Out-takes from a seminar

[Caveat Lector! This is a compilation of bits and pieces from the TLL seminar, full of speculation and incomplete arguments. It is **not** ready for prime time.]

§I, Some examples

Following [4 def 2.8], an affine variety $X/\mathbb{F}_1$ is defined by a triple $(\mathbf{X}, i, X_{\mathbb{Z}})$ consisting of a covariant functor $\mathbf{X}$ from finite abelian groups to sets, an affine scheme $X_{\mathbb{Z}}$ over the integers, and a natural transformation

$$i : \mathbf{X}(D) \to X_{\mathbb{Z}}(\mathbb{C}[D]) \ ,$$

all subject to further conditions which I'll omit here. I'll also neglect refinements involving monoids and gradings which won't come up in these elementary examples.

Ex. 1 The scheme $\mathfrak{A}_K$ of points at infinity of a number field K

If D is a finite abelian group, and K is a finite extension of $\mathbb{Q}$, with ring $\mathcal{O}_K$ of integers, then

$$(\text{Spec } \mathcal{O}_K)(\mathbb{C}[D]) \cong \text{Hom}_{\text{algs}}(\mathcal{O}_K, \mathbb{C}[D])$$

is isomorphic to $\text{Hom}_{\mathbb{C}-\text{algs}}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}[D])$; but both $\mathcal{O}_K \otimes \mathbb{C}$ and $\mathbb{C}[D]$ are finite-dimensional commutative Banach algebras, and so are both isomorphic to rings of functions on finite sets.

The first of these is the set

$$\text{Spec } K \otimes_{\mathbb{Q}} \mathbb{C} := E(K)$$

of Archimedean embeddings of K, while the second is the Pontrjagin dual $\hat{D}$ of D; so the functor

$$(\text{Spec } \mathcal{O}_K)(\mathbb{C}[D]) \cong \mathfrak{A}_K(D) := \text{Fns}(\hat{D}, E(K))$$

is a gadget represented by the scheme Spec $\mathcal{O}_K$ over $\mathbb{Z}$, which therefore has the universal property defining an affine scheme over $\mathbb{F}_1$. An obvious refinement of this construction maps D to

$$\text{Fns}(\hat{D}, E(\overline{K})) \in \text{Gal}(\overline{K}/K) - (\text{Sets}) \ .$$

Ex. 2 The free loopspace of a gadget

If G is a (discrete) group, then its classifying space can be constructed as a simplicial set

$$BG : k \mapsto G^k \in \Delta^{\rm op}(\text{Sets})$$

whose face and degeneracy maps involve group composition. If $G = A$ is abelian the addition map $A \times A \to A$ is a homomorphism, so in that case $BA \in \Delta^{\rm op}(\text{Ab})$ defines a topological abelian group $BA = H(A, 1)$.

If $\mathbf{X} : (\text{Ab}_{\rm fg}) \to (\text{Sets})$ is a covariant functor (eg a gadget), then

$$k \mapsto L\mathbf{X}(k) := \text{inv. lim}_n \, \mathbf{X}((\mathbb{Z}/n\mathbb{Z})^k) \in \Delta^{\rm op}(\text{Sets})$$

is a kind of approximation to $\mathbf{X}(H(\hat{\mathbb{Z}}, 1))$. This seems to be a natural analog of the free loopspace of $\mathbf{X}$: it has a natural action of $\hat{\mathbb{Z}}$, but I don't know if it is actually a cyclic object.

Composing this with the preceding example defines a candidate

$$k \mapsto \text{inv. lim}_n \, \text{Fns}((\widehat{\mathbb{Z}/nZ})^k, E) = \text{Fns}(\text{dir lim}_n (\widehat{\mathbb{Z}/nZ})^k, E)$$

$$= \text{Fns}(\text{inv.} \, \widehat{\text{lim}_n (\mathbb{Z}/nZ)}^k, E) = \text{Fns}(\hat{\mathbb{Z}}^k, E) \; ,$$

for the free loopspace of $\mathfrak{A}_K$. This is a contravariant functor of k, and so defines a simplicial set; but in spite of its apparent simplicity, I don't know what it is.

[Be careful: one standard notation for the set $\text{Fns}(A, B)$ of functions from A to B is A^B, but

$$\text{Fns}(\text{Fns}(A, B), C) \neq \text{Fns}(\text{Fns}(C, B), A) \; ;$$

they have different variances.]

3 Real gadgets

The Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, c\}$ acts on $\mathbf{X}_{\mathbb{Z}}(\mathbb{C}[D])$. I propose to call a natural involution $\mathbf{c}$ of $\mathbf{X}$, for which the diagram

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & X_{\mathbb{Z}}(\mathbb{C}[D]) \\ \downarrow \mathbf{c} & & \downarrow X(c) \\ \mathbf{X} & \longrightarrow & X_{\mathbb{Z}}(\mathbb{C}[D]) \end{array}$$

commutes, a **real** structure [1] on $\mathbf{X}$.

