Remarks and exercises from 29/09

1 There are several useful models for the (homotopy category of) spaces, eg cell complexes (a.k.a. CW spaces), which build up spaces cell-by-cell (and in particular allow for the definition of objects with finitely many cells . . .) and simplicial sets (ie functors from the category of finite ordered sets and nondecreasing maps to sets, which generalize the notion of simplicial complex). They all agree up to (suitably defined) homotopy, and they all have nice model category structures. I think the main advantage of $\mathbf{CGTop}_*$, aside from its familiarity, is that it is Cartesian closed (ie has internal Hom objects); that's where the 'compactly generated' comes in.

2 Here are some remarks toward answers to some issues raised by Eugene's questions:

A spectrum

$$\mathbf{Y} := \{\Sigma Y_k \to Y_{k+1}\}$$

defines a cohomology functor

$$Y^k(X) = \lim\, [\Sigma^n X, Y_{n+k}] \ \ (k \in \mathbb{Z})$$

(the limit is a direct limit over n; I haven't spelled out the maps of the system. Square brackets denote the set of homotopy classes of maps between pointed spaces, and Σ^n is the smash product with the n-sphere $S^n = S^1 \wedge \cdots \wedge S^1$ (n times).)

The based loopspace functor is adjoint

$$[\Sigma X, Z] = [X, \Omega Z]$$

to suspension, so

$$Y^k(X) = \lim\, [\Sigma^n X, Y_{n+k}] = \lim\, [X, \Omega^n Y_{n+k}] = [X, \Omega^\infty \mathbf{Y}] \ ,$$

(at least, provided X is a finite complex; here $\Omega^\infty \mathbf{Y} := \lim \Omega^n Y_{n+k}$ is a model for the spectrum $\mathbf{Y}$ as an infinite loop-space.

It's easy to confuse $\mathbf{Y}$ and $\Omega^\infty \mathbf{Y}$. For example, associated to any abelian group A there is an Eilenberg-MacLane space $H(A,k)$ (which used to be denoted $K(A,k)$, but K is getting overused ...), characterized (using obstruction theory) by the property

$$\pi_i(H(A,k)) = A \text{ if } i = k, \text{ otherwise } 0 \; .$$

Adjointness shows that

$$\pi_{i-1}(\Omega H(A,k)) \cong \pi_i H(A,k) \; ,$$

so there is a homotopy equivalence $H(A,k-1) \to \Omega H(A,k)$ whose adjoint map leads to the definition of a spectrum

$$\mathbf{H}(A) := \{\Sigma H(A,k-1) \to H(A,k)\}$$

with $H(A)_k = H(A,k)$.

3.1 Let BU be a (pointed) space representing the functor which assigns to a (pointed) space X, the abelian group

$$\widetilde{K}(X) = [X, BU] = \text{ker } [\text{dim} : K(X) \to \mathbb{Z}]$$

of stable vector bundles[1] (ie, of formal dimension zero) over X.

I'll use Borel & Hirzebruch's theory, which identifies

$$H^*(BU, \mathbb{Z}) = \text{dir lim } H^*(BT^n)^{\Sigma_n-\text{inv}} = \text{dir lim } \mathbb{Z}[c_i \mid i \geq 1]$$

with the polynomial ring of characteristic classes for complex vector bundles: the Chern classes c_i,

$$\sum c_i z^i = \prod (1 + t_n z)$$

are the elementary symmetric functions of the 'Chern roots' $t_n \in H^2(BT, \mathbb{Z})$ associated to a maximal torus $T^k \subset U(k)$, and the Chern character

$$\text{ch} \in H^*(BU, \mathbb{Q})[z] = \sum \exp(t_n z) = \sum p_i z^i / i!$$

[1]This is not to be confused with the stable bundles of algebraic geometry; that is a much more subtle notion.

of the universal bundle ξ over BU is expressed in terms of the power sums $p_i = \sum t_n^i$. I'll need the fact that the p_i are polynomial generators for the ring of symmetric functions over the rationals [Newton] and that the Chern character is additive under direct sum of vector bundles, and multiplicative under their tensor product [Atiyah-Hirzebruch].

3.2 Let $\mathbb{C}P^\infty = BT = BU(1) \subset BU$ represent the natural transformation which regards a line bundle as a stable vector bundle, let

$$S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty = BU(1)$$

classify the Hopf line bundle H over the projective line; then we can define an *ad hoc* Bott map

$$\beta : S^2 \wedge BU \to BU(1) \wedge BU \to BU$$

to be its composition with $(L, V) \mapsto L \otimes V$ sending a line bundle and a vector bundle to their tensor product. The homomorphism

$$\beta^* : H^*(BU, \mathbb{Q}) \to H^*(S^2 \wedge BU, \mathbb{Q}) \cong H^{*-2}(BU, \mathbb{Q})$$

induced on cohomology sends $\mathrm{ch}(\xi)$ to $\mathrm{ch}(\xi) \cdot \mathrm{ch}(H) = \sigma^2 \mathrm{ch}(\xi)$, i.e.

$$p_k \mapsto k p_{k-1} \ .$$

This is a ring homomorphism, but the target space of the underlying map is a suspension, and so has trivial products in its cohomology; the homomorphism therefore sends decomposable elements to zero.

Let $\widetilde{\mathbf{K}} = \{Y_{2k} = BU\}$ be the spectrum with $\beta : S^2 \wedge Y_{2k} \to Y_{2(k+1)}$ as its transition maps.

3.3 Homotopy classes of morphisms $\mathbf{E} \to \mathbf{F}$ between spectra are elements of the inverse limit

$$E^*(\mathbf{F}) := \text{inv lim} \, \{\cdots \to E^{*+k}(F_k) \to E^{*+k-1}(F_{k-1}) \to \cdots\}$$

so for example

$$H^{2j}(\widetilde{\mathbf{K}}, \mathbb{Q}) = \text{inv lim} \, \{\cdots \to H^{2(j+k)}(BU, \mathbb{Q}) \to H^{2(j+k-1)}(BU, \mathbb{Q}) \to \cdots\}$$

is one-dimensional, generated by the sequence

$$\{\cdots \mapsto \mathrm{ch}_{j+k} \mapsto ch_{j+k-1} \mapsto \cdots\} \ .$$

This is usually paraphrased as the assertion that there is essentially only one natural transformation

$$\mathrm{ch} : \widetilde{K}(X) \to H^{2*}(X, \mathbb{Q}) \ ,$$

which happens to be a ring homomorphism. A related argument shows that $H^*(\widetilde{\mathbf{K}}, \mathbb{Z})$ is trivial.

Extending this to the 'unreduced' K-spectrum based on $\mathbb{Z} \times BU = BGl^+(\mathbb{C})$ is left as an exercise.

Afterword: As hinted above, this is a pretty *ad hoc* argument. I was surprised not to be able to find a concise reference in the literature, but that doesn't mean I didn't overlook something. I hope what I've said above is more or less correct.