1. Some topological preliminaries: the stable homotopy category.

1.1. **Homotopy groups and stable homotopy groups.** We will write $\mathbf{CGTop}_*$ for the category of compactly generated spaces[1] equipped with basepoints, and continuous maps preserving the basepoint. This is a symmetric monoidal category, with binary operation given by *the smash product*: if X, Y are two pointed topological spaces, we define their smash product $X \wedge Y$ as $\frac{X \times Y}{X \vee Y}$, the Cartesian product of X and Y but with the subspace $X \vee Y$, the one-point union of X and Y, collapsed to a point. The unit of this operation is the one-point space (pt.), i.e., there are natural homeomorphisms $(\mathrm{pt.}) \wedge X \cong X \wedge (\mathrm{pt.}) \cong X$. More interestingly, we can smash a space X with the 1-sphere S^1, and we often write ΣX for this space $S^1 \wedge X$, which we call the *suspension* of X.

Now one can form the homotopy category $\mathcal{H}o\ \mathbf{CGTop}_*$, by formally identifying together any pair of morphisms in $\mathbf{CGTop}_*$ which are homotopic; this category inherits the smash product from $\mathbf{CGTop}_*$ and it remains symmetric monoidal. One often writes $[X, Y]$ for the set of basepoint-preserving homotopy classes of maps from a pointed topological space X to a pointed topological space Y; this is exactly the hom-set $\hom(X, Y)$ in $\mathcal{H}o\ \mathbf{CGTop}_*$. Each sphere S^n, with $n \geq 1$, is a cogroup object[2] in $\mathcal{H}o\ \mathbf{CGTop}_*$, and when $n > 1$ the sphere S^n is a co-abelian-group object in $\mathcal{H}o\ \mathbf{CGTop}_*$; in other words, $[S^n, X]$ is a group for $n \geq 1$ and when $n > 1$ that group is abelian. Now we define the ith (unstable) homotopy $\pi_i(X)$ of a pointed space X as $[S^n, X]$; when $i = 0$ this is a set, when $i = 1$ this is a group, and when $i > 1$ this is an abelian group. When a space X satisfies $\pi_i(X) = 0$ for $i \leq n$, we say that X is n-connected.

The homotopy groups of a space are a very powerful invariant; Whitehead's theorem (a good reference is [19]) states that, if X, Y are connected topological spaces of the homotopy type of a CW-complex (this is a very weak technical condition), then a continuous map $X \stackrel{f}{\longrightarrow} Y$ is a homotopy equivalence if and only if $\pi_i(X) \stackrel{\pi_i(f)}{\longrightarrow} \pi_i(Y)$ is an isomorphism for all $i > 0$. So homotopy groups really are the right tool for studying topological spaces up to homotopy equivalence; but they are unruly in some ways. The functor $X \mapsto \oplus_{i>0} \pi_i(X)$ fails to be a *generalized homology theory*, in the sense of Eilenberg-Steenrod [5]; for instance, given a space X and two subspaces $A, B \subseteq X$ with the closure of A in X contained in the interior of B, even when you have an isomorphism in relative homology

$$H_n(X \backslash A, B \backslash A) \cong H_n(X, B),$$

(which you do, by the excision axiom), you don't usually get an isomorphism

$$\pi_n(X \backslash A, B \backslash A) \cong \pi_n(X, B)$$

in relative homotopy. Instead, you have the Blakers-Massey excision theorem (see [19]), which tells you that you have such an isomorphism when $\pi_i(X, A) = 0$ for

[1] A space X is "compactly generated" iff each of its closed subspaces has closed intersection with each compact subspace of X. This is a relatively mild technical condition which just rules out the worst pathological behaviors; [19] has an excellent section on this topic.

[2] You can look up the cogroup structure map in [18], or just ask me before or after the seminar; it has a very simple geometric description, given by "pinching" an equator of S^n. Alternatively, you can get the cogroup structure on S^n by studying the suspension functor Σ: there are homotopy equivalences $\Sigma S^{n-1} \simeq S^n$ for all $n > 0$, and ΣX actually has a canonical cogroup structure map, for any space X.

