# 1 The derived Carayol attack (one approach to topological Langlands), part 1.

In this first part of these notes we discuss the topological Lubin-Tate and Drinfeld towers. In the next part we will discuss derived Shimura stacks, i.e., the spectra of topological automorphic forms, and relations between the derived Shimura stacks and the topological Lubin-Tate and Drinfeld towers. In a third part we will discuss applications.

By the "Carayol attack" I mean the particular approach to constructing a Langlands correspondence over a local number field which is due to Henri Carayol and which he called "non-abelian Lubin-Tate theory." The moduli objects used in Carayol's attack on local Langlands admit certain topological enrichments; in particular, their structure sheaves naturally lift to presheaves of spectra (the fundamental objects of study in stable homotopy theory), such that composing these presheaves with the homotopy groups functor $\pi_*$ recovers the original structure sheaves. These are examples of *derived moduli stacks*, although some of them fit into the formalism developed by Lurie and by Toen and Vezzosi, while others don't quite. A number of papers on the subject of derived algebraic geometry and derived moduli problems have appeared in the past five years; we recommend the paper [**?**] of Toen and Vezzosi and also the multi-volume, highly technical and complete but at times also very expository work by Jacob Lurie, "Derived Algebraic Geometry" (as I write this, preprints of the first five volumes, DAG I through DAG V, are available for download, and Lurie has announced that at least one more volume is in preparation).

In algebraic geometry one encounters moduli problems; one has, for instance, some functor

$$\mathbf{CAlg} \xrightarrow{F} \mathbf{Sets}$$

from commutative rings to sets; one could let $F(R)$ be the set of all elliptic curves over $R$, for example, or the set of formal group laws over $R$, or the set of multiplicative units in $R$. The problem here is to construct an object which co-represents $F$, i.e., some commutative algebra $X$ such that we get a natural isomorphism

$$\hom_{\mathbf{CAlg}}(X, R) \cong F(R).$$

It is often the case that this isn't possible, and we have to work in an enlargement of the category of commutative rings, such as the category of schemes, or a category of algebraic spaces, or a category of algebraic stacks, in order to find a (co-)representing object for $F$. These ideas have a long history in algebraic geometry.

Some of the moduli objects arising in algebraic geometry also bear extra topological structure; in particular, if $\mathcal{X}$ is an algebraic stack solving a certain moduli problem, one sometimes has a presheaf $\mathcal{F}$ of spectra—the basic objects of study in stable homotopy theory—such that $\pi_0(\mathcal{F})$ or $\pi_*(\mathcal{F})$ recovers the structure sheaf $\mathcal{O}_{\mathcal{X}}$ of the stack $\mathcal{X}$. For example, let $\mathcal{M}_{fg}$ be the moduli stack of formal groups; this is an fpqc stack over Spec $\mathbb{Z}$ which admits a canonical

$\mathbb{G}_m$-action, and it is also canonically a filtered $\mathbb{G}_m$-equivariant homotopy limit of fppf algebraic stacks over Spec $\mathbb{Z}$ (these fppf algebraic stacks are the stackifications of the moduli prestacks of formal group law "$n$-buds"). To any affine scheme Spec $R$ equipped with a $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant flat affine map Spec $R \to \mathcal{M}_{fg}$, the *Landweber exact functor theorem* of stable homotopy theory produces a spectrum $E_R$ with the property that

$$\pi_i(E_R) \cong H^0(\mathbb{G}_m, \Gamma(\text{Spec } R) \otimes_{\mathcal{O}_{\mathbb{G}_m}} \Omega^{i/2}_{\mathbb{G}_m/\text{Spec } \mathbb{Z}}),$$

or in other words, the $\mathbb{G}_m$-action on Spec $R$ is equivalent to a grading on $R$ concentrated in even dimensions (so that we do not get any anticommutativity due to the graded commutativity sign convention), and the homotopy groups of $E_R$ are isomorphic to $R$ as a graded abelian group. One may say that the fpqc structure sheaf $\mathcal{O}_{\mathcal{M}_{fg}}$ admits a sort of "stable-homotopical enrichment," the *Landweber presheaf* $\mathcal{L}$ of spectra, with the property that

