SOME BACKGROUND ON LANGLANDS CORRESPONDENCES

Introduction: Langlands' program is enormous, and many powerful mathematicians have pushed it in new directions. Its original form involved relations between certain infinite-dimensional representations of Lie groups over the adeles of number fields, and the Galois groups of those number fields, in a way which generalized classical relations between characters of these Galois groups and representations of Gl_1. There is a similar theory for fields of functions in finite characteristic, and more recently the 'geometric Langlands program' [14] proposes a version for function fields over $\mathbb{C}$. There are also generalizations [11] in the direction of the higher classfield theory of Kato, Parshin, and others.

There has been a great deal of progress: Lafforgue, for example, received the Fields medal in 2002 for his work [12] on Gl_n over function fields. What has recently caught the attention of topologists is work on the LL program over **local** number fields, by Carayol, Harris-Taylor, and others [4,5,9]. It is the connections between this restricted facet of the general program, and developments in modern stable homotopy theory, that I will try to sketch here.

1 The Langlands program is conventionally understood as part of representation theory, but we should keep in mind that it has equally deep connections with number theory, which was revolutionized in the fifties by cohomological methods introduced by Serre, Tate, and others, and the developments I'll summarize here fit into that tradition. However, for historical purposes it's useful to go all the way back to work of Schur, which laid the foundations for a theory [13] later elaborated by Weyl, of duality between representations of finite symmetric groups and compact unitary groups:

A complex vector space $V = \mathbb{C}^n$ is in an obvious way a representation of group $\mathrm{U}(n)$ of unitary $n \times n$ matrices, and its kth tensor power $V^{\otimes k}$ is a representation of the group Σ_k of permutations of its k components. This bimodule decomposes into a sum of tensor products of irreducible representations of $\mathrm{U}(n)$ and Σ_k; in fact, the whole tensor algebra of V can be similarly decomposed, in a way which pairs irreducible unitary representations with irreducible representations of the symmetric group (which are easily classified by Young diagrams).

More generally, decomposing a bimodule, ie an object which belongs simultaneously to two apparently very different categories, is a very flexible way to construct correspondences between objects in one category and another; but this method depends on finding an object with rich symmetries of distinct sorts.

2 Homotopy theorists (more precisely, **stable** homotopy theorists) are involved in questions with deep connections to **local** number theory. Topologists are interested in relatively coarse classification of geometric phenomena, and so deal with problems whose solution sets are often finite. This leads into number theory; but for various reasons, 'cross-talk' between the primes involved in geometric classification problems tends to decrease when the geometric dimension becomes large with respect to the primes. It is therefore often the case that interesting problems (eg classification of the differentiable structures on spheres) can be attacked prime by prime.

The p-adic Langlands program involves correspondences between

- Hilbert-space representations of (groups like) $\mathrm{Gl}_n(\mathbb{Q}_p)$,

- representations of the (profinite) Galois groups of local number fields (and their mild generalizations, eg Weil or Weil-Deligne groups), and

- representations of the (compact) unit groups of certain p-adic division algebras D (of $n \times n$ matrices).

It is the last item on this list which caught the attention of topologists [8,10]: these groups are in some sense the motivic groups for the stable homotopy category.

Much of the recent progress in this field amounts to understanding ideas of Drinfel'd [6] from thirty years ago. One way to frame that work is to say that it is better to approach the classical subject of modular forms not through the usual upper half-plane, but rather through the space

$$\mathbb{P}_1(\mathbb{C}) - \mathbb{P}_1(\mathbb{R})$$

of points on the projective line which are **not** defined over the reals: this is a more invariant viewpoint, because the resulting two copies of the UHP are exchanged by the Galois group of $\mathbb{C}$ over $\mathbb{R}$.

More generally, Drinfel'd considered the (rigid analytic, whatever that means) generalized upper half-space

$$\mathbb{P}_n(\overline{\mathbb{Q}}_p) - \mathbb{P}_n(\mathbb{Q}_p)$$

defined by points in n-dimensional projective space over the algebraically closed field $\overline{\mathbb{Q}}_p$, whose coordinates do not lie in the base field $\mathbb{Q}_p$: this space admits obvious actions of both $\mathrm{Gl}_n(\mathbb{Q}_p)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. This gets us two-thirds of the way to an object admitting all the symmetries in the list above. Drinfel'd goes the rest of the way by constructing a system $\mathbf{X}_\infty$ of covering spaces with the units of D as its deck-transformation groups. [This seems to be only a part of the fundamental group of Drinfel'd's space [2].] The (characteristic zero) cohomology of this object (somewhat more precisely, of its fiber over the residue field) decomposes into a sum of tensor products of representations of the three groups above, yielding the sought-after correspondence (in a way quite analogous to what the much simpler Schur-Weyl correspondence does).

