Notes on classical modular forms

§I Basic definitions

1.1 A matrix

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \gamma \in \mathrm{PGl}_2(\mathbb{R})$$

with $ad - bc \neq 0$ acts on the upper half-plane

$$\mathrm{UHP} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

by

$$z \mapsto [\gamma](z) := \frac{az+b}{cz+d} \; .$$

The isotropy group of $+i$ is

$$\mathrm{SO}(2) = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right] \; :$$

so

$$\frac{i\cos\theta - \sin\theta}{i\sin\theta + \cos\theta} = \frac{i \cdot e^{i\theta}}{e^{i\theta}} = i \; .$$

Corollary: $\mathrm{UHP} \cong \mathrm{SO}(2)\backslash\mathrm{Sl}_2(\mathbb{R})$.

The **classical** moduli space of elliptic curves is

$$\mathrm{UHP}/\mathrm{PGl}_2(\mathbb{Z}) \cong \mathrm{SO}(2)\backslash\mathrm{Sl}_2(\mathbb{R})/\mathrm{Sl}_2(\mathbb{Z}) \; .$$

[In fact the subgroup ± 1 is troublesome, and there is already reason at this point to work with Gl_2 instead; more about this later.] **Why** this quotient parameterizes elliptic curves is deferred until the next section.

Note that this action on the upper half-plane is **not** free: it has two orbifold points (with isotropy $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$), corresponding to the square and the hexagonal lattices. The 'cusp', which can be interpreted as the orbit either of $\mathbb{Q}$ or of the point $i\infty$, is not (properly speaking) a point of the moduli space at all.

1.2 The long exact homology sequence defined by the coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T} \to 1 \ ,$$

together with Poincaré duality, defines a homomorphism

$$c \mapsto [\alpha \mapsto \int_c \alpha] : H_1(E(\mathbb{C}), \mathbb{Z}) \to H^1_{\mathrm{dR}}(E(\mathbb{C}), \mathbb{R})^*$$

whose cokernel can be identified with the Albanese torus $H_1(E(\mathbb{C}), \mathbb{T})$; the Abel-Jacobi map

$$\lambda \mapsto \int_{[0,\lambda]} \ : \ E(\mathbb{C}) \to H_1(E(\mathbb{C}), \mathbb{T})$$

is an isomorphism of topological groups. There is a complex structure on the target, coming from the cohomology basis defined by the exact one-forms $Y^{-1}dX$ and $XY^{-1}dX$, cf. [2].

Corollary: $E(\mathbb{C})_{\mathrm{tors}} \cong H_1(E(\mathbb{C}), \mathbb{Q}/\mathbb{Z})$

For any sublattice $L \subset H_1(E(\mathbb{C}), \mathbb{Z})$ of finite index there is an extension

$$0 \to (\text{finite abelian}) \to \tilde{E} \to E \to 0 \ ,$$

which implies the existence of an **isogeny**

$$0 \to \cdots \to (\mathbb{Q}/\mathbb{Z})^2 \to (\mathbb{Q}/\mathbb{Z})^2 \to 0$$

of torsion subgroups, classified by an element of $M_2(\hat{\mathbb{Z}})$ with determinant $\neq 0$.

1.3 Definition: A full level structure on E is a choice of isomorphism $E(\mathbb{C})_{\mathrm{tors}} \cong (\mathbb{Q}/\mathbb{Z})^2$. The commutative diagram

$$\begin{tikzcd}
\mathrm{UHP} \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathrm{Sl}_2(\hat{\mathbb{Z}}) \arrow[r, "\cong"] \arrow[d] & (\text{ell. curves w. } \textbf{level str.}) \arrow[d] \\
\mathrm{UHP} \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathrm{pt} \arrow[r, "\cong"] & (\text{elliptic curves})
\end{tikzcd}$$

can be interpreted as an inverse limit of a system of coverings of the moduli space.