For example, the additive groupscheme $\mathbb{G}_a/\mathbb{F}_1$ (defined by $\mathbf{G}_a(D) = \mathbb{Z}[D],\ \mathbb{G}_a(A) = A$) has a real structure defined by $\mathbf{c}_{\mathbb{G}_a} =$ identity, because

$$\mathbf{G}_a(D) = \mathbb{Z}[D] \to \mathbb{G}_a(\mathbb{C}[D]) = \mathbb{C}[D]$$

maps to the conjugation-invariant subring $\mathbb{R}[D]$. The multiplicative group $\mathbb{G}_m/\mathbb{F}_1$ (defined by $\mathbf{G}_m(D) = D$) is similarly real.

The scheme

$$\mathfrak{A}(D) := \text{Func}\ (\hat{D}, \text{Spec}\ (K \otimes_{\mathbb{Q}} \mathbb{C}))$$

of Archimedean places of K has a real structure defined by the natural action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on the set $\text{Spec}(K \otimes_{\mathbb{Q}} \mathbb{C}) = E(K)$ of its Archimedean embeddings.

Ex. 3 Categorification of the BC system [5, 10]:

The fiber product

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \dashrightarrow & \mathcal{Y} \\ \vdots & & \downarrow \Psi \\ \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Z} \end{array}$$

of two categories has

$$\text{obj}\ \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} = \{X \in \mathcal{X}, Y \in \mathcal{Y},\ \zeta : \Phi(X) \cong \Psi(Y)\}\ .$$

The morphisms from (X, Y, ζ_0) to (X', Y', ζ_1) are pairs $f : X \to X',\ g : Y \to Y'$ of morphisms such that the diagram

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\zeta_0} & \Psi(Y) \\ \downarrow & & \downarrow \\ \Phi(X') & \xrightarrow{\zeta_1} & \Psi(Y') \end{array}$$

commutes.

Notation: If a group G (or, even better, a monoid) acts on a set X, then $[X/G]$ is its associated transformation category: X is its set of objects, and

$$\text{mor}_{[X/G]}(x, x') = \{g \in G \mid gx = x'\}\ .$$

Example: The multiplicative monoid $\mathbb{Z}^\times$ of nonzero integers acts on the multiplicative group(-valued functor) $\mathbb{G}_m$ by

$$(n,u) \mapsto u^n \ : \ \mathbb{Z}^\times \times \mathbb{G}_m \to \mathbb{G}_m$$

(u being a unit in some unspecified commutative ring (or monoid)).

This defines a kind of stack over $\mathbb{F}_1$ (and thus, by pullback, over $\mathbb{Z}$). It is not a classical stack, because its morphisms are rarely invertible. It has been proposed that such generalized stacks be called **piles**.]

Claim: $[\mathbb{G}_m/\mathbb{Z}^\times]$ is a symmetric monoidal object, in the category of categories over $[1/\mathbb{Z}^\times]$.

More precisely: There is a nice product functor

$$\bullet : [\mathbb{G}_m/\mathbb{Z}^\times] \times_{[1/\mathbb{Z}^\times]} [\mathbb{G}_m/\mathbb{Z}^\times] \to [\mathbb{G}_m/\mathbb{Z}^\times]$$

(exhibited below).

Proof: We first need to describe the fiber product. The category $[1/\mathbb{Z}^\times]$ has the single object 1, with $\mathbb{Z}^\times$ as its monoid of endomorphisms. Objects of the category $[\mathbb{G}_m/\mathbb{Z}^\times]$ are units, and a morphism in this category is a triple (n,u,u') such that $u' = u^n$. The functors Φ and Ψ are the same: both send all objects to 1, and (n,u,u') to $(n,1,1)$.

Objects of the fiber product category are therefore triples (u,v,ζ), where $\zeta \in \{\pm 1\}$. Morphisms from (u,v,ζ_0) to (u',v',ζ_1) consist of pairs (n,m) such that $u' = u^n$, $v' = v^m$, and $\zeta_0 n = \zeta_1 m$.

The composition operator sends (u,v,ζ) to uv^ζ, and the morphism (n,m) to n: the diagram

$$\begin{array}{ccc} (u,v,\zeta_0) & \xrightarrow{(n,m)} & (u',v',\zeta_1) \\ \downarrow \bullet & & \downarrow \bullet \\ uv^\zeta & \xrightarrow{\quad n \quad} & u'v'^{\zeta_1} \end{array}$$

commutes, because

$$(uv^{\zeta_0})^n = u^n v^{n\zeta_0} = u^n v^{m\zeta_1} = u'v'^{\zeta_1} \ .$$

A similar chase, which I omit, shows that $\bullet$ is associative, and commutative if we define the twist by

$$\tau(u,v,\zeta) = (v^\zeta, u^\zeta, \zeta) \ .$$

The inclusion $[1/\mathbb{Z}^\times] \to [\mathbb{G}_m/\mathbb{Z}^\times]$ defines a unit.

