Local Langlands and chromatic homotopy theory

[Notes from a talk by Paul Goerss at the Fields Institute, September 2004]

§1 A generalized overview of local Langlands

[marginalia: $\pi_* S^0$ & what appears to be a picture of a tiny choo-choo train?]

Let Γ be a formal group of finite height h over $\mathbb{F}_p \subset k \subset \mathbb{F}_p^{ac}$ and let

$$\mathcal{O} = \mathrm{End}(\Gamma) = W\langle S \rangle / (S^h - p) \ ,$$

where $W = W(k)$ is the ring of Witt vectors; the pointy brackets entail that $Sa = \phi(a)S$ if $a \in W(k)$, ϕ being induced by the action of the Frobenius endomorphism of the residue field. $\mathcal{O}[p^{-1}] = B$ is a division algebra containing $K = W[p^{-1}]$, with Hasse invariant $h^{-1}[\in \mathbb{Q}/\mathbb{Z} = \mathrm{Br}(K)]$. The group $\mathrm{Aut}(\Gamma) = \mathcal{O}^\times$ is the maximal compact subgroup of $B^\times$.

Topologists are primarily interested in representations of $\mathrm{Aut}(\Gamma)$.]

For practical purposes the Weil group W_K is the fiber product of the reduction-to-the-residue-field homomorphism

$$\mathrm{Gal}(K^{ac}/K) \to \mathrm{Gal}(k^{ac}/k) \cong \hat{\mathbb{Z}}$$

by the dense embedding $\mathbb{Z} \to \hat{\mathbb{Z}}$ (which sends a generator to the Frobenius element in $\mathrm{Gal}(k^{ac}/k)$). This product is a locally compact totally disconnected topological group with abelianization W_K^{ab} canonically isomorphic (by classfield theory) to the multiplicative group $K^\times$ of K.

A certain submonoid

$$A = A_K \subset \mathrm{Gl}_h(K) \times B^\times \times W_K$$

(see §4 for details) has an interesting representation defined over $\mathbb{Q}_l$ (with $l \neq p$: hereafter, an l-adic representation) which splits up to define a system of correspondences between (certain classes of) l-adic representations of each of the three groups $\mathrm{Gl}_h(K)$, $B^\times$, and W_K.

[More precisely (cf. §5): Jacquet and Langlands defined a correspondence between l-adic representations of $\mathrm{Gl}_h(K)$ and those of $B^\times$, and Langlands later conjectured the existence of a correspondence between (certain infinite-dimensional 'automorphic') representations of $\mathrm{Gl}_h(K)$ and h-dimensional representations W_K, generalizing the Artin isomorphism of classfield theory (corresponding to the case $h = 1$).]

Remarks: Much of this machinery is familiar in chromatic homotopy theory. The representation of A gives a very explicit dictionary for comparing various classes of (carefully defined, irreducible) representations of these various groups. Note that this is essentially rational information, so techniques are going to matter more than results. These correspondences can be formulated very explicitly in terms of characters and tensor products ...

§2 Morava modules and Lubin-Tate spaces

Let $G/R_{\Gamma,k}$ be the universal deformation of a formal group Γ/k of finite height: it is defined over a complete local ring

$$R_{\Gamma,k} \cong W(k)[[u_1, \ldots, u_{h-1}]]$$

(compact in the natural topology). From now on we fix the height h, and supress it). There is an associated periodic complex-oriented homology theory $MU_*(-) \otimes E_*$ with coefficient ring $E_* = R_{\Gamma,k}[u^{\pm}]$, where u is a bookkeeping indeterminate of degree ± 2.

Let $\mathbb{G}$ be the group of Hopf algebra automorphisms of Γ, i.e.

$$\mathrm{Aut}(\Gamma, k) := \{\phi : k \cong k, f : \Gamma \cong \phi^*\Gamma\} \cong \mathrm{Gal}(k/\mathbb{F}_p) \rtimes \mathcal{O}_n^\times \ ;$$

this is some sort of p-adic Lie group, which acts on E_* by naturality. The functor

$$Y \mapsto E_*Y := \pi_* L_{K(n)}(Y \wedge E)$$

is one of the building blocks for the chromatic tower; it's not quite a cohomology theory, since Bousfield localization $L_K(n)$ takes cofibrations to cofibrations, but has complicated limit properties ... That

$$E_*E = \mathrm{maps}_c(\mathbb{G}, E_*)$$

is a basic computation. It follows that E_*Y is

• a continuous E_*-module, with a continuous linear action of $\mathbb{G}$, such that if $a \in E_*,\ x \in E_*(X),\ g \in \mathbb{G}$, then $g(ax) = g(a)g(x)$, and that

• the action of $\mathbb{G}$ on $E_*(X) \otimes_R k$ is **discrete**.

This latter condition implies that the completed group ring $R\langle\langle\mathbb{G}\rangle\rangle$ is **not** of this type. Note also that u-multiplication defines a periodicity isomorphism on E_*Y; such a module is thus determined by its components in degrees 0 and -1, together with the action of $\mathbb{G}$ on E_2. We can therefore restrict our attention to such modules **without** grading.

