber of the morphism of mapping spaces is contractible.

Open sub ∞-topoi. An object U of $\mathfrak{X}$ is an open subset if it is a suboject of the final object $1_{\mathfrak{X}}$ (i.e. the the homotopy fiber of the morphism of Kan complexes $\mathrm{Hom}(Y, U) \to \mathrm{Hom}(Y, 1_{\mathfrak{X}})$ is (-2)-truncated, for any $Y \in \mathfrak{X}$). Consider the canonical geometric morphism $j : \mathfrak{X}_{/U} \to \mathfrak{X}$. Since $U \to 1_{\mathfrak{X}}$ is a monomorphism (in a simplicial model category), the functor $j_!$ is full and faithful, and therefore the double adjoint j_* is also full and faithful. In these terms the ∞-geometric morphism $j : \mathfrak{X}_{/U} \to \mathfrak{X}$ is an extension of topoi. An extension of topoi $\mathfrak{X}' \to \mathfrak{X}$ is an ∞*-open extension* if it is defined by an opne subset in $\mathfrak{X}$. A sub-topos $\mathfrak{X}'$ of an ∞-topos $\mathfrak{X}$ is *open sub-topos* if the inclusion $\mathfrak{X}' \to \mathfrak{X}$ is an open extension. The open set associated to the open extension is $j : \mathfrak{X}' \to \mathfrak{X}$ is determined by the subobject $j_!(1_{\mathfrak{X}'})$ of $1_{\mathfrak{X}}$. In the sense described in the previous discussion, an open sub ∞-topos is a full sub ∞-category of the simplicial model category $\mathfrak{X}$, which can be obtained as the essetial image of the functor $j_* : \mathfrak{X}_{/U} \to \mathfrak{X}$.

The closed complement of an open sub-∞-topos. Let U be an open set in the topos $\mathfrak{X}$. Consider the localization morphism $j : \mathfrak{U} = \mathfrak{X}_{/U} \to \mathfrak{X}$. For any object $E \in \mathfrak{X}$. Let E_{cU} be the mapping cone of the counit $j_!j^*E \to E$ associted to the Quillen adjuntion $j_! \dashv j^*$. This can also be seen as the homotopy colimit of the diagram $E_{cU} = \mathrm{hocolim}(1_{\mathfrak{X}} \leftarrow j_!j^*E \to E)$. Since the adjunction $j_!j^*E \to E$ is a Quillen, it is a cofibration, and $j_!j^*E \to 1_{\mathfrak{X}}$ is a fibration, and thus we can replace the hocolim with a colim. The Quillen adjunction also gives an equivalence of Kan complexes $\mathrm{Hom}_{\mathfrak{X}}(j_!j^*E, 1_{\mathfrak{X}}) \simeq \mathrm{Hom}_{\mathfrak{X}_{/U}}(j^*E, j^*1_{\mathfrak{X}}) = \mathrm{Hom}_{\mathfrak{X}_{/U}}(j^*E, (U \to U)) \simeq \mathrm{Hom}_{\mathfrak{X}}(U \times E, U)$. The final equivalence follows from the full-faithfulness of the functor $j_!$. Therefore we can define

$$E_{cU} = \mathrm{colim}\begin{pmatrix} j_!j^*E \longrightarrow E \\ \downarrow \\ 1_{\mathfrak{X}} \end{pmatrix} = \mathrm{hocolim}\begin{pmatrix} U \times_{\mathfrak{X}} E \longrightarrow E \\ \downarrow \\ U \end{pmatrix}$$

Define the closed complement of $\mathfrak{U}$ to be the full subcategory $\mathfrak{Y}$ of the simplicial model category $\mathfrak{X}$ consisting of the objects E for which the canonical morphism $E \to E_{cU}$ is an acyclic fibration, or equivalently, the pullback j^*E is the final object in the ∞-topos $\mathfrak{U}$. The inclusoin functor $i_* : \mathfrak{Y} \to \mathfrak{X}$, id the direct image functor of a geometric morphism of ∞-topoi $i : \mathfrak{Y} \to \mathfrak{X}$. The unit of the adjunction $i_* \dashv i^*$, $E \to i_*i^*E$, is the canonical morphism $E \to E_{cU}$. Conversely, a sub ∞-topos $\mathfrak{Y}$ of $\mathfrak{X}$ is closed if there is an open set U in $\mathfrak{X}$ such that $\mathfrak{Y}$ is the closed complement of $\mathfrak{X}_{/U}$. The object U is determined by $i_*(\emptyset_{\mathfrak{Y}})$, $\emptyset_{\mathfrak{Y}}$= initial object in $\mathfrak{Y}$.

Therefore there are geometric morphism of ∞-topoi $j : \mathfrak{U} \to \mathfrak{X} \leftarrow \mathfrak{Y} : i$, and a collection of adjoint functors $j_! \dashv j^* \dashv j_*$ and $i^* \dashv i_*$, i.e these are Quillen adjoint functors between simplicial model categories.

$$\mathfrak{U} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow j^* \longrightarrow}{\rightrightarrows}}} \mathfrak{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathfrak{Y}$$

The object $1_{\mathfrak{X}cU} \in \mathfrak{X}$ is the *Koszul* object in the ∞-topos $\mathfrak{X}$. It represents *homotopy $\mathfrak{Y}$-local sections*, i.e. sections of any object $F \in \mathfrak{X}$ with 'homotopy support' in $\mathfrak{Y}$. In notation $\Gamma_{\mathfrak{Y}}(\mathfrak{X}, F) = \mathrm{Hom}_{\mathfrak{X}}(1_{\mathfrak{X}cU}, F)$.

