ON THE NOTION OF GEOMETRY OVER $\mathbb{F}_1$

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- Chevalley groups
- The geometries of J. Tits
- Chevalley groups over finite fields
- Chevalley groups over $\mathbb{F}_1$ (after Tits)

- Graded gadgets and affine varieties over $\mathbb{F}_1$
- Chevalley schemes as graded varieties over $\mathbb{F}_1$
- Zeta-function of a Chevalley scheme over $\mathbb{F}_1$

The geometries of J. Tits

Reference: J. Tits (1956) “Sur les analogues algébriques des groupes
. semi-simples complexes”

Motivations: $\begin{cases} \text{geometric axiomatic} \\ \text{geometric interpretation of the} \\ \text{algebraic theory of Chevalley groups} \end{cases}$

Simple complex Lie groups: $\begin{cases} \text{classical: } A_n,\ B_n,\ C_n,\ D_n \\ \text{exceptional: } G_2,\ F_4,\ E_6,\ E_7,\ E_8 \end{cases}$

Over a field K: Jordan(1870), Dickson(1901), Dieudonné(1948)
but partial and ‘ad hoc’ constructions

C. Chevalley (1955): algebraic structure theorems for semi-simple complex Lie algebras $\mathfrak{g}$ and Lie groups to transfer simultaneously their ‘heart-structure’ over any field K

Reference: C. Chevalley (1955) “Sur certains groupes simples”

Main tool: definition of an integral model $\mathfrak{g}_{\mathbb{Z}}$
(integral version of the structure theorem)

$\mathfrak{g}$ complex, (semi)simple Lie algebra, $\boxed{\Phi = \text{roots set}}$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan algebra

$\Phi \ni r : \mathfrak{h} \to \mathbb{C}$; $0 \neq X_r \in \mathfrak{g}$ root elt $([H, X_r] = r(H)X_r,\ H \in \mathfrak{h})$

<u>Theorem</u> (Chevalley) If $r, s, r+s \in \Phi$, the roots elements can be chosen so that

$$[X_r, X_{-r}] = n_r \in \mathfrak{h};\ [X_r, X_s] = N_{r,s}X_{r+s},\ N_{r,s} \in \mathbb{Z}$$

$\mathfrak{g}$ has an integral basis: $\underbrace{n_{r_1}, n_{r_2}, \ldots}_{\text{co-roots } \in \mathfrak{h}}\ X_{r_1}, X_{r_2}, \ldots$

- the structure of $\mathfrak{g}_{\mathbb{Z}}$ as Lie algebra depends <u>only</u> on $\mathfrak{g}$

$$\mathcal{H} := < n_{r_i} \in \mathfrak{h} \mid r_i \in \Phi > \subset \mathfrak{h}, \qquad r(n_r) = 2$$

<u>choose</u> a basis: $n_1, \ldots, n_\ell;\ X_1, \ldots, X_\nu \in \mathfrak{g}_{\mathbb{Z}}$

$$\mathfrak{g}_K := K \otimes_{\mathbb{Z}} \mathfrak{g}; \qquad \mathfrak{h}_K = K \otimes \mathcal{H} = < n_i^* = 1 \otimes n_{r_i} >$$

$$\mathfrak{g}_K = \mathfrak{h}_K \oplus < X_j^* = 1 \otimes X_j; j = 1, \ldots, \nu >$$

$$\boxed{L = \mathbb{Z} \cdot \Phi \text{ lattice}}, \quad \text{rk } L = \ell = \text{rk } \mathfrak{g}$$

$$\boxed{\text{Hom}(L, K^*) \xrightarrow{\sim} \mathfrak{H} \subset \text{Aut}(\mathfrak{g}_K)} \quad \chi \mapsto h(\chi)$$

$$h(\chi)(n_r^*) = n_r^*; \quad h(\chi)(X_r^*) = \chi(r)X_r^*$$

$$r \in \Phi \quad \phi_r : \mathsf{SL}(2,K) \to \mathsf{Aut}(\mathfrak{g}_K) \quad \text{group hom}$$

$$\phi_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h(\chi_r), \quad \chi_r(s) = t^{s(n_r)}; \quad \phi_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \omega_r$$

$$\phi_r \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x^*_{-r}(t), \quad \phi_r \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x^*_r(t)$$

$$(x_r(t) = \exp t(\mathsf{ad X_r}), \quad t \in \mathbb{C})$$

$$\boxed{\mathfrak{X}_r = \{x^*_r(t) : t \in K\} \subset \mathsf{Aut}(\mathfrak{g}_K)}$$