$i < a$, and $\pi_i(X, B) = 0$ for $i < b$, and $n < a + b - 2$. In other words, the excision axiom—which any generalized homology theory must satisfy—only works for homotopy groups when all the spaces involved are highly connected.

We will now "fix" homotopy groups so that they form a generalized homology theory. Given a homotopy class $f \in [X, Y]$, we can take the suspension and get a unique homotopy class $\Sigma f \in [\Sigma X, \Sigma Y]$. The map $f \mapsto \Sigma f$ turns out to be a group homomorphism when X is a cogroup object in $\mathcal{H}o\ \mathbf{CGTop}_*$, and we define the nth stable homotopy group $\pi_n^S(X)$ of a space X as the direct limit of the diagram of abelian groups

$$[S^{n+1}, \Sigma X] \to [S^{n+2}, \Sigma^2 X] \to [S^{n+3}, \Sigma^3 X] \to \dots. \tag{1}$$

The functor $X \mapsto \oplus_{n \geq 0} \pi_n^S(X)$ turns out to be a generalized homology theory: the direct limit above turns out to take homotopy groups and force them to obey the excision axiom, which fails without taking that limit. We can evaluate π_*^S on the zero-sphere S^0, and the resulting graded abelian group $\pi_*^S(S^0)$ has a commutative ring structure as well: given stable classes $\hat{f} \in [S^a, S^b]$ and $\hat{g} \in [S^c, S^d]$ representing elements $f \in \pi_{a-b}^S(S^0)$ and $g \in \pi_{c-d}^S(S^0)$, respectively, the element $fg \in \pi_{a-b+c-d}^S(S^0)$ is the image of the class $(\Sigma^{c-b} f) \circ g \in [S^{a+c-b}, S^d]$ under the limit of diagram 1. One can fill in the details to make this a well-defined graded commutative product.

Now the graded commutative ring $\pi_*^S(S^0)$ is called the *stable homotopy of spheres*, and acts on $\pi_*^S(X)$ for every space X; in fact it acts on much more than this, as we will see shortly. Some properties of this ring: $\pi_0^S(S^0) \cong \mathbb{Z}$ and in each dimension $n > 0$ it is a finite group, i.e., $\pi_n^S(S^0)$ is finite for every $n > 0$; for each prime p, there is no number n such that $\pi_i^*(S^0)$ has no p-torsion for $i > n$ (in other words, not only is there no dimension such that stable homotopy is trivial above that dimension, there is not even a prime p such that the p-local part of stable homotopy is trivial above some dimension); and $\pi_*^S(S^0)$ is nilpotent above dimension zero, i.e., for any element $f \in \pi_n^S(S^0)$ with $n > 0$, there is a positive integer i with $f^i = 0$.

1.2. **The category of spectra.** Now the homotopy classes of maps from X to Y form the hom-set $[X, Y] = \hom_{\mathcal{H}o\ \mathbf{CGTop}_*}(X, Y)$ in $\mathcal{H}o\ \mathbf{CGTop}_*$; we would like to form some category $\mathcal{H}o\ \mathcal{S}$ in which $\hom_{\mathcal{H}o\ \mathcal{S}}(X, Y)$ consists of the *stable* homotopy classes of maps from X to Y, in other words, the limit of the diagram 1. There is a notational objection to be made here: we are talking about $\mathcal{H}o\ \mathcal{S}$ without any mention of a model[3] category $\mathcal{S}$ whose homotopy category is $\mathcal{H}o\ \mathcal{S}$. This is because there many candidates[4] for $\mathcal{S}$ but essentially only one candidate for $\mathcal{H}o\ \mathcal{S}$.