Note that this **Shimura variety** can be expressed as

$$\mathrm{SO}(2)\backslash\mathrm{Sl}_2(\mathbb{R}) \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathrm{Sl}_2(\hat{\mathbb{Z}}) \;\cong\; \mathrm{SO}(2)\backslash\mathrm{Sl}_2(\mathbb{A}_{\mathbb{Q}})/\mathrm{Sl}_2(\mathbb{Q})$$

or (better) as

$$(\text{max connected comp})\backslash\mathrm{Gl}_2(\mathbb{A}_{\mathbb{Q}})/\mathrm{Gl}_2(\mathbb{Q})$$

where the adéle group $\mathrm{Gl}_2(\mathbb{A}_{\mathbb{Q}})$ is the product of $\mathrm{Gl}_2(\mathbb{R})$ with the restricted product $\coprod_p \mathrm{Gl}_2(\mathbb{Q}_p)$ – which can be defined as the union, over all collections F of finitely many primes, of the products

$$\prod_{p\in F} \mathrm{Gl}_2(\mathbb{Q}_p) \times \prod_{p\notin F} \mathrm{Gl}_2(\mathbb{Z}_p) \; .$$

General problem: Suppose G is some nice groupscheme (eg $\mathrm{U}(1,n-1)$) over $\ldots$ ($\mathbb{F}_1$? $\mathbb{S}$?): can one define a generalized Shimura variety

$$(\text{connected})\backslash G(\text{adeles})/G(\text{base})$$

over $\ldots$?

§II The classical moduli space

2.1 A classical elliptic curve $E(\mathbb{C}) \cong \mathbb{C}/L$ is determined by its **period lattice** L, generated by ω_1 and ω_2 in $\mathbb{C}$, chosen so $\tau = \omega_1/\omega_2 \in$ UHP. The edges c_1 and c_2 of the period lattice

[Insert Fig. 1]

define classes $[c_1], [c_2] \in \pi_1(E(\mathbb{C}), 0)$ and hence in H_1.

The Weierstrass function

$$\wp_L(z) = z^{-2} + \sum_{\lambda\neq 0\in L} [(z-\lambda)^{-2} - \lambda^{-2}]$$

of L can be rearranged [5 VII §2.3] as

$$z^{-2} + \sum_{k\geq 2} (2k-1) G_k(L)\, z^{2k-2} \; ,$$

where $G_k(L) = \sum_{\lambda \neq 0 \in L} \lambda^{-2k}$ is a (non-normalized) Eisenstein series of weight $2k$.

If $u \in \mathbb{C}^\times$ then

$$G_k(uL) = u^{-2k} G_k(L) \ ,$$

so we can think of G_k as a function the space of lattices modulo rescaling, or, better, as a section of a certain line bundle over the space of such lattices.

2.2 The transformation $[\gamma]$ has derivative

$$[\gamma]'(\tau) = (c\tau + d)^{-2}$$

so

$$\gamma, (\tau, z) \mapsto [\gamma](\tau, z) := ([\gamma](\tau), [\gamma]'(\tau) \cdot z)$$

defines an action of $\mathrm{Sl}_2(\mathbb{Z})$ on UHP $\times$ $\mathbb{C}$: for

$$[\gamma_0]([\gamma_1](\tau, z)) = [\gamma_0]([\gamma_1](\tau), [\gamma_1]'(\tau) \cdot z) =$$

$$([\gamma_0]([\gamma_1](\tau)), [\gamma_0]'([\gamma_1](\tau)) \cdot [\gamma_1]'(\tau) \cdot z) = ([\gamma_0 \cdot \gamma_1](\tau), [\gamma_0 \cdot \gamma_1]'(\tau) \cdot z) \ .$$

This defines a complex line bundle

$$\mathbb{L} := \mathrm{UHP} \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathbb{C} \to \mathrm{UHP} \times_{\mathrm{Sl}_2(\mathbb{Z})} \mathrm{pt}$$

over the moduli space, which is in fact isomorphic to its cotangent bundle. The tensor power $\mathbb{L}^{\otimes k}$ corresponds to the action

$$\gamma, (\tau, z) \mapsto ([\gamma](\tau), ([\gamma]'(\tau))^k \cdot z) \ .$$

A **modular form** f of weight $2k$ is a section of $\mathbb{L}^{\otimes k}$; for example,

$$G_k(\gamma L) = (c\tau + d)^{-2k} G_k(L) \ .$$

One reason for the popularity of modular forms is that this transformation condition is in practice easy to check: a function f holomorphic on the upper half-plane is modular of weight $2k$ iff it is periodic ($f(\tau + 1) = f(\tau)$) and satisfies the condition

$$f(-\tau^{-1}) = \tau^{2k} f(\tau) \ .$$

If we define $q = \exp(2\pi i \tau)$[1] then G_k can be rewritten [5 VII §4 Prop 8] as $2\zeta(2k)E_k(q)$, where

$$E_k(q) := 1 + (-1)^k \frac{4k}{B_k} 4k \sum \sigma_{2k-1}(n) \ q^n \ ,$$

---

[1] Quantum group folks prefer $q = \exp(-1/\hbar)$, cf $\tau \mapsto -\tau^{-1} \ldots$

with B_k the kth Bernoulli number, and $\sigma_k(n) = \sum_{d|n,d\geq 1} d^k$.