$G_K := < \mathfrak{H}, \mathfrak{X}_r : r \in \Phi >$ **<u>CHEVALLEY GROUP</u>**

$\Phi^o = \{a_1, \ldots, a_\ell\} \subset \Phi$ <u>fundamental</u> (simple) roots

<u>Theorem</u> (Chevalley)

$$G_K = < \mathfrak{H}, \mathfrak{X}_{\pm a} : a \in \Phi^o >$$

Abstract Root System: $(L, \Phi, n_r; r \in \Phi)$

$L =$ lattice (group of weights)

$\Phi \subset L$ finite set (lin. indep); $n_r : L \to \mathbb{Z}$ (r co-root)

ax1. $L \otimes \mathbb{Q}$ is generated by Φ and $\cap_{r \in \Phi} \mathrm{Ker}(n_r)$

ax2. $n_r(r) = 2, \ \forall\ r \in \Phi$

ax3. $r \in \Phi$, $ar \in \Phi$, $a \in \mathbb{Q}$, $\Rightarrow$ $a = \pm 1$

ax4. $r, s \in \Phi$, $\Rightarrow$ $w_s(r) := r - n_s(r)s \in \Phi$

$w_s : L \overset{\sim}{\to} L$, $w_s(x) = x - n_s(x)s$ reflection w.r.t $\underline{s}$

$$\phi_s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (r) = \omega_s(r) = w_s(r) \qquad (w_s^2 = 1)$$

($\omega_s \in \mathrm{Aut}(\mathfrak{g}_K)$, $\omega_s(X_r^*) = \pm X_{w_s(r)}^*$, $\omega_s(n_r^*) = n_{w_s(r)}^*$

automorphism corresponding to the symmetry w_s)

Theorem (Chevalley, Grothendieck/Demazure)

$(L, \Phi, n_r) \rightsquigarrow \mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$ reductive group scheme$_{/\mathbb{Z}}$

$$\mathfrak{G}(K) = G_K$$

- $\mathcal{T} \subset \mathfrak{G}$ maximal split torus, $\mathcal{T}(K) = \mathfrak{H}$
- $\mathcal{N} = \mathcal{N}_{\mathfrak{G}}(\mathcal{T})$, normalizer of $\mathcal{T}$ in $\mathfrak{G}$

$$\boxed{\mathcal{N}/\mathcal{T} \simeq W(M) \quad \textbf{Coxeter/Weyl group}}$$

$$W(M) := < r_i \in \Phi^o;\ (r_i r_j)^{m_{ij}} = 1 > \quad \simeq \quad < w_{r_i} : L \xrightarrow{\sim} L >$$

$$M = (m_{ij}) = (m_{ji}) \quad (m_{ii} = 1) \quad \underline{\textbf{Coxeter matrix}}$$

$$2m_{ij} = \#\{r \in \Phi : r = c_i r_i + c_j r_j,\ r_i, r_j \in \Phi^o; c_i, c_j \in \mathbb{Z}\}$$

- $\mathcal{U} := < \mathfrak{X}_r : r \in \Phi^+ > \subset \mathfrak{G}$, $\mathfrak{X}_r := \mathrm{Im}(x_r : \mathbb{G}_{a/\mathbb{Z}} \to \mathfrak{G})$
- $w \in W(M)$, $\Phi_w = \{r \in \Phi^+ : w(r) \in \Phi^-\}$

$$\mathcal{U}_w := < \mathfrak{X}_r : r \in \Phi_w >$$

Theorem (Chevalley) If K is a field

$$\mathfrak{G}(K) = \coprod_{w \in W} \mathcal{U}(K)\mathcal{T}(K) n_w \mathcal{U}_w(K)$$

$$< \mathfrak{H}, \omega_r : r \in \Phi > \ni n_w \in \mathcal{N}(K) \twoheadrightarrow N(K)/\mathcal{T}(K) \quad \omega_r \mapsto w_r$$

Geometries of the Chevalley groups (Tits)

$\mathfrak{G}$ Chevalley group scheme$/_{\mathbb{Z}}$

$$\Phi^o = \{a_1, \ldots, a_\ell\},\ \mathcal{A}_i := \{r \in \Phi : r = \sum_{\substack{j=1 \\ j\neq i}}^{\ell} c_j a_j,\ c_j \in \mathbb{Z}\}$$