[3]A *model category* is a category equipped with a distinguished class of morphisms called weak equivalences, a distinguished class of morphisms called fibrations, and a distinguished class of morphisms called cofibrations, satisfying appropriate axioms which are modelled on the properties of weak equivalences, cofibrations, and Serre fibrations or Hurewicz fibrations in $\mathbf{CGTop}_*$. From a model category structure, one gets a kind of homological algebra of additive categories which are not abelian; sometimes this is called "homotopical algebra." The idea is due to Quillen in [14] but the modern definitions can be found in [9]. Any model category $\mathcal{C}$ also has an associated homotopy category $\mathcal{H}o\ \mathcal{C}$, obtained by inverting the weak equivalences, and this is one way of getting $\mathcal{H}o\ \mathbf{CGTop}_*$ from $\mathbf{CGTop}_*$, which actually has a couple of natural model category structures, depending on whether you use Serre fibrations or Hurewicz fibrations.

[4]The original construction was Boardman's PhD thesis [2], and the reference that had to suffice for many years was [1], and our brief treatment here is based on it; unfortunately the smash product on Adams' category of spectra is "hand-crafted" and not quite a canonical construction, only well-defined up to homotopy, and there are problems with its associativity. The book [6] was

Our first try can proceed as follows: let a *spectrum* be a choice of pointed topological space X_i together with a basepoint-preserving continuous map $\Sigma X_i \to X_{i+1}$ for each $i \in \mathbb{Z}$. For example, to any topological space X we can associate the *suspension spectrum* $\Sigma^\infty X$ whose ith space is the one-point space (pt.) for $i < 0$ and is $\Sigma^i X$ for $i \geq 0$, and whose structure maps $\Sigma(\Sigma^{i-1}X) \to \Sigma^i X$ are just homeomorphisms, for $i > 0$. We let spectra be the objects of a category $\mathcal{S}$. Now, the morphisms from a spectrum X to a spectrum Y in this category need to be slightly more general than just maps $X_i \to Y_i$ for each i that commute with the structure morphisms; we need to have all those morphisms but also arbitrary desuspensions of them. We refer the reader to [1] for the (formidable) technical details of the morphisms and extending the smash product of topological smash to a smash product in $\mathcal{S}$. In the end, we get that $\mathcal{H}o\, \mathcal{S}$, the *homotopy category of spectra* or colloquially "the stable homotopy category," is a symmetric monoidal category (the monoidal operation is the smash product). Here are some of its most cogent properties:

- $\mathcal{H}o\, \mathcal{S}$ is an additive category.
- The suspension spectrum $\Sigma^\infty S^0$ of the zero-sphere, called the *sphere spectrum*, which plays such a fundamental role here that we write S for it, co-represents a functor $\mathcal{H}o\, \mathbf{CGTop}_* \longrightarrow \mathrm{Ab}$ for which we write π_0; in other words, if X is a spectrum, we write $\pi_0(X)$ for the abelian group $[S, X] \cong \hom_{\mathcal{H}o\, \mathcal{S}}(S, X)$. We also write $\pi_i(X)$ for the abelian group $[\Sigma^{-i}S, X]$, and $\pi_*(X) \cong \oplus_{i\in\mathbb{Z}}\pi_i(X)$ for the entire *stable homotopy* of X.
- The sphere spectrum S is the unit of the smash product operation on $\mathcal{H}o\, \mathcal{S}$, i.e., $X \wedge S \cong S \wedge X \cong X$.
- We have a colimit-preserving functor $\mathcal{H}o\, \mathbf{CGTop}_* \xrightarrow{\Sigma^\infty} \mathcal{H}o\, \mathcal{S}$, given by sending a space to its suspension spectrum, with the property that $\pi_*(\Sigma^\infty X) \cong \pi_*^S(X)$ for every topological space X. In other words, once we map $\mathbf{CGTop}_*$ into $\mathcal{H}o\, \mathcal{S}$, the functor taking a space to its stable homotopy groups is corepresentable.
- Any spectrum E represents a functor $\mathcal{H}o\, \mathbf{CGTop}_*^{\mathrm{op}} \xrightarrow{E^i} \mathrm{Ab}$ for each $i \in \mathbb{Z}$ which is given by $E^i(X) \cong [\Sigma^i X, E]$, which we call *$E$-cohomology in dimension i*. It also gives rise to a functor $\mathcal{H}o\, \mathbf{CGTop}_* \xrightarrow{E_i} \mathrm{Ab}$ for each $i \in \mathbb{Z}$ which is given by $E_i(X) \cong \pi_i(E \wedge X) \cong [\Sigma^{-i}S, X \wedge E]$, which we call *$E$-homology in dimension i*. Precomposing E-homology with the suspension spectrum functor always produces a generalized homology theory; in other words, $X \mapsto E_*(\Sigma^\infty X)$ is a generalized homology theory, in the sense of Eilenberg-Steenrod, on topological spaces. Precomposing E-cohomology with the suspension spectrum functor always produces a generalized cohomology theory, as well.
- *ALL* generalized homology and cohomology theories arise in this way; this is the content of the (stable) Brown representability theorem [4]. Loosely,

