Some notes on Toen & Vaquié

In these notes

• $(\mathbf{C}, \otimes, 1)$ will be a symmetric monoidal category, which is **closed**, complete, and cocomplete; perhaps with

• a monoidal **model category** structure [5 §5.1 (p. 34)], including Hom objects in $\Delta^{\mathrm{op}}(\mathrm{Sets})$.

Examples:

• (Sets) and (Abelian groups): in which case the model category structure is trivial or essentially irrelevant, or

• Simplicial sets ($\Delta^{\mathrm{op}}(\mathrm{Sets})$) and very special Γ-spaces (which are objects in $\Gamma^{\mathrm{op}}(\mathrm{Spaces})$, where by (Spaces) I mean simplicial sets!)

Recollection: Γ is a small category with the same objects as Δ, but more morphisms; so $X \in \Gamma^{\mathrm{op}}(\mathrm{Spaces})$ defines a simplicial set by restriction. There are natural maps

$$X[n] \to X[1] \times_{X[0]} \cdots \times_{X[0]} X[1] \ \ (n \text{ times})$$

and X is special if these maps are homotopy equivalences; it is **very** special if $X[0] = *$ is a point. Such vs-Γ-spaces provide a very nice model for abelian monoids in the homotopy category, ie **connected spectra**.

• $\mathrm{Comm}(\mathbf{C})$ will be the category of commutative monoids in $\mathbf{C}$ (in the first of the two cases above; in the second, it's really the category of cofibrant E_∞ objects!).

If $A \in \mathrm{Comm}(\mathbf{C})$ there is a category $(A-\mathrm{Mod})$ of objects of $\mathbf{C}$ with A-action, and if $A \to B$ is a morphism of monoids, then there is a functor

$$X \mapsto B \otimes_A X : (A - \mathrm{Mod}) \to (B - \mathrm{Mod}) \ .$$

Under the hypotheses above, these categories inherit reasonable model structures.

- $\text{Aff}(\mathbf{C}) = \text{Comm}(\mathbf{C})^{\text{op}}$ is the category of affine schemes over $\mathbf{C}$. It has a natural **Zariski topology**:

A map $f : A \to B$ in $\mathbf{C}$ corresponds to the inclusion of a Zariski open iff

1) $\text{Hom}(B, Z) \to \text{Hom}(A, Z)$ is injective $\forall Z$,

2) $\text{Hom}_A(B,\ \text{colim} Z_i) = \text{colim}\ \text{Hom}_A(B, Z_i)$, and

3) $Z \mapsto B \otimes_A Z$ is **exact**.

[In the **homotopical** examples (p. 35), these conditions are suppposed to hold up to homotopy (in a suitable sense): eg 3) becomes the condition that the associated derived functor commutes with homotopy fiber products; 1) becomes the condition that

$$\pi_* \text{Hom}(B, Z) \to \pi_* \text{Hom}(A, Z)$$

is injective for $* = 0$ and an isomorphism for $* > 0$, and 2) is a homotopy equivalence after cofibrant replacement of A.]

- The Zariski topology is **sub-canonical**[1]: representable functors are sheaves in that topology. The category of **schemes over C** is the category of sheaves on $\text{Aff}(\mathbf{C})$ which are locally isomorphic to affine objects.

It is only terminology at this point, but it is convenient to define Spec $\mathbf{C}$ to be the opposite of the category of schemes over $\mathbf{C}$.

Examples: If $\mathbf{C}$ is the category of abelian groups, then $\text{Comm}(\mathbf{C})$ is the category of commutative rings, and $\text{Schemes}(\mathbf{C})$ is (some version of) the classical category of schemes over $\mathbb{Z}$ in algebraic geometry.

If $\mathbf{C} = (\text{Sets})$, $\otimes = \times$, and $1 = *$ then $\text{Comm}(\mathbf{C})$ is the category of abelian monoids, and $\text{Schemes}(\mathbf{C})$ is some version of a category of schemes over $\mathbb{F}_1$.