It can be shown that the algebra

$$\oplus_k \Gamma(\mathcal{M}_{\rm ell}, \mathbb{L}^{\otimes k})$$

of modular forms over an **integral model** of the moduli space is polynomial over $\mathbb{Z}$ on two generators E_4 and E_6, **away from the prime 6**. To understand what happens at the primes two and three requires the theory of **topological** modular forms: for some reason (involving positivity in the classical case), the derived functors $R^*\Gamma_{\mathcal{M}}$ vanish for $* > 0$, except for contributions torsion contributions coming from the cohomology of the isotropy groups.

2.3 The Weierstrass function is doubly periodic, and satisfies the equation

$$\wp'_L(z)^2 = 4\wp^3_L(z) - g_2\wp_L(z) - g_3 \ ,$$

where $g_2 = 60G_2,\ g_3 = 140G_3$. It follows that

$$z \mapsto [\wp_L(z) : \wp'_L(z)] \in \mathbb{C}P_2$$

embeds the elliptic curve as

$$E(\mathbb{C}) = \mathbb{C}/L \to \{[X : Y] \in \mathbb{C}P_2 \mid Y^2 = 4X^3 - g_2(L)X - g_3(L)\} \ ;$$

it is **nonsingular** iff the **discriminant** $\Delta = g_2^3 - 27g_3^2 \neq 0$.

2.4 The determinant defines an isomorphism

$$\mathrm{Gl}_2(\mathbb{R})/\mathrm{Gl}_2(\mathbb{Z}) \to (\mathrm{Sl}_2(\mathbb{R})/\mathrm{Sl}_2(\mathbb{Z})) \times \mathbb{R}^\times_+$$

with the product of the space of lattices of volume one and the non-negative reals. The function

$$L \mapsto \mathbf{g}(L) = (3g_2(L), g_3(L)) \in \mathbb{C}^2 - 0$$

scales by

$$L, t \in \mathbb{R}^\times_+ \mapsto \mathbf{g}(tL) = (3t^{-4}g_2(L), t^{-6}g_3(L)) \ ,$$

so it follows [1] that there is a unique $t > 0$ such that $\mathbf{g}(L)$ lies on the sphere of radius $\sqrt{2}$: the equation

$$|\mathbf{g}(tL)|^2 = 9|g_2|^2t^{-8} + |g_3|^2t^{-12} = 2$$

can be rewritten (with $T = t^4$) as $2T^3 = 9|g_2|^2 T + |g_3|^2$, which has a unique positive solution.

The resulting embedding

$$\mathrm{Sl}_2(\mathbb{R})/\mathrm{Sl}_2(\mathbb{Z}) \to S^3(\sqrt{}2)$$

omits the one-dimensional locus $z_1^3 = z_2^2$ where $\Delta = 0$, which lies on the torus $|z_1| = |z_2| = 1$. In other words, the space of lattices of volume one maps homeomorphically to the complement of the torus knot

$$K : x \in [0,1] \mapsto (\exp(4\pi i x), \exp(6\pi i x)) \in S^3(\sqrt{}2) \;:$$

[Insert Fig. 2]

We can rephrase this [see [1 §?]] to say that SO(2) acts on $\mathrm{Sl}_2(\mathbb{R})/\mathrm{Sl}_2(\mathbb{Z}) \cong S^3 - K$ with quotient $\mathrm{UHP}/\mathrm{Sl}_2(\mathbb{Z}) \cong S^2 - i\infty$, by a twisted analog of the Hopf fibration.

2.4 This displays the elliptic moduli space as a kind of orbifold. In fact a theorem of T. Kawasaki [3] asserts that an orbifold has a smooth frame bundle: that is, that the frame bundle of an orbifold is a smooth manifold with an action by a Lie group having finite isotropy. The argument above identifies the frame bundle of the elliptic moduli space with the complement of the trefoil knot.