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a breakthrough in this area, producing a category of spectra, called "S-modules," with a well-defined and associative smash product, as well as many new constructions which allowed most of the constructions of algebra to be made in the category of spectra; Shipley, Schwede, and Smith developed the theory of symmetric spectra at around the same time, producing a different (but equivalent) category of spectra which also fixed the problems with the older spectra and their "hand-crafted" smash products. Finally, a new and very important version of the category of spectra is chapter 9 of [12], which constructs the category of spectra as a stable ∞-category and shows that it has a certain universal property among all stable ∞-categories.

> $\mathcal{H}o\,\mathcal{S}$ is the minimal category in which all generalized homology and cohomology theories on topological spaces become representable.

A priori, a generalized cohomology theory is just an additive theory; it does not have a cup product. A ring object in the additive category $\mathcal{H}o\,\mathcal{S}$, i.e., a spectrum E equipped with morphisms of spectra $E \wedge E \to E$ (a product map) and $S \to E$ (a unit map) satisfying appropriate unicity and associativity and distributivity axioms, is called a *ring spectrum*. A ring spectrum E represents a generalized cohomology theory E_* with a cup product, and any generalized cohomology theory E_* with a cup product is represented by a ring spectrum E.

Many constructions from algebra can be mimicked in $\mathcal{H}o\,\mathcal{S}$, by regarding spectra as abelian groups, ring spectra as rings, and the smash product as the tensor product: for instance, when E is a ring spectrum, we can let an *E-module* be a spectrum X equipped with a map $E \wedge X \to X$ satisfying certain axioms completely analogous to those that the structure map $R \otimes_{\mathbb{Z}} M \to M$ of a module M over a commutative ring R must satisfy. When E is a ring spectrum, we even have a relative smash product of spectra, $X \wedge_E Y$, analogous to the tensor product $M \otimes_R N$ over a ring R larger than the integers. We recover the usual smash product as the relative smash product over the sphere spectrum: $X \wedge_S Y \simeq X \wedge Y$. This is due to the sphere spectrum S being the initial ring object in $\mathcal{H}o\,\mathcal{S}$; the role of S in stable homotopy theory is analogous to the role of $\mathbb{Z}$ in commutative algebra.

The fact that S is the initial object in ring spectra means that we have a natural transformation $\pi_* \to E^*$ for any generalized cohomology theory E_* with a cup product; in other words, if E^* is any generalized cohomology theory with a cup product, and X is any space or spectrum, $E^*(X)$ is a module over the ring $\pi_*(S)$. This means that $\pi_*(S)$ is the underlying coefficient ring of essentially any construction one can conceivably make in algebraic topology, and one of the outstanding conjectures of the field, the Generating Hypothesis, essentially states that the homotopy category of suspension spectra of finite cell complexes is precisely the category of modules over $\pi_*(S)$; see [10] for details. This is some additional evidence that $\pi_*(S)$ is an important object.