If $\mathbf{C}$ is the category of vs-Γ-spaces, $\text{Comm}(\mathbf{C})$ is (Quillen equivalent to) the model category of connected E_∞ ring-spectra, which can also be described either as a category of very nice (multiplicative) cohomology theories, or as a category of modules[2] over the sphere spectrum $\mathbb{S}$. When $\mathbf{C}$ is the category

[1] Thanks to AS for the vocabulary lesson

[2] connective, ie without negative-dimensional homotopy groups

of simplicial sets (with Cartesian product as monoidal functor) the resulting category of monoids is essentially the category of (plain vanilla) Γ-spaces. In this case Schemes($\mathbf{C}$) is described as the category of schemes over '$\mathbb{S}_1$'.

In all cases of interest these definitions have various enrichments; eg Soulé and others (eg [1 §2.8], [4 §1.7.2]) attach some kind of Archimedean data to the bare bones above. Similarly, topological (Atiyah-Hirzebruch) K-theory is periodic, hence not bounded below, and so lies in a more general category of $\mathbb{S}$-modules. Toen and Vaquié also discuss other topologies beyond Zariski's, but I've left them out of this summary.

I'd argue that T & V's framework allows us to compare at least the outlines of these various constructions side-by-side. In particular, they show [5 §2.22, 5.12] that a monoidal functor $\mathbf{C} \to \mathbf{D}$ with a suitable right adjoint defines a functor from Spec $\mathbf{C} \to$ Spec$\mathbf{D}$. [This requires, in particular, some discussion of the flat topology, which I'm going to use as an excuse to omit details.] For example:

- The base change functor

$$M \mapsto \mathbb{Z}[M] : \text{Comm(Sets)} \to \text{Comm(Ab)}$$

maps schemes over $\mathbb{F}_1$ to schemes over $\mathbb{Z}$, defining a map

$$\text{Spec } \mathbb{Z} \to \text{Spec } \mathbb{F}_1 \ .$$

Similarly, if X is a simplicial set (ie a space) then $\mathbb{S}[Q(X)]$, with

$$Q(X) = \lim \ \Omega^n \Sigma^n X \ ,$$

is an E_∞ ring-spectrum (analogous to the group ring of a monoid). This defines a base-change functor

$$\text{Spec } \mathbb{S} \to \text{Spec } \mathbb{S}_1 \ .$$

Finally, the functor $X \mapsto \pi_0 X$ maps simplicial sets to sets, and (interpreted as $X \mapsto \pi_0 X[1]$) also takes commutative ring-spectra to commutative rings.

These constructions can be compiled into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\,\mathbb{Z} & \longrightarrow & \mathrm{Spec}\,\mathbb{F}_1 \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbb{S} & \longrightarrow & \mathrm{Spec}\,\mathbb{S}_1 \end{array}$$

which suggests a

Homework problem: Is there a factorization

$$\mathrm{Spec}\,\mathbb{Z} \to \mathrm{Spec}\,\mathbb{S} \times_{\mathrm{Spec}\,\mathbb{S}_1} \mathrm{Spec}\,\mathbb{F}_1$$

of the maps in the upper left corner of the diagram, through some sort of categorical fiber product?

[I should say that I don't know how to define the fiber product on the right: that's part of the problem!]

Some references:

1. A. Connes, C. Consani, On the notion of geometry over $\mathbb{F}_1$, `arXiv:0809.2926`

2. N. Durov, New approach to Arakelov geometry, `arXiv:0704.2030`

3. M. Marcolli, Cyclotomy and endomotives, `arXiv:0901.3167`

4. Y. Manin, Cyclotomy and analytic geometry over $\mathbb{F}_1$, `arXiv:0809.1564`

5. B. Toen, M. Vaquié, Under Spec $\mathbb{Z}$, `arXiv:math/0509684`