$\mathfrak{G}(K) \rightsquigarrow G(G_1, \ldots, G_\ell) = \Gamma(G; G_i)$ **collection of index** ℓ

$G_i := < \mathcal{U}(K), \mathfrak{H}, \mathfrak{X}_{-r}, r \in \mathcal{A}_i >; \quad \Gamma(G; G_i) = \Gamma(\mathcal{E}; \mathcal{F}_i; \iota; A)$

$$\mathcal{E} = \cup_i \mathcal{F}_i, \quad \mathcal{F}_i = G/G_i, \quad A \simeq G/\bigcap_{g\in G, i} g^{-1} G_i g$$

$\Gamma(G; G_i) \rightsquigarrow \Sigma = \{\Gamma^{(j)} = \Gamma(G_j; G_i \cap G_j, i \neq j), j = 1 \ldots \ell\}$

complete system of geometries of type $\Gamma(G; G_i)$

$\Sigma(G_K) = \{\Gamma^{(j)}, j = 1 \ldots \ell\} \leftrightsquigarrow S(G_K)$ **"scheme"**

Example: $\mathfrak{G}(K) = G_K = PGL_{\ell+1}(K)$

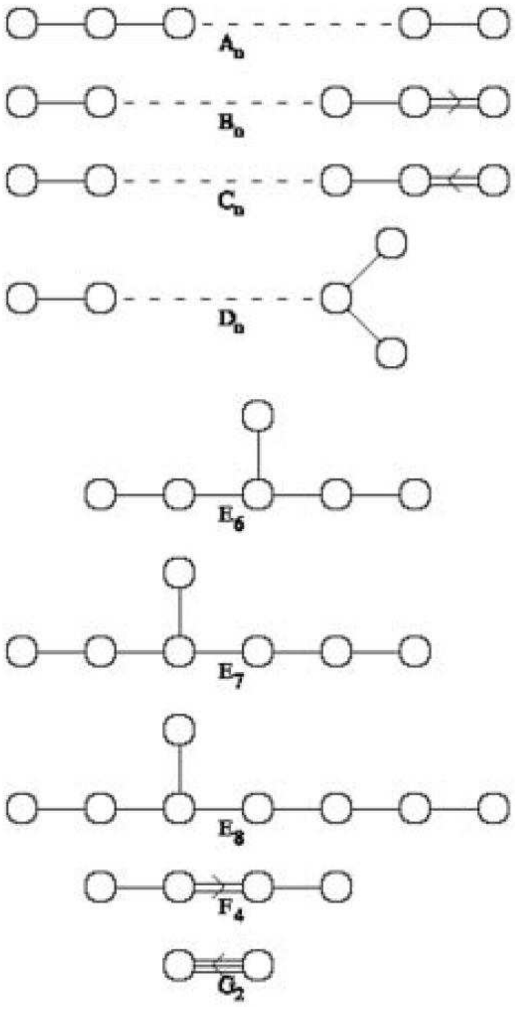
$\Sigma(G_K) =$ ℓ-dim. projective geometry over K

$S(G_K) =$ **Coxeter** (Witt-Dynkin) **diagram** A_ℓ

The definition of $\Gamma(G; G_i)$ depends on K

(Tits) **The scheme $S(G_K)$ is universal!!**

Witt-Dynkin schemes



(Tits) **There are** 4 **elementary types: geometries over** K **are entirely characterized by those corresponding to**:

$A_1 \times A_1$ (no link), A_2 $(\circ - \circ)$, B_2 $(\circ = \circ)$, G_2 $(\circ \equiv \circ)$

Chevalley groups over $\mathbb{F}_q$

$\mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$ Chevalley group scheme

$$\mathfrak{G}(\mathbb{C}) = G$$

K any field:

$$\begin{aligned} |\mathfrak{G}(K)| &= \sum_{w \in W} |\mathcal{U}(K)\mathcal{T}(K) n_w \mathcal{U}_w(K)| \\ &= |\mathcal{U}(K)||\mathcal{T}(K)| \sum_{w \in W} |\mathcal{U}_w(K)| \\ &= |\mathbb{A}^N(K)||\mathbb{G}_m(K)^\ell| \sum_{w \in W} |\mathbb{A}^{N_w}(K)| \end{aligned}$$

$\ell = \text{rk } \mathfrak{g}$, $N = \#\Phi^+$, $N_w = \#\Phi_w$; $2N + \ell = \dim \mathfrak{g}$