$$\pi_*(\mathcal{L}(\text{Spec } R)) \cong \mathcal{O}_{\mathcal{M}_{fg}}(\text{Spec } R) \cong R$$

as graded abelian groups, with Spec $R \to \mathcal{M}_{fg}$ any $\mathbb{G}_m$-equivariant fpqc open. We will call any such presheaf of spectra a *derived* $\mathcal{O}_{\mathcal{M}_{fg}}$*-module* (in slight defiance of standard terminology). The Landweber presheaf is *not* a presheaf of ring spectra, i.e., not a derived $\mathcal{O}_{\mathcal{M}_{fg}}$-algebra; there is no canonical way to produce a ring structure, $E_\infty$ or otherwise, on the spectrum $\mathcal{L}(\text{Spec } R)$ (and indeed there is a "folk counterexample" of a $\mathbb{G}_m$-equivariant fpqc open such that the spectrum $\mathcal{L}(\text{Spec } R)$ cannot be given a ring structure).

Here is another example: fix a prime $p$, and let $F_n$ be the universal height $n$ formal group law over $\mathbb{F}_{p^n}$ (one has that formal group laws are classified by height over $\overline{\mathbb{F}_{p^n}}$, but in fact all height $n$ formal group laws become isomorphic under base change to $\mathbb{F}_{p^n}$). Consider the functor $\text{Def}_{F_n}$ on characteristic 0 local artin rings given by sending the ring $R$ to the set of deformations of $F_n$ to $R$, modulo the equivalence relation given by letting two deformation be equivalent if they agree modulo the maximal ideal (this equivalence relation is called "$\star$-isomorphism"); by general deformation theory methods (e.g. Schlessinger's theorem) one has that $\text{Def}_{F_n}$ is pro-representable, and Lubin and Tate computed the pro-representing object, which we will call $\text{lifts}_n$:

$$\text{lifts}_n \cong W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]].$$

One extends the functor $\text{Def}_{F_n}$ to a sheaf on the category of schemes by letting $\text{Def}_{F_n}$ be pro-represented by Spf $\text{lifts}_n$. We call Spf $\text{lifts}_n$ *height $n$ Lubin-Tate space.*

As just a functor from schemes to sets, $\text{Def}_{F_n}$ is sort of stupid; it has extra structure, though, which is of great importance and which we now describe. First of all, the strict automorphism group $\text{Aut}(F_n)$, which stable homotopy theorists call the *Morava stabilizer group*, acts functorially on the set of $\star$-isomorphism classes of deformations of $F_n$ to $R$ for any $R$, and so $\text{Aut}(F_n)$ acts on Spf $\text{lifts}_n$.

In the Carayol approach to the Langlands correspondences over $\mathbb{Z}_p$, one considers not only the moduli of deformations of $F_n$, but the moduli of deformations of $F_n$ with a Drinfeld level $p^i$-structure (again, modulo $\star$-isomorphism); this moduli problem is also pro-representable, by a flat (but not étale) cover $\mathrm{Spf}\ \mathrm{lifts}_{n,i}$ of $\mathrm{Spf}\ \mathrm{lifts}_n$. Now one has a tower of flat covers

$$\begin{array}{c} \cdots \\ \downarrow \\ \mathrm{Spf}\ \mathrm{lifts}_{n,3} \\ \downarrow \\ \mathrm{Spf}\ \mathrm{lifts}_{n,2} \\ \downarrow \\ \mathrm{Spf}\ \mathrm{lifts}_n \end{array}$$

sometimes called *the Lubin-Tate tower.* We have an action of $GL_n(\mathbb{Z}/p^i\mathbb{Z})$ on $\mathrm{Spf}\ \mathrm{lifts}_{n,i}$ for each $i$, given by permuting the level structure, and on the limit of the Lubin-Tate tower, $\lim_i \mathrm{Spf}\ \mathrm{lifts}_{n,i}$, we have a corresponding action of $GL_n(\hat{\mathbb{Z}}_p)$, which commutes with the action by $\mathrm{Aut}(F_n)$. One can base change to $\overline{\mathbb{Q}_p}$ to get an action of the Galois group $G_{\mathbb{Q}_p}$ on all involved spaces.