Given any ring spectrum E, we also have a *Bousfield localization* functor $L_E : \mathcal{H}o\,\mathcal{S} \to \mathcal{H}o\,\mathcal{S}$, which is again analogous to localization in commutative algebra. The idea is as follows: we will call a map $X \xrightarrow{f} Y$ of spectra an E_*-equivalence if $E_*(X) \xrightarrow{E_*(f)} E_*(Y)$ is an isomorphism, and then, given a spectrum X, we want $L_E X$ to be a spectrum equipped with a map $X \to L_E X$ such that every E_*-equivalence $X \to Y$ extends to $L_E X$. Bousfield showed in [3] that $L_E X$ always exists, and that the localization process has nice functorial properties. One can put a partial ordering on ring spectra by saying that $E \leq F$ iff $F_*(X) \xrightarrow{F_*(f)} F_*(Y)$ implies $E_*(X) \xrightarrow{E_*(f)} E_*(Y)$ for every map $X \xrightarrow{f} Y$ of spectra; in other words, $E \leq F$ iff every F_*-equivalence is also an E_*-equivalence. The equivalence class of a ring spectrum E under this partial ordering is sometimes called the *Bousfield class* of E. It is an easy consequence of Bousfield's methods that the sphere spectrum S lies in the maximal Bousfield class, so in some very strong sense, stable homotopy π_* is the strongest, most data-rich possible generalized homology theory. This is further evidence that the coefficient ring $\pi_*(S)$ of stable homotopy is an important object.

1.3. **The E-Adams spectral sequence.** We hope that we have convinced the reader of the importance of $\pi_*(S)$. The single most useful tool for computing $\pi_*(S)$ is the family of spectral sequences called the *E-Adams spectral sequences*; the E here is supposed to be a ring spectrum.

To explain what the E-Adams SS is, we will first quickly detour into stack theory[5]. From the choice of a ring spectrum E, we have maps $S \xrightarrow{\eta} E$ and $E \wedge E \xrightarrow{\nabla} E$ which give the ring structure of E; we also have maps

$$E \xrightarrow{\simeq} S \wedge E \xrightarrow{\eta\wedge E} E \wedge E,$$
$$E \xrightarrow{\simeq} E \wedge S \xrightarrow{E\wedge\eta} E \wedge E,$$

and the factor swap map $E \wedge E \to E \wedge E$. Assume for a moment that $E_*(E)$ is flat and smooth as a $\pi_*(E)$-algebra, that $\pi_i(E) \cong 0$ for i odd, and that $E_i(E)$ is finitely generated as an $\pi_0(E)$-algebra for each i (these are technical conditions which are milder than they sound, and can be relaxed in some circumstances). We take the homotopy groups π_* of all these spectra and maps between, and apply Spec to get a groupoid object in the category of affine schemes:

$$\text{Spec } \pi_*(E \wedge E) \times_{\text{Spec } \pi_*(E)} \text{Spec } \pi_*(E \wedge E) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \text{Spec } \pi_*(E \wedge E) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \text{Spec } \pi_*(E)$$

As a category fibered in groupoids over affine schemes, this is a prestack in the fpqc (faithfully flat, quasicompact) topology; we stackify it (see [11] or the forthcoming [17]) to get a stack $\mathcal{X}_E$ equipped with an fpqc cover $\text{Spec } \pi_*(E) \to \mathcal{X}_E$; the cover is a filtered homotopy limit[6]of fppf (faithfully flat, finitely presented) algebraic stacks $(\mathcal{X}_E)_i$ equipped with affine presentations $(X_E)_i \to (\mathcal{X}_E)_i$:

$$\begin{array}{ccccccccc}
\text{Spec } \pi_*(E) & \xrightarrow{\simeq} & \text{holim}_i (X_E)_i & \longrightarrow & \cdots & \longrightarrow & (X_E)_2 & \longrightarrow & (X_E)_1 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{X}_E & \xrightarrow{\simeq} & \text{holim}_i (\mathcal{X}_E)_i & \longrightarrow & \cdots & \longrightarrow & (\mathcal{X}_E)_2 & \longrightarrow & (\mathcal{X}_E)_1 .
\end{array}$$