Let $(\mathfrak{X}, \mathcal{A})$ be an ∞-ringed topos. For $E \in \mathfrak{X}$ let $j_E : \mathfrak{X}/E \to \mathfrak{X}$ be the localization morphism. Then $j_E^*(\mathcal{A})$ is canonically equipped with the structure of an E_∞-ring. Denote this ring by $\mathcal{A}|E$. The induced ∞-topos $\mathfrak{X}_{/E}$ is ringed by $\mathcal{A}|E$. $j_{E*}j_E^*\mathcal{A}$ is also canonically a ring and there is a map of E_∞-rings $\mathcal{A} \to j_{E*}j_E^*\mathcal{A}$.

If $\mathcal{M}$ is an $\mathcal{A}$-module ($\mathcal{A}$ is regarded as a sheaf of connective commutative S-algebra and M a connective $\mathcal{A}$-module spectrum), then $j_E^*(\mathcal{M})$ has the structure of a $\mathcal{A}|E$-module. Therefore the functor $j_E^* : (\mathfrak{X}, \mathcal{A})$-modules $\to$ $(\mathfrak{X}_{/E}, \mathcal{A}|E)$-modules. Given a $\mathcal{A}|E$-module $\mathcal{N}$, $j_{E*}\mathcal{N}$ has the structure of a $j_{E*}(\mathcal{A}|E)$-module and is therefore a $\mathcal{A}$-module via the adjunction unit $\mathcal{A} \to j_{E*}\mathcal{A}|E$. This defines a functor $(\mathfrak{X}/E, \mathcal{A}|E)$-modules $\to (\mathfrak{X}, \mathcal{A})$-modules. For a $\mathcal{A}$-module $\mathcal{M}$ and a $\mathcal{A}|E$-module $\mathcal{N}$ there is an equivalence of spectra, $\mathrm{Hom}_{\mathcal{A}|E}(j_E^*\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, j_{E*}\mathcal{N})$. $j_{E*} \dashv j_E^*$ is a Quillen adjunction(??).

The functor $j_{E!} : \mathfrak{X}_{/E} \to \mathfrak{X}$ induces a left adjoint $j_{E!}$ to $j_E^* : \mathcal{A}|E$-modules $\to \mathcal{A}$-modules called *extension by zero*. If $\mathfrak{X}$ is the ∞-topos of presheaves of spaces on a simplicial cite $\mathcal{C}$, i.e $\mathfrak{X} \simeq [\mathcal{C}^{opp}, \mathrm{SSet}]_{inj,global}$ as simplicial model category, and $E \in \mathcal{C}$ is an object of the site $\mathcal{C}$, then $j_{E!}$ has a very explicit description. For every $\mathcal{A}|E$-module M,

$$j_{E!}M(U) = \bigvee_{u \in \pi_0 \mathrm{Hom}_{\mathcal{C}}(U,E)} M(u)$$

If $\mathfrak{X}$ is modeled as the category of *sheaves* of spaces over $\mathcal{C}$, the The functor $j_{E!}$ is obtained as the the extension by zero on presheaves followed by *sheafification* i.e. fibrant replacement in the $[\mathcal{C}^{opp}, \mathrm{SSet}]_{inj,local}$.

Let $(\mathfrak{X}, \mathcal{A})$ be a ringed ∞-topos and E an object in $\mathfrak{X}$. The $\mathcal{A}$-module $j_{E!}\mathcal{A}|E$ is denoted $\mathcal{A}|E$. This is the $\mathcal{A}$-module freely generated by E; for any (connective) $\mathcal{A}$-module M, there is a cononical eqivalence of spaces $\mathrm{Hom}_{\mathfrak{X}}(E, M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}_E, M)$.

The ∞-Koszul complex Let $(\mathfrak{X}, \mathcal{A})$ be a ringed ∞-topos, U a sub-object of $1_{\mathfrak{X}}$. Let $\mathfrak{U}$ be the open sub-∞-topos defined by U. Let $\mathfrak{Y}$ be the complement closed sub-∞-topos. There are canonical geometric moprphism of ∞-topoi; $j : \mathfrak{U} \to \mathfrak{X}$ and $i : \mathfrak{Y} \to \mathfrak{X}$. $\mathfrak{U}$ is ringed by the $\mathcal{A}|U$, and $\mathfrak{Y}$ is ringed by $i^*\mathcal{A}$, denote by $\mathfrak{A}|\mathfrak{Y}$. The morphisms j and i are now morphisms of ringed ∞-topoi. These define the usual Quillen adjoint functors between the (simplicial model) categories of modules over the ringed ∞-topoi.

$$(\mathfrak{U}, \mathcal{A}|U)\text{-modules} \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{j^*}}}}} (\mathfrak{X}, \mathcal{A})\text{-modules} \underset{i_*}{\overset{i^*}{\rightleftarrows}} (\mathfrak{Y}, \mathcal{A}|\mathfrak{Y})\text{-modules}$$

The E_∞-ring object $K_\bullet^\infty(\mathfrak{Y}) = \mathcal{A}_{cU}$ is the ∞-Koszul complex associated to the closed sub ∞-topos $\mathfrak{Y}$. $K_\bullet^\infty(\mathfrak{Y})$ can be identified with *hocolim* $\left(\begin{array}{ccc} \mathcal{A}_U & \longrightarrow & \mathcal{A}_U \\ \downarrow & & \\ \mathcal{A} & & \end{array}\right)$ where the vertical map is the couniit of adjunction $j_{U!} \dashv j_U^*; \mathcal{A}_U = j_{U!}j_U^*\mathcal{A} \to \mathcal{A}$.