<u>If $K = \mathbb{F}_q$</u>: $|\mathfrak{G}(\mathbb{F}_q)| = (q-1)^\ell q^N \sum_w q^{N_w}$

<u>Thm</u> (R. Bott) $$\sum_{w \in W} q^{N_w} = \prod_{i=1}^{\ell} \frac{q^{d_i} - 1}{q - 1}$$

$d_i \in \mathbb{Z}_{\geq 0}$ **exponents of the Weyl group**

$$|\mathfrak{G}(\mathbb{F}_q)| = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1)$$

(Tits) $P_{\mathfrak{g}}(q) := \prod_{i=1}^{\ell}(q^{d_i}-1), \quad |G_K| = q^N P_{\mathfrak{g}}(q)$

$E(\mathfrak{g}) = \{d_i\}$ <u>determined</u> by the action of $W(M)$ on L associated to the Witt-Dynkin scheme of G

$$|W(M)| = \prod_i d_i$$

$$\mathfrak{G}(K) = G_K \rightsquigarrow G(G_1, \ldots, G_n) = \Gamma(G; G_i)$$

$\Gamma(G; G_i) = \Gamma(\mathcal{E}, \mathcal{F}_i, \iota, G_K)$ associated geometry

<u>FACT</u> $|\mathcal{F}_i| = |G_K/G_i| = \frac{P_{\mathfrak{g}}(q)}{(q-1)P_{\mathfrak{g}_i}(q)} = Q_{\mathfrak{g},i}(q)$

$$(q-1) \nmid Q_{\mathfrak{g},i}(q)$$

$E(\mathfrak{g}_i) = \{d_{i_t}\}$ $(i \leftrightarrow \mathcal{F}_i)$ associated to the Witt-Dynkin scheme: $S(G_K) \setminus \{i\}$

Chevalley groups over $\mathbb{F}_1$

Main Fact $$Q_{\mathfrak{g},i}(1) = \frac{|W(M)|}{\prod\limits_{d_{i_t} \in E(\mathfrak{g}_i)} d_{i_t}} \in \mathbb{Z}_{\geq 0}$$

Tits interprets $Q_{\mathfrak{g},i}(1)$ as the number of elements of the family $\mathcal{F}_i^*$ ($\mathcal{F}_i \overset{q \to 1}{\to} \mathcal{F}_i^*$) belonging to a limiting geometric structure over $\mathbb{F}_1$

$$\Gamma_{\mathbb{F}_q}(\mathcal{E}, \mathcal{F}_i, \iota, G_K) \xrightarrow{q \to 1} \Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W(M))$$

- $W(M)$ is the group of symmetries of the limiting geometry
- $W(M)$ is the skeleton of G

$\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W)$ **"compositum" of the limiting version of the 4 "elementary" geometries**:

i.e. union of polygons with 2,3,4 and 6 sides

Graded gadgets over $\mathbb{F}_1$

Tits' construction produces a notion of a "geometry" over $\mathbb{F}_1$

$$\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W(M))$$

associated to any Chevalley group scheme $\mathfrak{G}_{/\mathbb{Z}}$

Example: $G = PGL_{\ell+1}(\mathbb{C})$, $W(M) = S_{\ell+1}$

$\mathcal{E}^* = \mathcal{P}_\ell$ finite set of $\ell + 1$ points

$\mathcal{F}_i^* \subset \mathcal{E}^*$ set of $i + 1$ points

Question: Can $\mathfrak{G}_{/\mathbb{Z}}$ be obtained by base-change from a variety G defined over $\mathbb{F}_1$

$$\mathfrak{G}_{/\mathbb{Z}} = G \times_{\mathrm{Spec}(\mathbb{F}_1)} \mathrm{Spec}(\mathbb{Z}) \ ?$$

Goal: Algebraic-geometric definition of G (over $\mathbb{F}_1$) compatible with Tits' geometry $\Gamma_{\mathbb{F}_1}(\mathcal{E}^*, \mathcal{F}_i^*, \iota^*, W)$

$X = (\underline{X}, X_{\mathbb{C}}, e_X)$ **(graded) gadget over $\mathbb{F}_1$**

- $\underline{X} = (\coprod_{k \geq 0} \underline{X}^{(k)}) : \mathcal{F}_{ab} \to \mathcal{S}ets$ covariant funct