3 The object $\mathbf{X}_\infty$ underlying this construction can be interpreted as a moduli space for formal groups of a certain sort, endowed with level structures. [Roughly speaking, a level structure on an abelian group object is a kind of system of generators for its torsion points; in our case this amounts to a choice of isomorphism of that subgroup with $(\mathbb{Q}/\mathbb{Z}_{(p)})^n$.] The compact group $\mathrm{Gl}_n(\mathbb{Z}_p)$ acts naturally on this moduli object, through automorphisms of the level structure.

Now since Quillen topologists have recognized in formal groups a subject which has one foot in arithmetic geometry and another in homotopy theory. The prime ideal spectrum of the complex cobordism ring can be interpreted as an enrichment, from a moduli object for one-dimensional formal groups to a ringed space with a structure sheaf of commutative symmetric ring-spectra. From this fundamental object one can construct moduli objects for formal groups of various heights, which can then be endowed with level structures (using ideas of Hopkins).

In fact the relevant homotopy-theoretic object is best be interpreted as a structure slightly more general than a stack, in that its underlying category naturally contains non-invertible morphisms (coming from isogenies of formal groups, rather than just isomorphisms). Ando [1], Rezk and others use such ideas to extend the $\mathrm{Gl}_n(\mathbb{Z}_p)$-action described above to provide a homotopy-theoretic interpretation of the action of the (now only locally compact) group $\mathrm{Gl}_n(\mathbb{Q}_p)$ on this moduli object.

A further technical issue involves the rigid-analytic structure. Objects in this class are inconvenient for many topological questions, because they are modelled locally by p-adic algebras allowing certain tightly-controlled denominators, which can wipe out delicate p-torsion phenomena. Relatively recently, however, Faltings [7] showed that one can work equally well with closely related formal Lubin-Tate schemes more familiar (and more convenient) in homotopy theory.

Behrens and Lawson's recent book [3] summarizes the current state of the art (modulo level structure, maybe): they construct enrichments of the moduli spaces of Abelian varieties used by Harris and Taylor in their work on the local LL correspondence, to schemes in the category of symmetric spectra, ie over Spec $\mathbb{S}$; or, equivalently, that these stacks support a natural nice sheaf of cohomology theories.

4 Maybe the main point of this note that any object Z (such as the complex cobordism spectrum **MU**, or the enriched moduli stack of elliptic curves underlying elliptic cohomology, or more generally the enriched Shimura varieties of Behrens-Lawson) which belongs simultaneously to two very different worlds $\mathcal{W}, \mathcal{W}'$ defines a **correspondence**

$$X \mapsto X \times_{\mathcal{W}} Z : \mathcal{W} \to \mathcal{W}'$$

between those worlds, which provides a tool for analyzing the objects of $\mathcal{W}$ by the techniques of $\mathcal{W}'$. Representation-theorists are interested in certain (vanishing-cycle) cohomology groups of things like Drinfel'd's upper-half-space, whereas homotopy theorists are interested in their coherent-sheaf cohomology. But these are very different functors, as different as simplicial homology is from automorphic cohomology.

The moduli spaces defining these correspondences are among the most beautiful and symmetrical known to mathematics. Schur's thesis dates from 1901, and it seems likely that the gadgets we're studying now will be similarly important for a good part of this coming century. At the present early stage of exploration, it would be a mistake to expect the questions and tools of representation theory to define the direction topological investigations should take (and conversely). These moduli objects are highly multifaceted, and deserve to be looked at from every conceivable viewpoint.

It is probably too early to talk about a topological Langlands **program**, because that phrase suggests we have a guess at where the subject is going. It might be fairer to say that at present there is a meaningful topological Langlands **correspondence**, whose properties we only beginning to understand.

Appendix: Problem for Extra Credit

Make some kind of sense of

$$\mathbb{P}_1(\mathbb{C})//\mathrm{Sl}_2(\mathbb{Z})\dots$$

(eg, as a derived or enriched scheme of some sort, or perhaps as a moduli object, eg as a noncommutative compactification of the space of elliptic curves (by adding some kind of noncommutative toruses along the boundary)? Note that T. Ono's Introduction to Algebraic Number Theory contains a very detailed analysis of $\mathbb{P}_1(\mathbb{R})/\mathrm{Sl}_2(\mathbb{Z})$ as a set ...).

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13 See . `http://en.wikipedia.org/wiki/Schur-Weyl_duality`

14. See the geometric Langlands seminar:

$$\texttt{http : //www.math.uchicago.edu/} \sim \texttt{mitya/langlands.html}$$