4 Brown Representability for triangulated categories

Let $\mathcal{S}$ be a triangulated category. Suppose that all small coporoducts exist in $\mathcal{S}$. An object x in $\mathcal{S}$ is *compact* if $\mathrm{Hom}(x, -)$ respects coproducts, i.e. $\mathrm{Hom}(x, \coprod_\alpha y_\alpha) = \coprod_\alpha \mathrm{Hom}(x, y_\alpha)$. The full subcategory of $\mathcal{S}$ consisting of compact objects is denoted by $\mathcal{S}^c$. A triangulated category $\mathcal{S}$ is *compactly generated* if there is a set of compact objects S in $\mathcal{S}$ such that if y is an object of $\mathcal{S}$ and $\mathrm{Hom}(s, y) = 0$ for all $s \in S$, then $y = 0$.

The Brown representability theorem, in one of many variations, states:

Theorem. *Let $\mathcal{S}$ be a compactly generated triangulated category, $\mathcal{T}$ any triangulated category with small coproducts. Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. Suppose F respects coproducts; the natural map $\coprod_\alpha F(s_\alpha) \to F(\coprod_\alpha s_\alpha)$ is an isomorphism. Then there exists a right adjoint functor $G : \mathcal{T} \to \mathcal{S}$ for F,i.e. there is a natural isomorphism $Hom_{\mathcal{S}}(x, Gy) = Hom_{\mathcal{T}}(Fx, y)$.*

Our duality theorem is a direct application of Brown representability theorem. Let's consider Grothendieck duality for schemes. Suppose $f : X \to Y$ is a proper morphism of quasi-cpmpact and separated schemes. Then Grothendieck showed that there is a natural isomorphism

$$Rf_* RHom_X(x, f^! y) \simeq RHom_Y(Rf_* x, y)$$

of objects in the bounded derived category of quasi-coherent sheaves. Here $Rf_* : D^+_{qc}(X) \to D^+_{qc}(Y)$ is the right derived functor of f_*. The statement above asserts that Rf_* has a right adjoint $f^!$, and it behaves well with respect to pullbacks by open immersions.

Grothendieck's original proof of the exixtence of $f^!$ is constructive. An abstract way to the existence of $f^!$ was first given by Deligne. In the context of Brown representability one is forced to work with unbounded derived categories of quasi-coherent sheaves, since the triangulated categories under consideration must be closed under direct sums. In order to see Grothendieck duality for schemes as a consequence of Brown representability, one has to show two things:

1. Let X be a quasi-compact, separated scheme. The category $D_{qc}(X)$ is compactly generated
2. Let $f : X \to Y$ be a separated morphism of quasi-compact, separated schemes. The right derived functor $Rf_* : D_{qc}(X) \to D_{qc}(Y)$ is triangulated and respects coproducts.

Classically, the question of compact generation of $D_{qc}(X)$ was settled by making an assumption that the scheme admits a family of ample line bundles. However, this becomes unnecessary after the work of Thomason and Troubough on localization theorems for algebraic K-theory, and Neeman's generalization for triangulated categories. We shall discuss, in the following section, the ideas behind the localization theorems and eventually apply them to proving compact generation of $D_{qc}(X)$, where X is derived stack.

We need to recall a few more notions from triangulated categories. A triangulated category is said to have direct sums if it has catgeorical direct sums, and direct sums of triangles are triangles.

Let $\mathcal{S}$ be a triangulated category closed under arbitraty direct sums. A full sub-triangulated category L is *localizing* if evry direct summand of an object in L is in L, and every deirect sum of objects in L is in L. One may form the quotient category $\mathcal{S}/L$ where objects in L are identified with 0.

4.1 Neeman-Thomason localization theorem

Let X be a quasi-compact, separated scheme. Suppose X admits an ample family of line bundles. Let U be an open subscheme, and let $X - U$ be the complement. Let A be the abelian category of quasi-coherent sheaves on X, whose support is contained inside $X - U$. Let B be the abelian category of quasicoherent sheaves on X and let C be the abelian category of quasi coherent sheaves on U. Then it is well known that A is a Serre subcategory of B, and the quotient B/A is C.

Let $\mathcal{S} = D(\mathsf{B})$, $\mathcal{T} = D(\mathsf{C})$ and $\mathcal{R} = D_{\mathsf{A}}(\mathsf{B})$, the category of chain complexes of B with A-cohomology. Then $\mathcal{S}$ is closed with respect to coproducts, and $\mathcal{R}$ is a localizing subcategory of $\mathcal{S}$, and $\mathcal{T}$ is the Verdier quotient $\mathcal{S}/\mathcal{R}$.

The theorem of Thomason and Trobaugh states that the map $\mathcal{R} \to \mathcal{S}$ takes $\mathcal{R}^c$ to $\mathcal{S}^c$, the map $\mathcal{S} \to \mathcal{T}$ takes $\mathcal{S}^c$ to $\mathcal{T}^c$, that the induced map $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithfull, and that every object in $\mathcal{T}^c$ is a direct summand of an object in $\mathcal{S}^c/\mathcal{R}^c$, i.e. the smallest localizing subcategory of $\mathcal{T}^c$ containing $\mathcal{S}^c/\mathcal{R}^c$ is $\mathcal{T}^c$.

Using ideas of Bousfield and Ravenel, Neeman produced a vast generalization of Thomason-Trobaugh's localization theorem to tringulated categories.