  $\mathcal{F}_{ab} =$ (finite) abelian groups

- $X_{\mathbb{C}}$ algebraic variety over $\mathbb{C}$
- $e_X : \underline{X} \to \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[-], X_{\mathbb{C}})$ natural transf

$\phi = (\underline{\phi}, \phi_{\mathbb{C}}) : X \to Y$ **morphism of gadgets**

- $\underline{\phi} : \underline{X} \to \underline{Y}$ natural transformation
- $\phi : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ morphism of algebraic varieties
- the following diagram commutes $\forall\ D \in \mathcal{F}_{ab}$

$$\begin{tikzcd}
\underline{X}(D) \arrow[r, "\phi(D)"] \arrow[d, "e_X(D)"] & \underline{Y}(D) \arrow[d, "e_Y(D)"] \\
\mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[D]), X_{\mathbb{C}}) \arrow[r, "\phi_{\mathbb{C}}"] & \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[D]), Y_{\mathbb{C}})
\end{tikzcd}$$

$\phi : X \hookrightarrow Y$ **immersion of gadgets**

- $\phi(D) : \underline{X}(D) \hookrightarrow \underline{Y}(D)$ injective $\forall D \in \mathcal{F}_{ab}$
- $\phi_{\mathbb{C}}$ embedding

<u>Example</u> (gadget) $V_{\mathbb{Z}} = \mathrm{Spec}(A)$ defines a gadget:

$$X = \mathcal{G}(V) = (\underline{X}, X_{\mathbb{C}}, e_X)$$

$\underline{X}(D) = \mathrm{Hom}(A, \mathbb{Z}[D]), \quad \forall D \in \mathcal{F}_{ab}$

$X_{\mathbb{C}} = V_{\mathbb{C}} = V \otimes \mathbb{C}$

$e_X(D) : \mathrm{Hom}(A, \mathbb{Z}[D]) \to \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[D]), V_{\mathbb{C}})$

$$e_X(D)(f : A \to \mathbb{Z}[D]) = \mathrm{Spec}(f \otimes 1_{\mathbb{C}})$$

Affine varieties over $\mathbb{F}_1$

$X = (\underline{X}, X_{\mathbb{C}}, e_X)$ **finite, graded gadget**

- $\exists\ X_{\mathbb{Z}}$ affine variety
- $\exists\ i : X \hookrightarrow \mathcal{G}(X_{\mathbb{Z}})$ immersion of gadgets such that:

$\forall\ V_{\mathbb{Z}} = \mathrm{Spec}(A)$ and

$\forall\ \varphi : X \to \mathcal{G}(V_{\mathbb{Z}})$ morphism of gadgets

$\exists!\ \varphi_{\mathbb{Z}} : X_{\mathbb{Z}} \to V_{\mathbb{Z}}$ s.t.

$$X \overset{i}{\hookrightarrow} \mathcal{G}(X_{\mathbb{Z}}) \overset{\mathcal{G}(\varphi_{\mathbb{Z}})}{\to} \mathcal{G}(V_{\mathbb{Z}}) \quad \varphi = \mathcal{G}(\varphi_{\mathbb{Z}}) \circ i$$

Guiding principles

(for a meaningful definition of a graded gadget)

- $\underline{X} = \coprod_{k \geq 0} \underline{X}^{(k)} : \mathcal{F}_{ab} \to \mathcal{S}ets$ ought to contain enough points, so that together with $X_{\mathbb{C}}$, it characterizes X

- $|\underline{X}(\mathbb{F}_{1^n})| = N(n)$ (polynomial) function s.t.

$$N(q) = |X_{\mathbb{Z}}(\mathbb{F}_q)|, \quad \text{if } n = q-1$$

(Soulé) $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} := \mathbb{Z}[T]/(T^n - 1)$

$N(q) = \sum_k a_k (q-1)^k, \quad a_k \in \mathbb{Z}$ (Taylor exp)

$$a_k (q-1)^k = |\underline{X}^{(k)}(D)|, \quad q-1 = |D|$$

Examples

$|\mathbb{G}_m(\mathbb{F}_q)| = N(q) = q - 1$

$|\mathbb{A}^d(\mathbb{F}_q)| = N(q) = q^d$

$|\mathfrak{G}(\mathbb{F}_q)| = N(q) = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1)$