Now we come to the stable homotopy theory: there exists a universal deformation of $F_n$ to $\mathrm{Spf}\ \mathrm{lifts}_n$, so we can consider the map $\mathrm{Spf}\ \mathrm{lifts}_n \rightarrow \mathcal{M}_{fg}$ classifying this formal group. Unfortunately $\mathrm{Spf}\ \mathrm{lifts}_n$ doesn't have an interesting $\mathbb{G}_m$-action, so instead we consider the moduli space $\mathrm{Spf}\ \mathrm{lifts}_n^*$ of deformations equipped with a choice of 1-form, again up to $\star$-isomorphism; $\mathbb{G}_m$ permutes the 1-forms and so we have a (free) $\mathbb{G}_m$-action on $\mathrm{Spf}\ \mathrm{lifts}_n^*$, and a $\mathbb{G}_m$-equivariant, flat, (pro-)affine map $\mathrm{Spf}\ \mathrm{lifts}_n^* \xrightarrow{f} \mathcal{M}_{fg}$, so the Landweber presheaf $\mathcal{L}$ gives us a spectrum $E_n$, called *Morava E-theory,* with the property that

$$(E_n)_{2i} \cong W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$$

for each integer $i$. A priori, we don't have a ring structure on $E_n$, but in fact there exists an $E_\infty$-ring structure $E_n$, and the action of $\mathrm{Aut}(F_n)$ on $\mathrm{Spf}\ \mathrm{lifts}_n^*$ actually lifts to an action of $\mathrm{Aut}(F_n)$ on $E_n$ by $E_\infty$-ring automorphisms.

**Question 1.** We have a pullback presheaf $f^*\mathcal{L}$ of spectra, i.e., $f^*\mathcal{L}$ is a derived $\mathcal{O}_{\mathcal{M}_{fg}}$-module. Can the $\mathbb{G}_m$-equivariant fpqc opens of $\mathrm{Spf}\ \mathrm{lifts}_n^*$ be given canonical ring structures on their Landweber spectra, i.e., is $f^*\mathcal{L}$ actually a derived $\mathcal{O}_{\mathrm{Spf}\ \mathrm{lifts}_n^*}$-algebra? By Lurie's derived Artin criterion (the theorem which has become known as "Jacob's machine") then the answer is yes, $f^*\mathcal{L}$ is a derived $\mathcal{O}_{\mathrm{Spf}\ \mathrm{lifts}_n^*}$-algebra but with $\mathcal{O}_{\mathrm{Spf}\ \mathrm{lifts}_n^*}$ the structure sheaf of the local étale site on $\mathrm{Spf}\ \mathrm{lifts}_n^*$, *not* the local fpqc site.

**Question 2.** We can construct the limit of the Lubin-Tate tower of moduli of deformations with level structures paired with 1-forms to get a free $\mathbb{G}_m$-action on a space $\mathrm{Spf}\ \mathrm{lifts}_{n,*}^*$. The Landweber presheaf now gives us a spectrum

$\mathrm{lev}_{p^i} E_n := \mathcal{L}(\mathrm{Spf\ lifts}^*_{n,i})$ with

$$\pi_{2j}(\mathrm{lev}_{p^i} E_n) \cong \mathrm{lifts}_{n,i}$$

for each integer $j$; so we have a tower of spectra, the *topological Lubin-Tate tower*:

$$\begin{array}{c} \cdots \\ \downarrow \\ \mathrm{lev}_{p^i} E_n \\ \downarrow \\ \mathrm{lev}_{p^{i-1}} E_n \\ \downarrow \\ \cdots \\ \downarrow \\ \mathrm{lev}_{p^2} E_n \\ \downarrow \\ E_n, \end{array}$$