Theorem. *Suppose $\mathcal{S}$ is any triangulated category closed with respect to arbitrary coproducts. Suppose the subcategory $\mathcal{S}^c$ of compact objects is small, and $\mathcal{S}$ is the smallest localizing category containing $\mathcal{S}^c$. Suppose R is a set of objects in $\mathcal{S}^c$ close with respect to suspension functor and $\mathcal{R}$ is the smallest localizing subcategory containing R. Let $\mathcal{T}$ be the Verdier quotient category $\mathcal{S}/\mathcal{R}$. Then*

1. *$\mathcal{R} \to \mathcal{S}$ carries $\mathcal{R}^c$ to $\mathcal{S}^c$,*
2. *the map $\mathcal{S} \to \mathcal{T}$ carries $\mathcal{S}^c$ to $\mathcal{T}^c$,*
3. *the natural functor $\mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is fully faithful, and $\mathcal{T}^c$ is the smallest localizing category containing the image.*

The hypothesis of the theorem implies that $\mathcal{S}$ is a compactly generated category and if R is a generating set for all of $\mathcal{S}$, then $\mathcal{R} = \mathcal{S}$. Moreover if R is closed under the formation of triangles and direct summand, then it is all of $\mathcal{R}^c$.

The content of the localization theorem can be explained in the following way. Suppose t is a compact object of $\mathcal{T}$. Then there is an object $t' \in \mathcal{T}^c$ and an object $s \in \mathcal{S}^c$ and an isomorphism in $\mathcal{T}$; $s \simeq t \oplus t'$. Therefore, t may not be isomorphic, in $\mathcal{T}$ to a compact object in $\mathcal{S}$, but it is a direct summand of an object isomorphic in $\mathcal{T}$ to a compact object in $\mathcal{S}$.

If $\mathcal{S}$ is the stable homotopy catagory, i.e. the category of S-modules localized at stable weak equivalence, then category of compact objects $\mathcal{S}^c$ is the category of finite spectra. The collection of spectra given by $\{S^n | n \in \mathbb{Z}\}$ is a set of compact generators for the stable homotopy category. Suppose E is a commutative ring spectrum, i.e a commuative ring object in the stable homotopy category. Let R be the category of finite spectra that are E-acyclic. R is closed under suspensions. The Verdier quotient category $\mathcal{S}/\mathcal{R}$ is isomorphic to the stable homotopy category of E-local spectra. Moreover, since R is closed under formation of cofibers (triangles) and direct wedge summands, the compact objects of $\mathcal{R}$ are exactly the finite E-acyclic spectra. The localization theorem then tells us that the compact objects in the triangulated category of E-local spectra are wedge summands of finite spectra.

Suppose X is a quasi-compact separated scheme. Then the full subcategory of compact objects $D_{qc}(X)^c$ of the derived category of quasi-coherent sheaves is the category of all perfect complexes on X. It is known that perfect complexe are compact and form a set of generators for $D_{qc}(X)$. Moreover, perfect complexes are closed with respect to triangles and direct summands. This means $D_{qc}(X)^c = Perf(X)$. The same arguement works as well when X is a quasi-compact, separated algebraic stack. We want to use the localizatio theorem to show that

Proposition. *The derived category of quasi-coherent modules over a quasi-compact, sepatrated scheme is compactly generated.*

We intend to show the following is true. Let X be a quasi-compact, separated scheme. Let $U \to X$ open affine. Let x be an arbitrary object of $D_{qc}(X)$, and let u be a perfect complex in $D_{qc}(U)$. Suppose there is map in $D_{qc}(U)$ of the form $u \to x$. Then there is a perfect complex u' in $D_{qc}(X)$ so that the map $u \oplus u' \to u \to x$ lifts to $D_{qc}(X)$. There exists a perfect complex $\tilde{u}$ on X which restricts to $u \oplus u'$ on U, and a map $\tilde{u} \to x$ defined on X, which restricts to $u \oplus u' \to x$ on U.

This should imply compact generation of $D_{qc}(X)$. In order to see this, take U to be an affine scheme. The U admits an ample line bundle, the trivial bundle in ample. Therefore U is compactly generated and the compact objects are the perfect complexes. if x is an object in $D_{qc}(X)$ and the restriction of x to U is non-zero, then there is a perfect complex x on U and a non-zero map $u \to x$ on U. By the above claim, this map can be extended to a non-zero map $\tilde{u} \to x$ on all of X, where $\tilde{u}$ is perfect. Thus unless the restriction of x to every open representable $U \to X$ is zero, there is a non-zero map from some perfect complex to x. But if the restriction of x to every U is zero, then x is zero (by Zariski descent).

Proof of ?. Suppose that $X = \mathrm{Spec}R$. Then X has an ample line bundle, the trivial bundle is ample. Theerfore $D_{qc}(X)$ is compactly generated; $\{\mathcal{O}_x[\mathbb{Z}]\}$ is a collection of compact generators. Futhermore, the compacts are precisely the perfect complexes. Let $D_{qc}^{X-U}(X) \subset D_{qc}(X)$ be the full subcategory of complexes supported on $X - U$. That is, all complexes whose restriction to U is acyclic. Now suppose $U = \bigcup_{i=1}^n \mathrm{Spec}R[f_i^{-1}]$. Consider the Koszul complex $B = \otimes_{i=1}^n (f_i : R \to R)$. Then $A \in D_{qc}(X)$ is supported on U if and only if $\mathrm{Hom}(B[n], A) = 0$ for all n. In other words, the suspension of the one compact object B in $D_{qc}^{X-U}(X)$ is a set of compact generators.

In Neeman's localization theorem let $\mathcal{S}$ be the category $D_{qc}(X)$, and let R be a generating set for $D_{qc}^{X-U}(X)$, for example $R = \{\Sigma^n B\}$. This makes $\mathcal{R} = D_{qc}^{X-U}(X)$. The triangulated category quotient $\mathcal{T} = D_{qc}(U)$. Thomason localization theorem implies that any perfect complex $u \in D_{qc}(U)$ is a direct summand of some perfect complex on X; there is a perfect complex u' on U such that $u \oplus u'$ lifts to a perfect comlex $\tilde{u} \in D_{qc}(X)$. MOreover, the map $u \oplus u' \to x$ can be lifted to $\tilde{u} \to x$ on all of X, for some lift $\tilde{u}$ of $u \oplus u'$.