$|\mathbb{P}^d(\mathbb{F}_q)| = N(q) = 1 + q + \cdots + q^d$

The multiplicative group $\mathbb{G}_m$

$$N(q) = q - 1 = 0 + (q-1), \quad q = p^r$$

$$\mathbb{G}_m = (\underline{\mathbb{G}_m}, \mathbb{C}^*, e_m)$$

$$\underline{\mathbb{G}_m} = \coprod_{k \geq 0} \underline{\mathbb{G}_m}^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\underline{\mathbb{G}}_m(D)^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ D & \text{if } k = 1. \end{cases}$$

<u>Example</u>:

$$\underline{\mathbb{G}}_m(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \emptyset & \text{if } k \in \mathbb{Z}_{\geq 0} \setminus \{1\} \\ \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1. \end{cases}$$

$$e_m(D) : \underline{\mathbb{G}}_m(D) \to \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[D], \mathbb{C}^*)$$

$$e_m(D)(g) = \chi(g), \quad \chi : \mathbb{C}[D] \to \mathbb{C}, \quad g \in D$$

$$(\mathbb{G}_m)_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T^{\pm 1}])$$

The affine space $\mathbb{A}^d$

$$N(q) = q^d = (q-1)^d + d(q-1)^{d-1} + \cdots + d(q-1) + 1$$

$$\mathbb{A}^d = (\underline{\mathbb{A}}^d, \mathbb{C}^d, e_d)$$

$$\underline{\mathbb{A}}^d = \coprod_{k \geq 0} (\underline{\mathbb{A}}^d)^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\underline{\mathbb{A}}^d(D)^{(k)} = \coprod_{\substack{Y \subset \{1,\ldots,d\} \\ |Y| = k}} D^Y$$

Example:

$$\underline{\mathbb{A}}^2(\mathbb{F}_{1^n})^{(k)} = \begin{cases} \{0\} & \text{if } k = 0 \\ \mathbb{Z}/n\mathbb{Z} \coprod \mathbb{Z}/n\mathbb{Z} & \text{if } k = 1 \\ (\mathbb{Z}/n\mathbb{Z})^{\{1,2\}} = (\mathbb{Z}/n\mathbb{Z})^2 & \text{if } k = 2 \\ \emptyset & \text{if } k \geq 3 \end{cases}$$

$$|\underline{\mathbb{A}}^2(\mathbb{F}_{1^n})| = n^2 + 2n + 1$$

$$e_d(D) : \underline{\mathbb{A}}^d(D) \to \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D], \mathbb{C}^d)$$

$$e_d(D)((g_j)_{j \in Y}) = (\xi_j)_{j \in \{1,\ldots d\}}, \quad \xi_j = \begin{cases} \chi(g_j) & \text{if } j \in Y; \\ 0 & \text{if } j \notin Y. \end{cases}$$

$$\mathbb{A}^d_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T])$$

The projective space $\mathbb{P}^d$ (as graded functor)

$$\underline{\mathbb{P}}^d = \coprod_{k\geq 0} (\underline{\mathbb{P}}^d)^{(k)} : \mathcal{F}_{ab} \longrightarrow \mathcal{S}ets$$

$$\underline{\mathbb{P}}^d(D)^{(k)} = \coprod_{\substack{Y\subset\{1,2,\ldots,d+1\}\\ |Y|=k+1}} D^Y/D$$

the right action of D is the diagonal action

$$\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)} = \{1,2,\ldots,d+1\}$$

$$|\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)}| = d+1$$

$\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})$ coincides in degree zero with the $d+1$ points of the set $\mathcal{P}_d$ on which Tits' defines a projective geometry of dimension d over $\mathbb{F}_1$

$$|\mathbb{P}^d(\mathbb{F}_q)| = N(q) = 1+q+\cdots q^d \xrightarrow{q\to 1} |\underline{\mathbb{P}}^d(\mathbb{F}_{1^n})^{(0)}|$$