Example 4 Kapranov's stack [8]

There is a natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$, defined over $\mathbb{F}_1$ by

$$\delta, \sum a_i\delta_i \mapsto \sum a_i\delta \cdot \delta_i : D \times \mathbb{Z}[D] \to \mathbb{Z}[D] \ .$$

The functor sending D to the transformation groupoid $[\mathbb{Z}[D]/D]$ defines a kind of affine stack over $\mathbb{F}_1$, which I'll denote $[\mathbb{G}_a/\mathbb{G}_m]$, and the transformation

$$\tau \circ (\mathbf{c} \times \mathbf{c}) : [\mathbb{G}_a/\mathbb{G}_m]^2 \to [\mathbb{G}_a/\mathbb{G}_m]^2$$

(where $\tau(x, y) = (y, x)$ is the twist map) defines an analog

$$\mathsf{Hdg} := [\mathbb{G}_a/\mathbb{G}_m] \times_{\mathbf{c}} [\mathbb{G}_a/\mathbb{G}_m] \to \mathrm{Spec}\ \mathbb{F}_1$$

of Kapranov's $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant model for a stack of the form $[\mathbb{C}/\mathbb{C}^\times]$.

§2, Some questions:

1: Cohomology with coefficients in $\mathbb{F}_1$?

There is a sheaf of abelian groups

$$[(\mathbb{C}_\delta \times \mathbb{C})/\mathbb{C}^\times] \to [\mathbb{C}/\mathbb{C}^\times]$$

(where $\mathbb{C}_\delta$ is the complex numbers with the discrete topology; the group $\mathbb{C}^\times$ acts via the diagonal action in the top line). This is a special case of a more general diagram

$$\begin{array}{ccc} [(X \times \mathbb{C}_\delta \times \mathbb{C})/\mathbb{C}^\times] & & \\ \downarrow & & \\ [(X \times \mathbb{C})/\mathbb{C}^\times] & \xrightarrow{\pi} & [\mathbb{C}/\mathbb{C}^\times] \end{array}$$

defining a similar sheaf above the product of a topological space X with the stack $[\mathbb{C}/\mathbb{C}^\times]$.

When X is an algebraic variety, Hodge-de Rham gives the topological direct image sheaf

$$R^*\pi(\mathbb{C}_\delta) \cong [(H^*(X(\mathbb{C}), \mathbb{C}_\delta) \times \mathbb{C})/\mathbb{C}^\times]$$

the structure of an algebraic vector bundle over $[\mathbb{C}/\mathbb{C}^\times]$ (maybe even with a connection) [2, 3, 6, 7].

I suspect there may be some analog of this for schemes over $\mathbb{F}_1$, with $[\mathbb{C}/\mathbb{C}^\times]$ replaced by $[\mathbb{G}_a/\mathbb{G}_m]$; this might be a plausible kind of cohomology with coefficients in $\mathbb{F}_1$. Toric varieties could provide an accessible test of this suggestion.

2 Blowing up infinity?

Work of Rognes [11] can be interpreted in the language of [12] as implying the existence of a pullback diagram

$$\begin{array}{ccc} \mathbf{FGLs} & \longrightarrow & \text{Spec } MU \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{Z} & \longrightarrow & \text{Spec } \mathbb{S} \end{array}$$

the object in the upper left corner being the stack of one-dimensional formal group laws. On the other hand, Ex. 4 above implies the existence of a pullback diagram of the form

$$\begin{array}{ccc} [\mathbb{C}/\mathbb{C}^\times] & \longrightarrow & [\mathbb{G}_a/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{Z} & \longrightarrow & \text{Spec } \mathbb{F}_1 \ . \end{array}$$

It would be nice to combine these constructions in a product stack over

$$\text{Spec } \mathbb{S} \times_{\text{Spec } \mathbb{S}_1} \text{Spec } \mathbb{F}_1$$

and to extend complex cobordism and Hodge cohomology simultaneously to a functor taking values in sheaves over this product. [Here $\mathbb{S}_1$ is a version [12] of **un**stable homotopy theory.]

Some references:

1. M.F. Atiyah, K-theory and reality. Quart. J. Math. Oxford 17 (1966) 367–386

2. C. Deninger, On the Γ-factors attached to motives, Invent. Math. 104 (1991) 245–261

3. ——, On the Γ-factors of motives. II. Doc. Math. 6 (2001) 69–97

4. A. Connes, C. Consani On the notion of geometry over $\mathbb{F}_1$, `arXiv:0809.2926`

5. ——, ——, M. Marcolli, Fun with $\mathbb{F}_1$, `arXiv:0806.2401`

6. C. Consani, Double complexes and Euler L-factors, Compositio Math. 111 (1998) 323–358.

7. ——, M. Marcolli, Archimedean cohomology revisited, `arXiv:math/0407480`

8. M. Kapranov, Real mixed Hodge structures, `arXiv:0802.0215`

9. Y. Manin, Cyclotomy and analytic geometry over $\mathbb{F}_1$, `arXiv:0809.1564`

10. M. Marcolli, Cyclotomy and endomotives, `arXiv:0901.3167`

11. J. Rognes, Galois extensions of structured ring spectra, `arXiv:math/0502183`

12. B. Toen, M. Vaquié, Under Spec $\mathbb{Z}$, `arXiv:math/0509684`