The fundamental group

$$\pi_1(S^3 - K) \cong \{a, b \mid aba = bab\} \cong \mathrm{Br}_3$$

of the trefoil complement is isomorphic to the braid group on three strands, which fits in an exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Br}_3 \to \mathrm{Sl}_2(\mathbb{Z}) \to 1 \;.$$

This can be interpeted as the homotopy exact sequence of the Borel fibration

$$(S^3 - K) \to |[(S^3 - K)//\mathrm{SO}(2)]| \to B\mathrm{SO}(2) \sim H(\mathbb{Z}, 2)$$

defined by the orbifold; thus

$$\pi_1^{\mathrm{orb}} |[(S^3 - K)//\mathrm{SO}(2)]| \cong \mathrm{Sl}_2(\mathbb{Z}) \;.$$

This is consistent with the orbifold interpretation of the classical description of the moduli space, which has a Borel fibration

$$\mathrm{UHP} \to |[\mathrm{UHP}/\mathrm{Sl}_2(\mathbb{Z})]| \to B\mathrm{Sl}_2(\mathbb{Z})$$

with contractible fiber.

The braid group Br_n can be defined as the fundamental group of the space $(\mathbb{C}^n - \mathbf{\Delta})/\Sigma_n$ of **unordered** configurations of n **distinct** points in the plane. When $n = 3$ this space has six real dimensions; $(S^3 - K)$ is a three-dimensional retract. The map between them is a version of the classical description of an elliptic curve by an equation of the form

$$Y^2 = \prod_{1 \leq i \leq 3} (X - e_i) \; .$$

2.5 The group of real fractional linear transformations acts on the real points of $\mathbb{C}$ with noncompact isotropy, isomorphic to the group of upper triangular 2×2 matrices (or to the group of affine translations, according to taste).

The elliptic moduli space is often described as looking like a three-corned hat, having two orbifold points with isotropy $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, together with the 'cusp', which is often described as corresponding to a degenerate elliptic curve in which one of the basic cycles has been collapsed, defining a two-sphere with 0 and ∞ glued together. This suggests that the isotropy group at the cusp ought to be something like $\mathbb{Z}$ (corresponding to the vanishing cycle on the curve at ∞); but in fact the picture seems to be more complicated.

[Insert Fig. 3]

The quotient $\mathbb{C}P_1/\mathrm{Sl}_2(\mathbb{Z})$ looks to me more like a tricorn hat with a feather: in this picture the cusp corresponds to $\mathbb{R}P_1/\mathrm{Sl}_2(\mathbb{Z})$, which is a quite fuzzy object. Its structure involves the representation of real numbers by continued fractions in a very beautiful way, explained towards the end of [4]. The quotient (a kind of zero-dimensional non-Hausdorff orbifold) is isomorphic to the set of all maps from $\mathbb{Z}$ to itself, modulo the obvious action of $\mathbb{Z}$ by translation. Periodic sequences have nontrivial isotropy (related to the groups of units in an associated real quadratic extension of the integers). A lot of attention has been paid to this object by noncommutative geometers;

here I just want to remark that there is an interesting diagram

$$\begin{tikzcd}
(\mathbb{R}^2 - 0)/\mathrm{Sl}_2(\mathbb{Z}) \arrow[r] \arrow[d, "\mathbb{R}^\times"] & (\mathbb{C}^2 - 0)/\mathrm{Sl}_2(\mathbb{Z}) \arrow[d, "\mathbb{C}^\times"] \\
\mathbb{R}P_1/\mathrm{Sl}_2(\mathbb{Z}) \arrow[r] & \mathbb{C}P_1/\mathrm{Sl}_2(\mathbb{Z})
\end{tikzcd} .$$

I'd like to propose understanding $(\mathbb{R}^2 - 0)/\mathrm{Sl}_2(\mathbb{Z})$ as a **homework problem**.

$\mathrm{Sl}_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 - 0$: for example, γ as in §1 sends $\mathbf{e}_1 = (1, 0)$ to (c, d). The isotropy subgroup of $\mathbf{e}_1$ is thus the group of upper triangular matrices, so the isotropy group for the action of $\mathrm{Sl}_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, acting through

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} .$$

This suggests that the map across the top of the diagram above induces an isomorphism on π_1^{orb}.

§III IOU re Hecke operations

The original purpose of this talk was to describe the action of Hecke's algebra on modular forms, but I have run out of time, and will have to return to this later. However, the framework has been set up.

The starting point of the theory is that by regarding a modular form of weight k as a function of lattices with with suitable scaling behavior(as in §2.1 above), we can define interesting operations

$$f \mapsto T_n(f)(L) = n^{2k-1} \sum_{[L:L']=n} f(L') .$$

The basic properties of these operations are explained beautifully in Serre [VII §5]. For our purposes the next topic of interest is how to reformulate these operations in terms of the action of $M_2(\hat{\mathbb{Z}})$ on the Shimura varieties displayed in §1.2-3 above ...