Extended Coxeter & Weyl groups

Abstract Root System: $(L, \Phi, n_r; r \in \Phi)$

$\Phi^o \subset \Phi^+$ fundamental roots
$\Pi = \{1, \ldots \ell\}, \ |\Pi| = |\Phi^o|$
$M = (m_{ij})$ Coxeter matrix $m_{ij} = m_{ji}$

$$W(M) \simeq \ < r_i \in \Phi^o; \ (r_i r_j)^{m_{ij}} = 1 >$$

Coxeter group

$$B(M) = < q_i : (q_i q_j)^{m_{ij}} = (q_j q_i)^{m_{ij}}; i, j \in \Pi >$$

Braid group of M

$$X(M) = \mathrm{Ker}(B(M) \twoheadrightarrow W(M)) \quad q_i \mapsto r_i$$

$$V(M) = B(M)/[X(M), X(M)]$$

extended Coxeter group

<u>**Theorem**</u> (Tits) $V(M) = B(M)/[X(M), X(M)]$ is the ‘universal extension’ of $W(M)$

$$1 \to U(M) \to V(M) \xrightarrow{f} W(M) \to 1$$

by

$$U(M) = X(M)/[X(M), X(M)]$$

<u>free abelian</u> group generated by a set of elements $\{g(s), s \in S\}$ in <u>bijective</u> correspondence with the set $S \subset W(M)$ of reflections
($S \ni s \ \leftrightsquigarrow \ r_s \in \Phi^+$ is conjugate to a $r \in \Phi^o$)

The definition of the <u>extended Weyl group</u> is implemented ‘over’ the construction of $V(M)$

root system (L, Φ, n_r) $\quad \leftrightsquigarrow \quad$ $\mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$

$\mathcal{T} \subset \mathfrak{G}$ maximal, split torus
$\mathcal{N} = \mathcal{N}_{\mathfrak{G}}(\mathcal{T})$ normalizer of $\mathcal{T}$ in $\mathfrak{G}$
$\mathcal{N}/\mathcal{T} \simeq W(M) = W$ **Weyl** (Coxeter) **group**

(D, ϵ) $\quad D =$ abelian group, $\epsilon \in D$, $\quad \epsilon^2 = 1$

<u>**Proposition**</u> (Tits) $\quad (D, \epsilon) \to \mathcal{N}_{D,\epsilon} = \mathcal{N}_{D,\epsilon}(L, \Phi)$

$$1 \to T \to \mathcal{N}_{D,\epsilon}(L, \Phi) \xrightarrow{p} W \to 1$$

<u>canonical</u> extension of W by $T = \mathrm{Hom}(L, D)$,
<u>functorial</u> in (D, ϵ)

extended Weyl group

$$\mathcal{N}_{D,\epsilon} = (V(M) \times T)/\mathrm{Graph}(U(M) \xrightarrow{g(s_r) \to h_s^{-1}} T)$$
$$h_s(x) = \epsilon^{n_r(x)}$$

A commutative ring with 1

Theorem (Tits) The group extension

$$1 \to \mathcal{T}(A) \to \mathcal{N}(A) \xrightarrow{p} W \to 1$$

is canonically isomorphic to the group extension

$$1 \to \mathrm{Hom}(L, A^*) \to \mathcal{N}_{A^*,-1}(L, \Phi) \xrightarrow{p} W \to 1$$

$\mathcal{U}(A) =< x_r(a), r \in \Phi^+,\ a \in A >$

$\mathcal{U}_w(a) =< x_r(a), r \in \Phi_w,\ a \in A >$

Theorem (Chevalley)

$\psi : A^{\Phi^+} \xrightarrow{\simeq} \mathcal{U}(A), \quad (t_r)_{r\in\Phi^+} \mapsto \prod_{r\in\Phi^+} x_r(t_r)$

$\psi_w : A^{\Phi_w} \xrightarrow{\simeq} \mathcal{U}_w(A), \quad (t_r)_{r\in\Phi_w} \mapsto \prod_{r\in\Phi_w} x_r(t_r)$

Chevalley schemes as graded gadgets

$$\mathfrak{g}_{\mathbb{C}} \leftrightsquigarrow (L, \Phi, n_r) \quad \leftrightsquigarrow \quad \mathfrak{G} = \mathfrak{G}(L, \Phi, n_r)$$

$$G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$$

$$\underline{G} = \textstyle\coprod_{k \geq 0} \underline{G}^{(k)} : \mathcal{F}^{(2)}_{ab} \longrightarrow \mathcal{S}ets$$

$$\mathcal{F}^{(2)}_{ab} \ni (D, \epsilon), \quad \epsilon \in D,\ \epsilon^2 = 1$$

$$\underline{G}(D, \epsilon) = \underline{\mathbb{A}}^{\Phi^+}(D) \times \coprod_{w \in W} (p^{-1}(w) \times \underline{\mathbb{A}}^{\Phi_w}(D))$$