Now flat cohomology $H^*_{\text{fl}}(\mathcal{X};\mathcal{F})$ is defined as the derived functors of global sections on the category of $\mathcal{O}_{\mathcal{X}}$-modules, where $\mathcal{O}_{\mathcal{X}}$ is the structure ring sheaf of the[7] local

[5]The same results can be accomplished with Hopf algebroids, which are cogroupoid objects in commutative rings; but this requires just as much explanation as algebraic stacks, while stacks are at least familiar to algebraic geometers. The reference for Hopf algebroids and this point of view is [16].

[6]Stacks do not just form an ordinary category; they also form what is called a $(1,1)$-category, which is a 2-category in which all the 2-morphisms are 2-isomorphisms—so they look like homotopies. As a result we have not only limits but (usually more useful) homotopy limits in the category of stacks. Typically when we write $X \times_Y Z$ and any of X, Y, Z are stacks, we mean the homotopy fiber product rather than the strict fiber product, which does not give quite the right answers for stacks; and so on.

[7]It does not matter for cohomological purposes whether we use the big or the small fpqc site, as the change-of-sites spectral sequence collapses in this case. We use the fpqc topology as defined in [7]. This should eliminate any indeterminacy in speaking of "the" fpqc site.

Let S be a base scheme. We will describe the local big fpqc site $\mathcal{X}_{fpqc}$ on an fpqc S-stack $\mathcal{X}$ admitting a schematic fpqc epimorphism $X \to \mathcal{X}$ with X an affine S-scheme: the objects in $\mathcal{X}_{fpqc}$ consist of all affine S-schemes U equipped with morphisms $U \xrightarrow{f} \mathcal{X}$, a morphism from (U, f) to (V, g) consists of a pair (ϕ, θ) with $U \xrightarrow{\phi} V$ a morphism of S-schemes and $f \xrightarrow{\theta} g \circ \phi$ a 2-isomorphism, and a family (U_i, f_i) is a covering if the pullback family $\{U_i \times_{\mathcal{X}} X, f_i \times_{\mathcal{X}} X\}$ is an fpqc covering of X.

fpqc site on $\mathcal{X}$. The presentation Spec $\pi_*(E) \to \mathcal{X}_E$ is an affine presentation by an affine scheme, so by inspection of the associated Čech nerve, we have that the Čech-to-derived-functor-cohomology spectral sequence collapses, and we have a natural isomorphism:

$$H^*_{\mathrm{fl}}(\mathcal{X}_E; \mathcal{F}) \cong \check{H}^*_{\mathrm{fl}}((\mathrm{Spec}\ \pi_* E)/\mathcal{X}_E; \mathcal{F}),$$

so we have the Čech complex available[8] for computing the flat cohomology of $\mathcal{X}_E$.

Every space or spectrum X gives rise to a $\mathcal{O}_{\mathcal{X}_E}$-module $E_*(X)^{\sim}$. The fact that $\pi_*(E)$ and $E_*(E)$ are graded means we have additional structure on the stack $\mathcal{X}_E$, namely, we have a $\mathbb{G}_m$-action on it, and a compatible $\mathbb{G}_m$-action on its affine cover Spec $\pi_*(E)$; the fixed-points $(\mathrm{Spec}\ \pi_*(E))^{\mathbb{G}_m}$ recover Spec $\pi_0(E)$, for example, while the fixed points $(\mathrm{Spec}\ \pi_*(E) \times_{\mathcal{X}_E} \mathrm{Spec}\ \pi_*(E))^{\mathbb{G}_m}$ recover Spec $E_0(E)$.