Some references:

1. J. Milnor, **Introduction to Algebraic K-Theory**

2. N.M. Katz, Appendix to p-adic properties of modular schemes and modular forms, in **Modular functions of one variable III**, Springer LNM 350 (1973)

3. T. Kawasaki, The index of elliptic operators over V-manifolds, Nagoya Math. J. 84 (1981)

4. T. Ono, **Introduction to Algebraic number theory**

5. J.P. Serre, **A Course in Arithmetic**

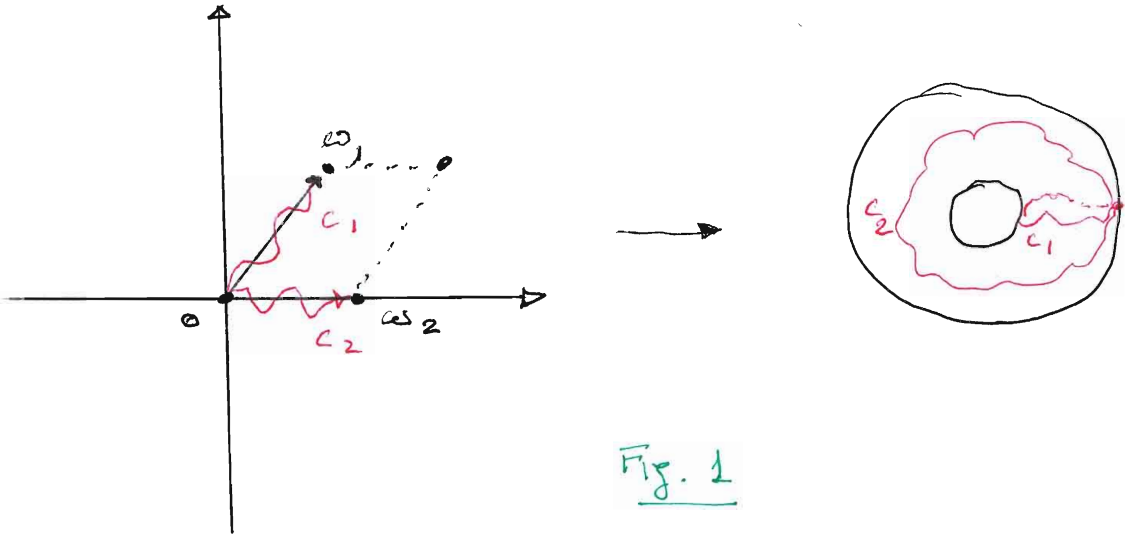

Fig. 1

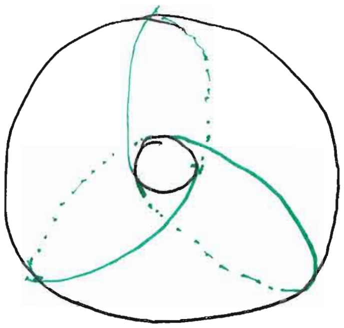

Fig 2

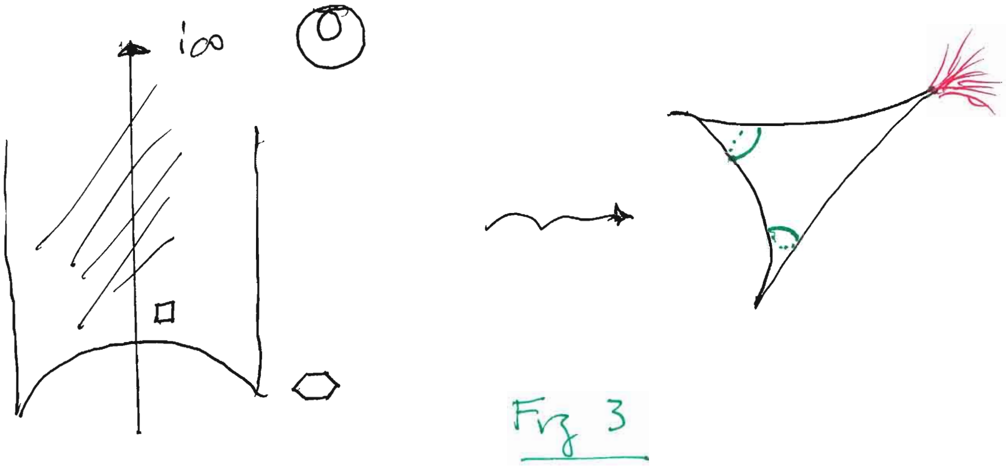

Fig 3