

Review Questions (for the upcoming quiz)

I. Linear algebra

1. If V, W are (finite-dimensional complex) vector spaces then there is a natural isomorphism

$$\Lambda^k(V \oplus W) \cong \bigoplus_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W),$$

of exterior powers, which implies the identity

$$\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F) \in K(X)[[t]]$$

for the total exterior power operation

$$E \mapsto \sum_{k \geq 0} [\Lambda^k(E)] = 1 + [E]t + [\Lambda^2 E]t^2 + \dots$$

in (classical topological) K -theory.

Ex: i) Calculate $\lambda_t(n)$, $n \in \mathbb{Z}$.

ii) Calculate $\lambda_t(E) \cdot \sigma_{-t}(E)$ (where σ_t is the analogous total **symmetric** power).

2. Define the total Adams operation

$$\psi_t(E) = \frac{\lambda'_t(E)}{\lambda_t(E)} = \sum_{k \geq 0} \psi^k(E) \in K(X)[[t]].$$

Ex: i) Show that $\psi^k(E \oplus F) = \psi^k(E) + \psi^k(F)$.

ii) If L is a line bundle, calculate $\psi^k(L)$.

iii) Show that if $E = L_1 \oplus \dots \oplus L_n$ and $F = L'_1 \oplus \dots \oplus L'_m$ are sums of line bundles, then $\psi^k(E \otimes F) = \psi^k(E) \cdot \psi^k(F)$.

iv) Let T^n be a maximal torus of the unitary group $U(n)$. Show that the map

$$K(BU(n)) \rightarrow K(EU(n)/T^n)$$

is injective; where $U(n)/T^n \rightarrow EU(n)/T^n \rightarrow BU(n)$ is a bundle over the classifying space for $U(n)$, with fiber the space of ‘flags’ in \mathbb{C}^n . Show further that the universal bundle over $BU(n)$ maps to a sum of line bundles over $EU(n)/T^n$.

[This is harder; it’s called the ‘splitting principle’ in Hirzebruch’s book. It’s unrealistic as an exercise; I’m just including it for completeness.]

v) Use this to show that the Adams operations are ring homomorphisms

$$\psi^k : K(X) \rightarrow K(X) ,$$

and that $\psi^k \circ \psi^l = \psi^{kl}$.

3. If X is a finite CW complex, define its p -adically completed K -theory to be $K(X)_p := K(X) \otimes \mathbb{Z}_p$, where \mathbb{Z}_p is the (compact, topological) ring of p -adic integers. For a general CW complex, define

$$K(X)_p = \text{inv lim } K(X_i)_p$$

to be the inverse limit taken over finite subcomplexes.

i) Show that this actually defines a cohomology theory.

ii) Show that if X is a finite complex, and $k_i \rightarrow \kappa$ is a sequence in \mathbb{Z} which converges (in the p -adic topology) to $\kappa \in \mathbb{Z}_p$, then for any $x \in K(X)_p$, the sequence $\{\psi^{k_i}(x)\}$ converges in the natural (compact) topology on $K(X)_p$, to an element $\psi^\kappa(x)$. [Hint: cf MF Atiyah, D. Tall: Group representations, λ -rings and the J -homomorphism, Topology 8 1969 253–297]

Recall (eg from Serre’s Course on Arithmetic) that the multiplicative group \mathbb{Z}_p^\times of p -adic integers is the product of the cyclic group (of order $p-1$) of roots of unity (which lift from the residue field \mathbb{F}_p), and a copy of the **additive** group of p -adic integers (using the logarithm and exponential series). Note also that when $p = 2$ things are trickier than one might think ...

iii) Show that the action

$$\alpha, x \mapsto \psi^\alpha(x) : \mathbb{Z}_p^\times \times K(X)_p \rightarrow K(X)_p$$

is continuous, thus making $K(X)_p$ into a topological module over the p -adic group ring $\mathbb{Z}_p[\mathbb{Z}_p^\times]$.

Remarks: i) This group ring is not complete, but it has a natural completion, which is isomorphic to a sum of $p - 1$ copies of the formal power series ring $\mathbb{Z}_p[[t]]$ (also known as the Iwasawa algebra ...).

ii) There is another route to this fact, based on the idea that \mathbb{Z}_p^\times can also be identified as the natural group of multiplicative automorphisms of the functor which assigns to a finite group G , the p -adic completion of its ring $R(G)$ of complex representations.

This is the basis for Quillen's (second) proof of the Adams conjecture. The idea goes back to Brauer (cf. Serre's Linear Representations of Finite Groups): any complex representation of a finite group can be defined over a **cyclotomic** subfield of \mathbb{C} , ie a field obtained by adjoining some finite set of roots of unity to \mathbb{Q} . In fact the full cyclotomic field (obtained by adjoining **all** roots of unity to \mathbb{Q}) is the maximal extension of the rationals with abelian Galois group:

$$\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times = \prod \mathbb{Z}_p^\times ,$$

which turns out to be the group of multiplicative automorphisms of the profinite completion of R . The resulting automorphisms can be identified with those coming from the Adams operations; this is worked out somewhere in the exercises in Serre.

II: Coming attractions:

We can regard the infinite orthogonal group $\lim O(n)$ as a submonoid of the space $\lim \Omega_1^n S^n$ of degree one self-maps of the sphere. On homotopy groups this inclusion induces (George) Whitehead's homomorphism

$$J : \pi_*(0) \rightarrow \lim \pi_{*+n}(S^n) = \pi_*^S(\text{pt}) .$$

Bott showed that the group on the left is infinite cyclic when $* = 4k - 1$, and its (necessarily cyclic) image is one of the larger chunks of the stable homotopy groups of spheres.

A generalization of this construction leads to the definition of a subgroup $J^*(X)$ of $\pi_*^S(X)$, and one of the triumphs of algebraic topology in the 70's was the proof (by Quillen and Sullivan) of some conjectures of Adams, which (roughly speaking, eg with various caveats at $p = 2$) identify $J^*(X)$ with the Galois cohomology

$$H_c^1(\hat{\mathbb{Z}}^\times, \hat{K}^*(X)) .$$

[This is actually part of the Adams spectral sequence for K -theory, but it is not much of a spectral sequence because $\hat{\mathbb{Z}}^\times$ has (more or less) cohomological dimension one. In fact the groups $J^*(X)$ fit more generally into the Bousfield localization story mentioned in AS's notes ...]

Further References:

DG Quillen, Some remarks on etale homotopy theory and a conjecture of Adams, *Topology* 7 (1968) 111–116,

——, The Adams conjecture, *Topology* 10 (1971) 67–80.

D. Sullivan, **Geometric Topology: Localization, Periodicity and Galois Symmetry**, 1970 MIT notes edited by Andrew Ranicki, available at

<http://www.maths.ed.ac.uk/aar/books/index.htm>