Now we can describe the E-Adams spectral sequence: let E be a ring spectrum and let X be a space or spectrum, and we assume that E is well-behaved in a number of technical ways (see [16] and [15] for the details). The *E-Adams spectral sequence* is a spectral sequence

$$E_2^{s,t} \cong \begin{cases} H^0(\mathbb{G}_m; H^s_{\mathrm{fl}}(\mathcal{X}_E; E_*(X)^{\sim}) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \Omega^{t/2}_{\mathbb{G}_m}) & \text{if } 2 \mid t \\ 0 & \text{if } 2 \nmid t \end{cases}$$

with $E_\infty^{s,t}$ a bigraded algebra corresponding to a certain filtration of $\pi_*(L_E X)$. In particular, $\pi_n(L_E X)$ comes from the $n = t - s$ line in the abutment[9].

When E and X are chosen well, $L_E X$ is a very meaningful object; for instance, when E is chosen to be the spectrum MU representing complex cobordism, we have $\pi_* L_{MU} S \cong \pi_* S$, and when E is chosen to be the Brown-Peterson spectrum BP for a particular prime p (the choice of prime is suppressed in the notation BP), we get $\pi_* L_{BP} S \cong (\pi_* S)_{(p)}$, the p-localization, in the sense of commutative algebra, of $\pi_* S$. The MU-Adams SS and the BP-Adams SS are of such great importance in computations of $\pi_* S$ that they are each called *the Adams-Novikov spectral sequences* or *ANSS*. At primes $p > 2$, of all the spectral sequences that converge to $(\pi_* S)_{(p)}$ that could be used for computations, the BP-Adams SS has by far the fewest differentials and by far the simplest E_2-term (which, however, is still complicated enough that we are nowhere near being able to describe the entire E_2).

The stacks whose cohomology give the Adams-Novikov E_2-terms are actually familiar stacks from algebraic number theory. We describe them now.

1.4. **Formal group laws, formal Lie groups, and the ANSS.** Let R be a commutative ring. A *one-dimensional commutative formal group law over R* (we will "abbreviate formal group law" as FGL) is a power series $F(X, Y) \in R[[X, Y]]$ satisfying:

In forming the site $\mathcal{X}_{fpqc}$ we have happily "squashed out" the 2-category structure and the topos of abelian sheaves on this site, or of modules on its structure ring sheaf, is a Grothendieck topos with typical properties, not a 2-topos.

[8] The fact that Čech cohomology agrees with derived functor cohomology for the stacks coming from these ring spectra E is equivalent to the fact that Ext in the category of E_*E-comodules is isomorphic to the derived functors of the cotensor product, and so the cohomology of the cobar complex is the cohomology of the Hopf algebroid (E_*, E_*E).

[9] The stack $\mathcal{X}_E$ is (something like a) Tannakian groupoid of the E-local stable homotopy category—in other words, the image of the functor $X \mapsto L_E X$ on spectra. I do not know Tannakian categories well enough to understand this perspective, but perhaps Jack will say more about it.

- $F(X, F(Y, Z)) = F(F(X, Y), Z)$ (associativity),
- $F(0, X) = F(X, 0) = X$ (unitality),
- there exists a power series $i(X) \in R[[X]]$ with $F(X, i(X)) = F(i(X), X) = X$ (existence of inverses), and
- $F(X, Y) = F(Y, X)$ (commutativity).