Suppose $X = U_1 \cup U_2$. Then X is a the quotient of the Zariski equivalence relation $[(j_1, j_2) : U_1 \cap U_2 \rightrightarrows U_1 \coprod U_2]$. Let u_{12} be a perfect complex on $U_1 \cap U_2$. $j_1 : U_1 \cap U_2 \to U_1$ is an open immersion. By the result in the previous paragraph, u_{12} lifts to a perfect complex u_1 over U_1. Precisely, $j_1^* u_1 \simeq u_{12} \oplus \Sigma u_{12} \in D_{qc}(U_1 \times_X U_2)$. Similarly u_{12} lifts to a perfect complex u_2 over U_2 via the open immersion j_2; or more precisely $j_{2*} u_2 \simeq u_{12} \oplus \Sigma u_{12} \in D_{qc}(U_1 \times_X U_2)$.

Let $p_1 : U_1 \to X$, $p_2 : U_2 \to X$ be the open immersions. $p_{12} = p_1 \circ j_1 = p_2 \circ j_2$ is the open immersion $U_1 \times_X U_2 \to X$. Then we have an isomorphism $p_{12}^* p_{2*} u_2 \simeq j_2^* u_2$ in $D_{qc}(U_1 \times_X U_2)$. This gives the adjoint map $p_{2*} u_2 \to p_{12*} j_2^* u_2$ in $D_{qc}(X)$. Similarly, the map $p_{1*} u_2 \to p_{12*} j_1^* u_1$ exists in $D_{qc}(X)$. Let $\tilde{u}$ be the homotopy fiber

$$\tilde{u} \to p_{1*} u_1 \oplus p_{2*} u_2 \to p_{12*}(u_{12} \oplus \Sigma u_{12})$$

obtained by completing the triangle in $D_{qc}(X)$. Suppose furthermore, given a complex $x \in D_{qc}(X)$, there is a map $u_{12} \to p_{12}^* x$ in $D_{qc}(U_1 \cap U_2)$. Then $u_{12} \oplus \Sigma u_{12} \to u_{12} \to x$ lifts to $D_{qc}(U_1)$. More precisely, there is a map $u_1 \to p_2^* x$ in $D_{qc} U_2$ that restricts to $u_{12} \oplus \Sigma u_{12} \to u_{12} \to x$. Similarly, this same map lifts to $D_{qc}(X)$. Therefore we obtain a map between homotopy fibers by completing the triangles of complexes ober X.

$$\begin{array}{ccccc} \tilde{u} & \longrightarrow & p_{1*}u_1 \oplus p_{2*}u_2 & \longrightarrow & p_{12*}(u_{12} \oplus \Sigma u_{12}) \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & p_{1*}p_1^*x \oplus p_{2*}p_2^*x & \longrightarrow & p_{12*}p_{12}^*x \end{array}$$

Now, if x is non-zero when restricted to $U_1 \cap U_2$, since $D_{qc}(U_1 \cap U_2)$ is compactly generated, we can assume the map $u_{12} \to x$ is non-zero. Then the lifts to U_1 and U_2 is non-zero. Therefore the map $\tilde{u} \to x$ is non-zero. It can be checked that $\tilde{u}$ is a perfect complex. If however the restriction of x to open affines is zero, then x is zero.

Since we have assumed that X is quasi-compact, there is a finite covering $\{U_i \to X\}_{i\in I}$ such that X is the quotient of the Zariski equivalence relation $[\coprod_{i,j\in I} U_i \cap U_j \rightrightarrows \coprod_{i\in I} U_i]$. The arguement can be extended easily to X. End.

Stable Koszul complex associated to a closed substack of a flat algebraic stack Let (A, Γ) be flat Hopf algebroid. Let X be the algebraic stack associated to the flat quotient $[\mathrm{Spec}\Gamma \rightrightarrows \mathrm{Spec}A]$ of affine schemes. Suppose $Y \subset X$ is a closed substack defined by an *invariant* ideal I in A. Let $U = X - Y$ be the open complement. Denote the inclusions by $j : U \hookrightarrow X$, $i : Y \to X$. Then for any object $A \in D(X)$ (a chain complex of Γ-comodules), there is a triangle

$$i_*i^!A \to A \to j_*j^*A \to \Sigma i_*i^!A$$

The functor $i_*i^!$ is the Grothendieck local cohomology functor. We can view the above traingle as Bousfield localization.

Let $L \subset D(X)$ be the full subcategory of $\mathcal{O}_X$-modules whose support is contained in Y. In terms of comodules, L is the full subcategory of $D(\Gamma\text{-comodules})$ which are I-torsion. Let $D_Y(X)$ be the category of chain complexes of comodules whose cohomology is supported in Y. Then $i_* : D_Y(X) \to D(X)$ is an isomorphism of $i_*D_Y(X)$ with L. Also, i_* commutes with coproducts and hence L is localizing.

An object A in $D(X)$ is L-local is $\mathrm{Hom}(K, A) = 0$ for any $K \in L$. Clearly, any object of the form j_*A is L-local, since $\mathrm{Hom}(i_*B, j_*A) = \mathrm{Hom}(j^*i_*B, A) = 0$. Therefore in the triangle

$$i_*i^!A \to A \to j_*j^*A \to \Sigma i_*i^!!A$$

the term $i_*i^!A$ is in L, and j_*j^*A is L-local. The map $A \to j_*j^*A$ is a Bousfield localization with respect to the localizing subcategory $L = i_*D_Y(X)$.