Let $X = X_{LT}$ denote the formal scheme Spf $R_{\Gamma,k}$: then we have an equivalence between the category of such modules and $\mathbb{G}$-equivariant quasicoherent sheaves on X_{LT}. (If Y is a finite complex?) there is a kind of Adams spectral sequence (when we talk about topology then maybe K better be $\mathbb{Q}_p$)

$$H_c^*(\mathbb{G}, E_*Y) \Rightarrow \mathrm{gr}^* \pi_* L_{K(n)} Y \ldots$$

Examples:

- Powers $\omega^{\otimes n}$ of the invariant differentials on the universal deformation G.
- The rigid-analytic theory of Hopkins and Gross defines an isomorphism

$$\Lambda^{h-1}\Omega_{X_{LT}/W} \cong \omega^{h(h-1)}$$

(where Ω is the Kähler differentials, or alternately the cotangent bundle, of X_{LT} over Spec W).

§3 Covers of X_{LT}

Let G be a deformation of Γ over some base object S thickening Spec k. We can think of G as a functor from nilpotent S-algebras to abelian groups, such that the kernel $G(n)$ of multiplication by p^n is a groupscheme noncanonically isomorphic (in some suitable topology?) to $(\mathbb{Z}/p^n\mathbb{Z})^h_S$.

A level n structure on G is the choice of such an isomorphism. Note that $G(n)$ is also a codimension one subobject of the formal group, which defines a divisor on G; taking this seriously leads to the notion of a Drinfel'd level structure ... A **full** level structure on G is the choice of a system of identifications of the p-torsion points of G with $(\mathbb{Q}/\mathbb{Z})^h(p)$.

Theorem: The functor from S to the (set of equivalence classes of) deformations of Γ with level n structure is represented by a Galois extension $D_{n,\Gamma}$ of R_Γ, with Galois group $\mathrm{Gl}_h(\mathbb{Z}/p^n\mathbb{Z})$.

Similarly, the moduli problem for full Drinfel'd level structures is represented by an extension with Galois group $\mathrm{Gl}_h(\mathbb{Z}_p)$. Thus $X_n = \mathrm{Spf}\ D_n$ is a system of étale (even Galois) covers

$$X_\infty \to \ldots \to X_2 \to X_1 \to X_0 = X_{LT} \ ,$$

and quasicoherent sheaves over X_{LT} can be recovered by descent from sheaves over X_n. Since D_n is étale and hence faithfully flat over R, $D_n \otimes E$ defines a cohomology theory, from which E can be recovered as the subring of $\mathrm{Gl}_h(\mathbb{Z}/p^n\mathbb{Z})$-invariant elements.

These constructions have found a use in homotopy theory, e.g. in the study of the analog of the Dyer-Lashof algebra for E ... By using formal W-modules instead of formal groups, we can define a more general class of level structures ...

§4 Actions on X_∞

We need to be more explicit about the object A_K of the introduction. For simplicity let's assume from now on that K is $\mathbb{Q}_p$.

Let $\nu : \mathbb{Q}_p^\times \to \mathbb{Q}$ be the p-adic order, normalized so $\nu(p) = 1$; then $\nu(x) \geq 0$ iff

x is a p-adic integer. Let

$$v_{Gl} = \nu \circ \det : \mathrm{Gl}_h(\mathbb{Q}_p) \to \mathbb{Q}$$

assign to a matrix, the valuation of its determinant. A division algebra splits over an algebraically closed field, so we can regard $B^\times$ as a group of invertible matrices over such a field; let v_B be the valuation of the determinant of such a realization, normalized so $v_B(S) = 1$. Thus $v_B(\phi) \geq 0$ iff $\phi \in \mathrm{End}(\Gamma)$. Finally, define a valuation v_W on the Weil group by composing $W^{ab}_{\mathbb{Q}} \cong \mathbb{Q}_p^\times$ with the standard p-adic valuation. With these definitions, then,

$$A_{\mathbb{Q}_p} := \{(g, b, \sigma) \mid v_{\mathrm{Gl}}(g) = v_B(\phi) + v_W(\sigma)\} \; .$$

Proposition: The group A acts naturally on X_∞, compatible with the action of $\mathcal{O}^\times$ on X_{LT} described above.

[This action has been previously employed by Ando, Hopkins, and Rezk (together and separately) ...]

For example: suppose that $g \in M_h(\mathbb{Z}_p)$ and $\phi \in B^\times$ with $v_{\mathrm{Gl}}(g) = v_B(\phi) = n \geq 0$; then ϕ is an endomorphism of Γ, and the Pontrjagin dual g^* of g is an endomorphism of $(\mathbb{Q}/\mathbb{Z})^h(p)$ with kernel H isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^h$. A full level structure $(\mathbb{Q}/\mathbb{Z})^h(p) \to G \in X_\infty(R)$ allows us to identify H with a finite subgroupscheme of G, and thus to define the quotient morphism, or isogeny, $G \to G/H$.