but none of the covers in the tower are étale, so we do not a priori have an $E_\infty$-ring structure on the limit $\mathrm{holim}_i \mathrm{lev}_{p^i} E_n$ of the topological Lubin-Tate tower, and we do not a priori have lifts of the Hecke operations of $GL_n(\hat{\mathbb{Z}}_p)$ on Spf $\mathrm{lifts}_{n,*}$ to topological Hecke operators on $\mathrm{holim}_i \mathrm{lev}_{p^i} E_n$. Is there some other way to get these topological structures on $\mathrm{holim}_i \mathrm{lev}_{p^i} E_n$? Also, we should know whether it is true that

$$\pi_{2j}(\mathrm{holim}_i \mathrm{lev}_{p^i} E_n) \cong \mathrm{colim}_i \mathrm{lifts}_{n,i}.$$

**Question 3.** Faltings constructed an equivalence between Lubin-Tate space and Drinfeld symmetric space; it was beefed up by Fargues to an equivariant equivalence of the entire Lubin-Tate tower with the entire Drinfeld tower. In the case of the Drinfeld tower we begin with the symmetric space $\Omega^d_{\mathbb{Q}_p}$ obtained as the complement of all $\mathbb{Q}_p$-linear hyperplanes in the dimension $d-1$ projective space $\mathbb{P}^{d-1}_{\overline{\mathbb{Q}_p}}$ over the algebraic closure $\overline{\mathbb{Q}_p}$. We automatically have an action of $G_{\overline{\mathbb{Q}_p}/\mathbb{Q}_p}$ and of $PGL_d(\mathbb{Q}_p)$ on $\Omega^d_{\mathbb{Q}_p}$, and Drinfeld constructed a tower of étale coverings

$$\cdots \rightarrow \Omega^{d,i}_{\mathbb{Q}_p} \rightarrow \Omega^{d,i-1}_{\mathbb{Q}_p} \rightarrow \cdots \rightarrow \Omega^{d,2}_{\mathbb{Q}_p} \rightarrow \Omega^{d}_{\mathbb{Q}_p}$$

such that $\lim_i \mathrm{Aut}(\Omega^{d,i}_{\mathbb{Q}_p}/\Omega^d_{\mathbb{Q}_p}) \cong \mathrm{Aut}(F_d)$, giving us an action of the three groups $\mathrm{Aut}(F_d)$, $PGL_d(\mathbb{Q}_p)$, and $G_{\mathbb{Q}_p}$ on the limit $\lim_i \Omega^{d,i}_{\mathbb{Q}_p}$ of the Drinfeld tower. Now the covers in the Drinfeld tower *are* étale, so if $\Omega^d_{\mathbb{Q}_p}$ is formally etale over the moduli stack of $p$-divisible groups with constant height $d$ and formal dimension 1 (are these conditions satisfied?) then not only do we have a "Drinfeld homology" represented by an $E_\infty$-ring spectrum $\mathrm{Dr}_d$, we have that the entire Drinfeld tower lifts to a tower of $E_\infty$-ring spectra, the *topological Drinfeld tower*

$$\begin{array}{c} \dots \\ \downarrow \\ \mathrm{Dr}_{n,i} \\ \downarrow \\ \mathrm{Dr}_{n,i-1} \\ \downarrow \\ \dots \\ \downarrow \\ \mathrm{Dr}_{n,2} \\ \downarrow \\ \mathrm{Dr}_n, \end{array}$$

with actions of the groups $\mathrm{Aut}(F_d), PGL_d(\mathbb{Q}_p)$, and $G_{\mathbb{Q}}$ by $E_\infty$-ring automorphisms.

If the topological Lubin-Tate tower can be given $E_\infty$-ring structures then one would like to have a *derived Faltings-Fargues equivalence*, an equivariant equivalence of the topological Lubin-Tate tower with the topological Drinfeld tower. Alternatively, one might be able to use the $E_\infty$-ring structures on the Drinfeld tower together with lifts of the maps from the (non-derived) Faltings-Fargues equivalence to the category of spectra, and transfer ring structures to the spectra in the topological Lubin-Tate tower along these maps.