One questionof interest is, when is the localization functor smashing? j_* commutes with direct sums since it is a left adjoint. Therefore j_*j^* is smashing when j_* preserves direct sums. This is true when U is quasi compact. We assume U is quasi compact.

Let $f \in A$, with $\eta_L(f) = \eta_R(f)$. An open *sub*stack of X of the form $X_f = [\mathrm{Spec}A[\frac{1}{f}] \times_X \mathrm{Spec}A[\frac{1}{f}] \rightrightarrows \mathrm{Spec}A[\frac{1}{f}]]$ is A is representable by the Hopf algebroid $(A, A[f^{-1}] \otimes_A \otimes\Gamma \otimes_A A[f^{-1}])$. If U is quasi-compact U is a finite union of these things, i.e. $U = \cup_{i=1}^n X_{f_i}$. X_f is the open complement of the closed substack $Y \subset X$ which is represented by the Hopf Algebroid $(A/I, A/I \otimes_A \Gamma_A A/I)$ where I is the invariant ideal of A generated by f. Since I is an invariant ideal, $A/I \otimes_A \Gamma \otimes_A A/I \simeq \Gamma/I$. Now we have a map of comodules $f : A \to A$. Define the *Koszul complex* for the invariant ideal I; $K_I = \otimes_{i=1}^n(f_i : A \to A)$. This is an object in the derived category of quasi-coherent sheaves over X, or equivalently the derived category of the abelian category of Γ-comodules.

Lemma *For any $A \in D_{qc}(X)$ is U-local if and only if $RHom(\Sigma^n K_U, A) = 0$ for all n.*

4.2 Comodules over étale Hopf algebroids

Proposition *Suppose (A, Γ) is a étale Hopf algebroid. The category of Γ-comodules is abelian. The derived category $D(\Gamma$-comodules$)$ of chain complexes of Γ-comodules is compactly generated.*

Let $A \rightrightarrows \Gamma$ be an étale Hopf algebroid. The étale topos associated to the cosimplicial ring

$$A_\bullet = \; A \rightrightarrows \Gamma \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \Gamma \otimes_A \Gamma \cdots$$

denoted $(A_\bullet)_{\acute{e}t}$ is the underlying topos for a Deligne Mumford stack X. The structure sheaf $\mathcal{O}_X$ is a ring object in $(A_\bullet)_{\acute{e}t}$. $\mathcal{O}_X$ is a family of rings $\mathcal{O}_{X_n}$ where $X_n = \mathrm{Spec}^{\acute{e}t}(\otimes_A^n\Gamma)$. There is an equivalence of categories (A, Γ)-comodules $\simeq$ quasi-coherent $- \mathcal{O}_X$-modules.

X has a affine covering $\mathrm{Spec}^{\acute{e}t}A \to X$. The homotopy(2-category) pullback $\mathrm{Spec}^{\acute{e}t}A \times_X \mathrm{Spec}^{\acute{e}t}A$ is affine. There is a pullback diagram of topoi

$$\begin{array}{ccc} \mathrm{Spec}^{\acute{e}t}\Gamma & \longrightarrow & \mathrm{Spec}^{\acute{e}t}A \\ \downarrow & & \downarrow \\ \mathrm{Spec}^{\acute{e}t}A & \longrightarrow & X \end{array}$$

The category of modules $\mathcal{O}_{X_0}$-modules is abelian; the derived category $D(\mathcal{O}_{X_0})$ is the derived category of A-modules, hence is compactly generated.

Let $\mathrm{Spec}^{\acute{e}t}B \to X$ be an étale presentation. Let x be an arbitrary object in $D_{qc}(X)$, and u a perfect complex in $D_{qc}(U)$. Suppose there is a morphism $u \to x$ in $D_{qc}(U)$. Then there is a perfect complex $u' \in D_{qc}(X)$ so that the morphism $u \oplus u' \to x$ lifts to $D_{qc}(X)$. That is, there exists a perfect complex $\tilde{u} \in D_{qc}(X)$ and a morphism $\tilde{u} \to x$ which when restricted to U is $u \oplus u' \to x$.

5 Grothendieck duality for derived stacks

5.1 The dualizing complex on ∞-topoi

Suppose X is a *compact* topological space. Let A be a E_∞-ring space. Consider the ∞-topos associated to X. That is, $X = [Op(X), \mathrm{SSet}]^{cf}_{inj,local}$, where $Op(X)$ is the simplicial-site associated to the site of open sets of X. We can consider the ringed ∞-topos $(X, \underline{A})$, where $\underline{A}$ is the constant sheaf over X. Then the category of $(X, \underline{A})$-modules is the category of $\underline{A}$-module spectra. This is a stable $(\infty, 1)$-category. It's homotopy category is a closed symmetric monoidal triangulated category.

The homotopy global sections of the structure sheaf $\mathcal{O}_X = \underline{A}$ over X; $\Gamma(X, \underline{A}) = \mathrm{Hom}_X(1_X, \underline{A}) = F(X_+, A)$. There one can think of the A-cohomology of the space X in terms of the derived cohomology of the structure sheaf of the ringed space $(X, \underline{A})$.

For a compact space X, the homotopy category $D(\underline{A}$-modules) is compactly generated. Given a map of ringed ∞-topoi $f : (X, A) \to (Y, B)$, the map induced of the homotopy category of modules $Rf_* : D(X, A)$-modules $\to$ $D(Y, B)$-modules, is tringulated and takes arbitrary coproducts to coproducts. It follows from Brown Representability of traingulated functors that Rf_* has a right adjoint. That is to say, there is a triangulated functor $f^! : D(Y, B) \to D(X, A)$ such that

$$Rf_* \mathrm{RHom}_A(M, f^! N) \simeq \mathrm{Rhom}_B(Rf_* M, N)$$

is a equivalence in the derived category of B-modules; M is a A-module and B is B-module.