$$\mathcal{N}_{D,\epsilon}(L, \Phi) \overset{p}{\twoheadrightarrow} W \to 1$$

$$e_G(D, \epsilon) : \underline{G} \to \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[D, \epsilon], \mathfrak{G}(\mathbb{C}))$$

$$e_G(D, \epsilon)(a, n, b) = \psi(e_{\Phi^+}(a))\, e_{\mathcal{N}}(n)\, \psi_w(e_{\Phi_w}(b))$$

$$\underline{\mathbb{A}}^{\Phi^+}(D) \overset{e_{\Phi^+}}{\rightarrow} \mathbb{C}^{\Phi^+} \overset{\psi \sim}{\rightarrow} \mathcal{U}(\mathbb{C}),\ \mathcal{N}_{D,\epsilon}(L, \Phi) \overset{e_{\mathcal{N}}}{\rightarrow} \mathcal{N}(\mathbb{C})$$

$$\underline{\mathbb{A}}^{\Phi_w}(D) \overset{e_{\Phi_w}}{\rightarrow} \mathbb{C}^{\Phi_w} \overset{\psi_w \sim}{\rightarrow} \mathcal{U}_w(\mathbb{C})$$

Chevalley schemes as graded varieties over $\mathbb{F}_{1^2}$

Theorem The graded gadget $G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$ defines a variety over $\mathbb{F}_{1^2}$.

$$G_{\mathbb{Z}} = \mathfrak{G}, \quad G \hookrightarrow \mathcal{G}(\mathfrak{G}) \text{ immersion of gadgets}$$
(by construction)

The universal property in the definition of (graded) affine varieties over $\mathbb{F}_{1^2}$ can be checked by applying

Proposition (Chevalley) $\mathcal{U} \times p^{-1}(w_o) \times \mathcal{U} \overset{\theta}{\hookrightarrow} \mathfrak{G}$
$$\theta(u, n, v) = unv$$

$\exists!\ w_o \in W, w_o(\Phi^+) = \Phi^-, \quad \mathcal{N} \ni \omega_o \overset{p}{\mapsto} w_o \in W$

θ open embedding onto $\Omega = \mathrm{Spec}(\mathcal{O}_\Omega) \subset \mathfrak{G}$
$$\mathcal{O}_\Omega = \mathcal{O}_{\mathfrak{G}}[d^{-1}], \quad d(\omega_o) = 1$$

Zeta function of some simple varieties over $\mathbb{F}_1$

(Weil) If $N(x) \in \mathbb{Z}[x]$

$$Z(q,T) := \exp\Big(\sum_{r\geq 1} N(q^r)\frac{T^r}{r}\Big)$$

(Soulé) $N(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x], \quad \chi := N(1)$

$$\lim_{q\to 1} Z(q, q^{-s})^{-1}(q-1)^{-\chi} = \prod_{i=0}^{d} (s-i)^{a_i} =: \zeta(s) \in \mathbb{Q}(s)$$

- $\zeta_{\mathbb{F}_1}(s) = s, \quad |\mathbb{F}_1| = 1$
- $\zeta_{\mathbb{G}_m}(s) = \frac{s-1}{s}, \quad |\mathbb{G}_m(\mathbb{F}_{1^n})| = N(n) = n$
- $\zeta_{\mathbb{A}^d}(s) = s-d, \quad |\mathbb{A}^d(\mathbb{F}_{1^n})| = N(n) = n^d$
- $\zeta_{\mathbb{P}^d}(s) = s(s-1)\cdots(s-d)$

$$N(n) = |\mathbb{P}^d(\mathbb{F}_{1^n})| = 1 + n + \cdots + n^d$$

Zeta function of $G = (\underline{G}, \mathfrak{G}(\mathbb{C}), e_G)$ over $\mathbb{F}_1$

Apply the general procedure to $\underline{G}$:

$$|\mathfrak{G}(\mathbb{F}_q)| = N(q) = q^N \prod_{i=1}^{\ell} (q^{d_i} - 1) = q^N P_{\mathfrak{g}}(q)$$

$$\boxed{\zeta_G(s) = \prod_{i=N}^{\dim\ \mathfrak{g}} (s - N - D_J)^{(-1)^{\ell+i}}, \qquad s \in \mathbb{R}}$$

$$D_J = \sum_{j \in J,\ J \subset \{1,\dots\ell\}} d_j, \quad \ell + N = \sum_{j=1}^{\ell} d_j, \quad N = \#\Phi^+$$

$$\dim\ \mathfrak{g} = 2N + \ell$$