[**unset diagram**]

Over the residue field, however, H has no points, and there is a unique isomorphism along the bottom of the commutative diagram

[**unset diagram**]

(where the quotient map $i : R \to R/\mathbf{m} = k$ represents inclusion of the closed point). This defines a new lift of Γ, together with a level structure on it.

For elements of A of the form $(g, 1, \sigma)$, we make a similar construction, with ϕ replaced by the appropriate power of the Frobenius element.

§5 A slightly more precise statement of local Langlands

Some history is relevant here. Jacquet and Langlands associated to an irreducible representation of $B^\times$ (in general, with center K), an irreducible ('essentially square-integrable) representation of $\mathrm{Gl}_h(K)$; this correspondence, which we'll denote by JL, has known image.

There is a different Langlands correspondence [BDKV,Rog?] which assigns to irreps of $\mathrm{Gl}_h(K)$, semisimple representations of the Weil group W_K; again the image of this correspondence is known. We'll denote it 'rec' (for reciprocity).

Both JL and rec are characterized by various formal properties of their characters and tensor products.

Now X_∞, regarded as an object over Spec$\mathbb{Z}_p$, has a generic fiber over the open point and a special fiber over the closed point:

[**unset diagram**]

There is an elaborate theory of 'vanishing' cycles (also called evanescent; there is a closely related notion of 'nearby' cycles) in algebraic geometry, going back to Picard and Lefschetz, where the role of Spec $\mathbb{Z}_p$ is played by the punctured disk $\mathbb{C}^\times$. These ideas were extended (largely by Deligne and Grothendieck?) to the present arithmetic context.

Of course higher derived functors are necessary. The module

$$R^q\Psi_\eta = i^{-1}R^q j_* j^{-1}\mathbb{Q}_l^{ac}$$

of derived nearby cycles is some kind of A-module by naturality. The local Langlands correspondence can now be formulated somewhat more precisely [cf. Harris & Taylor, or Carayol] as a kind of

pre-Theorem: If P is an irreducible l-adic representation of $B^\times$ then

$$\mathrm{Hom}_{\mathcal{O}^\times}(P, R^{h-1}\Psi_\eta) \cong \check{JL}(P) \otimes \mathrm{rec}(JL(P) \otimes |\det|^{1-h/2}) \; .$$

Here the check denotes a kind of dual, and the determinant is (I think?) related to the bundle ω back in §2. [I [JM] suspect that the other derived nearby cycle groups are zero?]

Topologists would rather know about $R^*\Psi(\mathbb{Z}/p^k\mathbb{Z})$. There is apparently some work on crystalline cohomology by Pascal Boyer which might be related but it doesn't seem to be available yet.

§6 Shimura varieties

The so-called 'simple' Shimura varieties relevant to local Langlands parametrize certain families of Abelian varieties over number fields. Formulating these moduli problems precisely involves a **lot** of extra structure which for the purposes of this talk are perhaps irrelevant. In particular one has a certain totally real field F of degree $d = [F : \mathbb{Q}]$, and one wants to represent a functor which assigns to a ring R, a set $\mathrm{Sh}_n(R)$ of (polarized) Abelian varieties A of dimension dn^2 over Spec$R$ endowed with level structure (to guarantee that the representing object be a scheme, rather than just a stack). The most important further condition involves a certain division algebra B (over a number field; its connection to the division algebra of isogenies of the relevant formal group leaves your amanuensis sorely confused) contained in the rationalized endomorphism algebra of the Abelian variety, which somehow entails that the Lie algebra of A, tensored with

$\mathbb{Q}_p$, contains a one-dimensional summand with special properties. The main consequence of this condition is that if p is nilpotent in R, then the p-divisible group $A[p^\infty]$ contains a natural one-dimensional infinitesimal subgroup G_A.

This construction defines a map $\bar{q}$ from the special fiber $\overline{\mathrm{Sh}}_n$ to the moduli stack of one-dimensional formal groups, which assigns $G_A \subset A_{\hat{e}}$ to A. This suggests the

Natural question: Does $\bar{q}$ extend to a (presumably flat?) morphism q defined in an open neighborhood of the special fiber; and, if so, does Lurie's technology apply to it?

There is a filtration of the special fiber by height (cf. Harris - Taylor):

$$\overline{\mathrm{Sh}}_n^{(1)} \subset \overline{\mathrm{Sh}}_n^{(2)} \subset \cdots \subset \overline{\mathrm{Sh}}_n^{(n)} = \overline{\mathrm{Sh}}_n \ ,$$

such that $\overline{\mathrm{Sh}}_n - \overline{\mathrm{Sh}}_n^{(n-1)}$ consists of finitely many components, each isomorphic to X_{LT}. Thus, the vanishing cycles for Sh_n can be studied stratum by stratum . . .

Background references:

Bernstein, Deligne, Kazhdan, Vigneras (89)

Carayol (Nonlinear Lubin-Tate, Bourbaki 859)

Deligne (Bourbaki 389)

Harris-Taylor (Annals studies 59?), Henniart . . .

Langlands SLN 349, [Ein Märchen ?]

Rogawski (Duke 83)