Let $\{pt\}$ be the *point* ∞-topos, and $f : (X, A) \to (\{pt\}, S)$ the canonical map. The right derived functor Rf_* is the homotopoy global sections functor. The derived right adjoint f^* exists and defines the dualizing complex for the ringed space (X, A) in the following way: $D_X = f^! S$. The *dual* complex/module associated to any (X, A)-module M is defined as the object in $D(A)$, $D_X(M) = \mathrm{Hom}_A(M, D_X)$.

Consider the ∞-point topos ringed by the constant sheaf $\underline{A}$; $(\{pt\}, \underline{A})$. Given any ∞-topos ringed by a sheaf of A-modules, there is a canonical morphism to the point $(\{pt\}, \underline{A})$. The derived right adjoint Rf_* is the homotopy global sections A-module. We can think of the ringed ∞-topos X,A as a derived 'scheme' obtained by pathching affines $(U_X, \underline{A}|U)$; the ring object $\underline{A}$ serves as the 'structure sheaf' $\mathcal{O}_X$. The derived sheaf cohomogy of $\mathcal{O}_X$, i.e the homotopy groups $\pi_i Rf_* \mathcal{O}_X$ can be identified with the A-cohomology of the space X; $\pi_i Rf_* \mathcal{O}_X = A^i(X_+)$. If the space X uderlying the ∞-topos under consideration is compact, the triangulated homotopy category of $\mathcal{O}_X$-modules is compactly generated. Therefore the triangulated functor $Rf_* : D(\mathcal{O}_{(X,A)}$-modules) $\to D(\mathcal{O}_{(\{pt\},A)}$-modules) has a right adjoint. What is $f^! \mathcal{O}_{\{pt\}}$? It follows from adjunction there is an equivalence in the derived category of A-modules,

$$Rf_* \mathrm{RHom}_{\underline{A}}(\mathcal{O}_X, f^! A) \simeq \mathrm{RHom}_A(Rf_* \mathcal{O}_X, A).$$

The homotopy global sections of $D^A_X = f^! A$ is the A-*homology* of X spectrum $A \wedge X_+$. This follows from the following sequence of weak equivalences of A-modules;

$$\Gamma(X, f^! A) = \mathrm{RHom}_A(Rf_* \mathcal{O}_X, A) \simeq \mathrm{Rhom}_A(\mathrm{Rhom}_S(X_+, A), A) \simeq \mathrm{Rhom}_A(DX_+ \wedge A, A) \simeq \mathrm{Rhom}_S(DX_+, A) \simeq \mathrm{Rhom}_S(S, X$$

Therefore we can identify the Grothendieck *trace map*, the counit of the adjunction $Rf_* \dashv f^!$, $Rf_* f^! \mathcal{O}_{\{pt\}} \to \mathcal{O}_{\{pt\}}$, with the map of A-modules, $X_+ \wedge A \to A$ associated to the canonical map (of spaces) to the point $X \to *$. The unit of the adjunction $f^* \dashv Rf_*$, $\mathcal{O}_{\{pt\}} \to Rf_* f^* \mathcal{O}_{\{pt\}}$, can be identified with the map of A-modules $A \to F(X_+, A)$, induced by the canonical map of spaces $X \to *$. Therefore we can think of the following diagram

$$\begin{array}{ccc} Rf_* f^! \mathcal{O}_{\{pt\}} & \xrightarrow{trace} & \mathcal{O}_{\{pt\}} \\ & \searrow & \downarrow \\ & & Rf_* f^* \mathcal{O}_{\{pt\}} \end{array}$$

as part of the part of the Tate diagram

$$\begin{array}{ccc} X_+ \wedge A & \longrightarrow & A \\ & \searrow^{Norm} & \downarrow \\ & & F(X_+, A) \end{array}$$

For a space X considered as an ∞-topos, there is an underlying ordinary topos $\pi_\bullet X \simeq Shv(\mathrm{Ho}(Op(X)))$. This captures the homotopy type of the space underlying X. The underlyinng ordinary topos is functorial; given a Quillen equivalence of model categories $[\mathcal{C}^{op}, \mathrm{SSet}]^{cf}_{local,inj} \simeq_Q [\mathcal{D}^{op}, \mathrm{SSet}]^{cf}_{local,inj}$ implies an equivalence of categories of sheaves of sets $Shv(\mathrm{Ho}\mathcal{C}) \simeq Shv(\mathrm{Ho}\mathcal{D})$.

The ordinary topos $\pi_0 X$ captures the 0-homotopy type of the space underlying space X. Define $\pi_0 X = Shv(\pi_0 Op(X))$, where the site $\pi_0 Op(X)$ is defined as follows; the category underlying $\pi_0 Op(X)$ has objects $\pi_0 U \to \pi_0 X$ for every object $U \to X$ in $Op(X)$, morphism are the obvious commuting triangles. A family $(\pi_0 U_i \to \pi_0 X)_{i \in I}$ is a covering family if $\coprod_{i \in I} \pi_0 U_i \to \pi_0 X$ is a surjection of sets.

For any ∞-topos X associated to a topological space, there is a geometric morphism of ordinary topoi $\pi_\bullet X \to \pi_0 X$ (map to connected components). Suppose $(X, \mathcal{O}_X)$ is a ringed ∞-topos. Then the topos $\pi_\bullet X$ is ringed by means of the same ring object $\mathcal{O}_X$ considered as sheaf on the ordinary site $\mathrm{Ho}Op(X)$ taking values in the homotopy category of simplicial sets; denote this by $\pi_\bullet \mathcal{O}_X$. Similarly, the ordinary topos capturing the 0-homotopy type, $pi_0 X$ is ringed by means of the sheaf of rings $(\pi_0 \mathcal{O}_X)(\pi_0 U \to \pi_0 X) = \pi_0 \mathcal{O}_X(U \to X)$.