Given two FGLs F, G over R, a *strict isomorphism* $F \xrightarrow{f} G$ is a power series $f(X) \in R[[X]]$ satisfying $f(F(X, Y)) = G(f(X), f(Y))$ and $f(X) \equiv X \mod (X^2)$. An FGL $F(X, Y) = \sum_{i,j \geq 1} \alpha_{i,j} X^i Y^j$ over R can be base-changed along a commutative ring morphism $R \xrightarrow{f} S$ to yield an FGL $f_* F(X, Y) = \sum_{i,j \geq 1} f(\alpha_{i,j}) X^i Y^j$. This data is enough to define a moduli problem: one wants to represent the groupoid-valued functor $\mathbf{Aff}/\text{Spec } \mathbb{Z} \xrightarrow{\theta} \mathbf{Groupoids}$ where $\theta(S)$ is the groupoid of FGLs over $\Gamma(S)$ (the morphisms in the groupoid are given by strict isomorphisms of FGLs). This moduli problem is solved by an fpqc prestack $\mathcal{X}_{FGL}$ over Spec $\mathbb{Z}$ which fails to be an fpqc stack. Lazard constructed (see [8]) a certain affine fpqc epimorphism Spec $L \rightarrow \mathcal{X}_{FGL}$ such that Spec L bears a $\mathbb{G}_m$-action compatible with a certain natural $\mathbb{G}_m$-action on $\mathcal{X}_{FGL}$.

We now define a related object: a *pointed one-dimensional formal Lie variety* over a base scheme S consists of a sheaf of sets X on the local fppf site of S together with a section $S \rightarrow X$, such that Zariski locally, $X \cong \text{Spf } \mathcal{O}_S[[T]]$ pointed by the zero-section[10]. A *one-dimensional commutative formal Lie group* (which we will abbreviate FG) is a commutative group object in pointed one-dimensional formal Lie varieties whose given section is the identity of the group operation. One sees that, Zariski locally, a formal group is determined simply by a choice of isomorphism with Spf $\mathcal{O}_S[[T]]$ (i.e., a choice of coordinate T) and a formal group law over the sections of that Zariski neighborhood. We call a morphism of FGs a *strict isomorphism* if it is an isomorphism inducing the identity map in the first infinitesimal neighborhoods of the FGs. This is enough information to construct a moduli problem, and this time, it is solved by an fpqc stack $\mathcal{X}_{FG}$ which is a filtered homotopy limit of fppf algebraic stacks.

Now we can state the important result. Let MU_* be complex bordism, the generalized homology theory; we will write $\mathcal{Y}_{MU}$ for the $\mathbb{G}_m$-equivariant prestack represented by (Spec $\pi_*(MU)$, Spec $MU_* MU$), and we will write $\mathcal{Y}^+$ for the stackification of a prestack $\mathcal{Y}$; so we have $\mathcal{Y}^+_{MU} \cong \mathcal{X}_{MU}$ by definition. Then we have a 2-commutative diagram of $\mathbb{G}_m$-equivariant fpqc prestacks over Spec $\mathbb{Z}$:

$$\begin{array}{ccccccccc}
\text{Spec } \pi_*(MU) & \xrightarrow{\cong} & \text{Spec } L & \xrightarrow{\cong} & \text{Spec } L & \xrightarrow{\cong} & \text{Spec } L & \xrightarrow{\cong} & \text{Spec } \pi_*(MU) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Y}_{MU} & \xrightarrow{\cong} & \mathcal{X}_{FGL} & \xrightarrow{i} & \mathcal{X}^+_{FGL} & \xrightarrow{\simeq} & \mathcal{X}_{FG} & \xrightarrow[f]{\simeq} & \mathcal{X}_{MU},
\end{array}$$

with the maps marked $\simeq$ being ($\mathbb{G}_m$-equivariant) fpqc homotopy equivalences; the map i is the stackification map. As a result the E_2-term of the Adams-Novikov spectral sequence has equivalent descriptions:

$$H^0(\mathbb{G}_m; H^s_{\text{fl}}(\mathcal{X}_{MU}; MU_*(X)^{\sim}) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \Omega^t_{\mathbb{G}_m}) \cong H^0(\mathbb{G}_m; H^s_{\text{fl}}(\mathcal{X}_{FG}; f^*(MU_*(X)^{\sim})) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \Omega^t_{\mathbb{G}_m}).$$

[10] See [13] for a more concrete, but much longer, definition.

In other words, the flat cohomology of the moduli stack of formal groups is a close approximation to the stable homotopy of spheres, and to compute $\pi_* S$, we must compute the flat cohomology of that stack.

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