[TLL notes from 06/09/08 (DRAFT, in progress!)]

§1 Moduli problems and derived algebraic geometry

The classification of families of objects with given properties has a long history (eg conic sections) in algebraic geometry. Such questions are usually formulated today in terms of a functor which associates to an object Z, the set $T(Z)$ of isomorphism classes of objects of type $\mathbb{T}$ parametrized by Z.

If this is a reasonable problem, then a map $Z' \to Z$ should associate to a family parametrized by Z, a family parametrized by Z', thus making $Z \mapsto T(Z)$ into a (contravariant) functor. One way to classify objects of type $\mathbb{T}$ is then to find an object $\mathcal{T}$ which **represents** this functor, in the sense that there is a (universal) family parametrized by $\mathcal{T}$, in the sense that $T(Z)$ naturally bijects with the set of maps from Z to $\mathcal{T}$. This is the case, for example, with topological vector bundles: a suitable Grassmannian parametrizes a universl family.

One standard way to construct such a universal family is construct a nice family X of such objects decorated with extra structures, and to then throw away the extra structures, eg by identifying objects whose extra data is isomorphic. In simple cases the isomorphisms of the extra data form a group G, and the universal object is a quotient of the decorated moduli space by this group of automorphisms; more generally, one is led to stacks. One disadvantage of this approach is that the resulting quotient X/G can be quite pathological.

There are various approaches to this whole area, and I will not try to be precise. I'll just say that it is often useful to seek a 'better' quotient $X//G$, which, for example, might resolve some of the singularities present in X/G. In topology this can be given a precise meaning, by defining

$$\pi : X//G := X \times_G EG \to X \times_G \mathrm{pt} = X/G$$

to be the 'homotopy-to-geometric' quotient map (with EG a contractible G-space with **free** G-action. In algebraic geometry, the classical approach (cf Mumford) tries to identify a subspace $B \subset X$ of pathological orbits, and defines

$$X//G := (X - B)/G\ .$$

For the purposes of this seminar, though, I'd like to sketch some ideas behind the approach proposed in [CK §2.2.1]: that one should allow as parametrizing objects, spaces endowed with a local structure sheaf of ring-like objects, those objects being differential graded (super)-commutative algebras.

[In the algebrogeometric context, such an object is a pair $(X, \mathcal{O}^\bullet_X)$, where $\mathcal{O}^\bullet_X$ is a sheaf of differential graded-commutative objects; such that

i) $(X, \mathcal{O}^0_X)$ is a scheme (so in particular, the degree zero part of $\mathcal{O}^\bullet_X$ is a sheaf of commutative local rings), and

ii) $\mathcal{O}^\bullet_X$ is a complex of quasi-coherent $\mathcal{O}^0_X$-modules.

A related idea comes up in the intersection theory of algebraic cycles, where it goes back to eg early work of Serre that intersection numbers can be calculated in terms of underlying sets when the intersections are transversal, but when they are not, it is necessary to correct them with data from higher derived Tors.]

I believe the roots of this idea go back to Grothendieck: in the last section of his 1957 Tohoku paper, he constructs a kind of Leray spectral sequence of the form

$$E_2 = H^*(X/G, R\pi_*(\mathcal{E})) \Rightarrow H^*(X//G, \mathcal{E}) \; ;$$

I'm being impressionistic here, but this assertion can be made precise in the case of the homotopy quotient map described above.] The point I'm trying to make is that the derived sheaf (with stalk

$$R\pi_*(\mathcal{E})_x = H^{-*}(\mathrm{Iso}(x), \mathcal{E}_x)$$

at x) can be interpreted as the homology of a DGA $\Pi^\bullet(\mathcal{E})$ (at least, when $\mathcal{E}$ itself is a sheaf of algebras), and that we might read this spectral sequence as asserting that the cohomology of the 'derived' quotient $X//G$ is naturally approximated by the cohomology of a 'derived' object $(X/G, \Pi^\bullet(\mathcal{E}))$.

If G acts on X with finite isotropy groups, then this can be said much more precisely using the language of stacks. Note that in that case, things become very simple if we're working with coefficients which are modules over a field of characteristic zero, because the cohomology of a finite group (eg the isotropy of a point) is torsion above degree zero, hence trivial with coefficients over a $\mathbb{Q}$-vector space.

If we want to work in arithmetic contexts, eg over $\mathbb{Z}$, then differential graded algebras can become technically clumsy. One standard route toward making things better is to replace chain complexes with (semi)simplicial objects.

[Aside: a (semi)simplicial object in a category $\mathcal{C}$ is a functor

$$C : \Delta^{\mathrm{op}} \to \mathcal{C} \ ,$$

where Δ is a kind of model for the standard simplex, interpreted as a category. [This turns out to be essentially just the category of finite ordered sets and order non-decreasing morphisms.] Such a simplicial object in an **additive** category defines a chain complex (constructed in a familiar way by summing up the face maps with appropriate signs), and the resulting objects are more or less equivalent to chain complexes. But in less abelian contexts, the simplicial approach works better ...]

Thus we can work with ringed spaces, where the structure sheaf is now a **simplicial** commutative algebra; cf. eg [JL]. At this point, however, it happens that pretty much any category whose Hom-objects are simplicial sets (as is the case for spaces ringed by commutative simplicial algebras) is naturally [RSS] enriched over the category of spectra; so it becomes pretty natural to take the next step, and consider ringed spaces whose structure sheaves are commutative ringspectra ...

To Be Continued: A natural test case is the moduli problem for elliptic curves. Classically (ie over $\mathbb{C}$)this is solved by the not-very-bad (ie orbifold) quotient of the upper half-plane by $\mathrm{PGl}_2(\mathbb{Z})$; but there is an arithmetic version, defined by the Deligne-Mumford stack $\overline{\mathcal{M}}_{1,1}(\mathbb{Z})$ of genus one algebraic curves with one marked smooth point.

It turns out that there is a natural sheaf of commutative ringspectra over this moduli space (maybe with curves with additive degeneration removed), whose homotopy groups can be naturally identified with the local sheaf of functions on the moduli space. [Removing the degenerate curves localizes the homotopy groups, making them periodic, so the grading is not really very visible.] That's a story [cf [NG]] to be told in more detail when we get a chance ...

Some references

[CK]: I. Ciocan-Fontanine, M. Kapranov Derived Quot schemes, arXiv:math/9905174

[NG] Nora Ganter's Talbot House elliptic cohomology web page,

http : //www.math.uiuc.edu/ ∼ ganter/talbot/index.html

[AG]: A. Grothendieck,Sur quelques points d'algbre homologique. Thoku Math. J. 9 (1957) 119–221.

[JL]: J. Lurie, Structured spaces (DAGV), available at

http : //math.mit.edu/ ∼ lurie/

[RSS]: C. Rezk, S. Schwede, B. Shipley, Simplicial structures on model categories and functors. Amer. J. Math. 123 (2001) 551–575.