There is a geometric morphism of ringed topoi $(\pi_\bullet X, \pi_\bullet \mathcal{O}_X) \to (\pi_0 X, \pi_0 \mathcal{O}_X)$. The induced functor on the category of modules

$$p : \pi_\bullet \mathcal{O}_X\text{-modules} \to \pi_0 \mathcal{O}_X\text{-modules}$$

Suppose X is the ∞-point, ringed by the E_∞-ring space A. The the functor p between the module categories

$$p : Ho(A\text{-modules}) \to \pi_0 A\text{-modules}$$

takes an A-modules M to $\pi_0 M$.

5.2 Generalized Tate cohomology

Consider a topos asscocoted to the continous site of a topological space X. Let the topos be ringed by the constant sheaf of A-modules $\underline{A}$, where A is an E_∞-ring space. Define the Tate A-cohology of X to be the homotopy cofiber of the norm map

$$X_+ \wedge A \to F(X_+, A) \to t_A(X).$$

Let M be a closed and orientable smooth n-manifold. Let $Sm(M)$ be the *smooth site* associated to M. The category underlying $Sm(M)$ has as objects smooth open maps $\phi : \mathbb{D}^n \to M$, and morphisms commutative triangles

$$\begin{array}{ccc} \mathbb{D}^n_\alpha & \xrightarrow{f_{\alpha\beta}} & \mathbb{D}^n_\beta \\ & {}_{\phi}\searrow \quad \swarrow_{\theta} & \\ & M & \end{array}$$

, $\theta \circ f_{\alpha\beta} = \phi$. A family of morphism $(\phi_i : U_i \to M)$ is a covering if $\coprod_{i \in I} U_i \to M$ is a smooth surjection. $Sm(M)$ is a simplicial site. Let M be the ∞-topos associated to $Sm(M)$; $M = [Sm(M)^{op}, \mathrm{SSet}]^{cf}_{inj,local}$. Let $(M, \mathcal{O}_M)$ be the ∞-topos ringed by the constant sheaf $\underline{H\mathbb{Z}}$. The dualizing complex $D^{H\mathbb{Z}}_M \mathcal{O}_{\{pt\}}$ is equivalent as a $H\mathbb{Z}$-module to $\Sigma^{-n} H\mathbb{Z}$.

5.3 Tate cohomology for algebraic stacks

Let X be an Deligne-Mumford stack with an étale Hopf algebroid atlas $X = [\mathrm{Spec}\Gamma \rightrightarrows \mathrm{Spec}A]$. Let us assume that the derived categoru of quasi-coherent sheves of $\mathcal{O}_X$-modules is a compactly generated triangulated category. The map to the the point $X \to \mathrm{Spec}\mathbb{Z}$ induces a triangulated cohomology functor $Rf_* : D_{qc}(\mathcal{O}_X) \to D(\mathbb{Z})$. The triangulated right adjoint implied by Brown representability, $f^! : D(\mathbb{Z}) \to D_{qc}(\mathcal{O}_X)$ defines the dualizing complex $f^!\mathbb{Z} \in D_{qc}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules over X. The Tate cohomology of X is represented in the derived category by the homotopy cofiber (triangulated) of the norm map in the following diagram in $D(\mathbb{Z})$.

$$\begin{array}{ccccc} Rf_*f^!\mathbb{Z} & \longrightarrow & \mathcal{O}_{\mathbb{Z}} & & \\ & {\scriptstyle Norm}\searrow & \downarrow & & \\ & & Rf_*f^*\mathbb{Z} & \longrightarrow & t(\mathcal{O}_X) \end{array}$$

The Tate object $t(\mathcal{O}_X)$ is a cochain complex of abelian groups and the i-th Tate cohomology $t^i(X, \mathcal{O}_X) = H^i(t(\mathcal{O}_X)$.

The affine group scheme $\mathrm{Spec}(\mathbb{Z}[G])$, G a finite group, has a trivial action on $\mathrm{Spec}\mathbb{Z}$; the quotient stack has an atlas by an étale Hopf Algebroid $[\mathbb{Z} \rightrightarrows \mathbb{Z}[G]]$. The quasi-coherent cohomology of the structure sheaf over the quotient stack $\mathrm{Spec}\mathbb{Z}//G$ is the group cohomolgy of the $\mathbb{Z}[G]$-module $\mathbb{Z}$, reprensented by the cochain complex of $\mathbb{Z}[G]$-comodules which has $\mathbb{Z}$ in degree 0 and zeroes elsewhere. The dualizing complex of $\mathrm{Spec}\mathbb{Z}//G$ is defined $f^!\mathbb{Z}$; right adjoint to $Rf_* : D_{qc}\mathrm{Spec}\mathbb{Z}//G \to D(\mathbb{Z})$ (induced by the morphism of Hopf algebroids $[\mathbb{Z}[G] \rightrightarrows \mathbb{Z}] \to [\mathbb{Z} \rightrightarrows \mathbb{Z}]$) in the triangulated category. The chain complex $Rf_*f^!\mathbb{Z} \in D(\mathbb{Z})$ represents the group homology functor $H_*(G;\mathbb{Z})$. The homotopy cofiber (mapping telescope in the triangulated category) of the composition $Rf_*f^!\mathbb{Z} \to \mathbb{Z}Rf_*f^*\mathbb{Z}$ represents the Tate cohomology $t^*(G;\mathbb{Z})$.

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