The method of moving planes.
A personal tribute to Louis Nirenberg:

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I first met Louis Nirenberg in 1972 when I became a Courant instructor. He was already a celebrated mathematician and a suave sophisticated New Yorker, even though he was born in Hamilton, Canada and grew up in Montreal. In this talk I will describe the geometric ideas related to his famous papers centered on the “method of moving planes”.
A central question in differential geometry has been to classify all possible "soap bubbles", or more formally put: Classify all compact constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$. Amazingly, the first such result was proven by J. H Jellett in 1853:

A starshaped compact surface $M$ in $\mathbb{R}^3$ with positive constant mean curvature is the standard round sphere.
A modern presentation of his proof, which extends to $\mathbb{R}^{n+1}$ goes as follows: Let $X: M^n \to \mathbb{R}^{n+1}$ be the position vector and $N$ the outward normal to $M$. We assume $X(M)$ has constant mean curvature $H > 0$ and is starshaped about the origin, i.e., $X \cdot N \geq 0$ on $M$. If $\Delta_M$ is the Laplace-Beltrami operator on $M$ and $A$ is the second fundamental form of $M$, then

\[ \Delta_M X = -nH N, \]  
\[ \Delta_M N = -|A|^2 N, \]  
\[ \Delta_M |X|^2/2 = n - nH N, \]  
\[ \Delta_M X \cdot N = nH - |A|^2 X \cdot N. \]  

Then (3), (4) imply

\[ \Delta_M (H|X|^2/2 - X \cdot N) = (|A|^2 - nH^2)X \cdot N. \]  

Integrate (5) over $M$ to conclude $|A|^2 = nH^2$, or $M$ is totally umbilic, so a sphere.
Heinz Hopf visits Stanford and lectures on this problem.

In his Stanford lectures, Hopf proves that if $M$ is an immersed surface of constant mean curvature and genus 0 in $\mathbb{R}^3$, then $M$ is a round sphere. His proof is ingenious but leads us away from our main story. In the same lectures he announced a new result by the Russian geometer A. D. Alexandrov and sketched the proof:

An embedded hypersurface $M^n$ in $\mathbb{R}^{n+1}$ of constant mean curvature is a round sphere.
Alexandrov’s method of moving planes.

Alexandrov’s idea is to show that an embedded hypersurface $S \subset \mathbb{R}^{n+1}$ has a plane of symmetry for any direction $e_1$ and thus must be a sphere. Note that any open set of directions suffices.
Reflecting as far as possible...

\[ S^*_x \subset \text{int}(S) \]

\[ S^*_x \] is tangent to \( S \) at \( p \neq T_x \)

\[ S^*_x \subset \text{int}(S) \]

\[ S^*_x \] is tangent to \( S \) at \( p \in T_x \)
The Hopf maximum principle and boundary point lemma implies symmetry.
Alexandrov’s generalized gradient map and contact set

Alexandrov introduced the normal mapping (subdifferential) of \( u \in C^0(\Omega) \), \( \Omega \in \mathbb{R}^n \):

\[
\partial u(x_0) = \{ p \in \mathbb{R}^n : u(x) \geq u(x_0) + p \cdot (x - x_0) \}
\]

If \( E \subset \Omega \), \( \partial u(E) = \bigcup_{x \in E} \partial u(x) \)

Of course \( \partial u(x_0) \) may be empty.

The (lower) contact set \( \Gamma_u = \{ x \in \Omega : \partial u(x) \neq \emptyset \} \)

If \( u \in C^1(\Omega) \) and \( x \in \Gamma_u \), then \( \partial u(x) = \nabla u(x) \)

If \( u \in C^2(\Omega) \) and \( x \in \Gamma_u \), then \( D^2 u(x) \geq 0 \)
Example. $\Omega = B_R(0)$, $u(x) = \frac{h}{R}(|x| - R)$. Then

$$\partial u(x) = \begin{cases} \frac{h}{R} \frac{x}{|x|} & x \neq 0 \\ B_{\frac{h}{R}}(0) & x = 0 \end{cases}$$

**Theorem.** If $u \in C^0(\Omega)$, then $S = \{ E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable} \}$ is a Borel $\sigma$ algebra.

The set function $\mathcal{M}u(E) = |\partial u(E)|$ is called the Monge-Ampere measure associated to $u$. If $u \in C^2(\Omega)$ is convex,

$$\mathcal{M}u(E) = \int_E \det(D^2 u)$$
Application to the Isoperimetric inequality

(Xavier Cabré) Given $\Omega$, solve the Neumann problem

$$\Delta u = 1 \text{ in } \Omega$$

$$u_\nu = c = \frac{|\Omega|}{|\partial \Omega|} \text{ on } \partial \Omega$$

Claim: $\Gamma_u$ contains a ball of radius $c$. To prove the claim, take any $p \in \mathbb{R}^n$ with $|p| < c$. Let $x_1$ be a point in $\overline{\Omega}$ such that

$$\min_{y \in \Omega} (u(y) - \langle p, y \rangle) = u(x_1) - \langle p, x_1 \rangle.$$ 

If $x_1 \in \partial \Omega$, then the exterior normal derivative

$$\frac{\partial}{\partial \nu}(u(y) - \langle p, y \rangle)(x_1) \leq 0,$$

which implies $c = u_\nu(x_1) \leq |p|$, a contradiction.
On the other hand if $x_1 \in \Omega$, then $p = \nabla u(x_1)$ and so $x \in \Gamma_u$ proving the claim. Therefore,

$$\omega_n c^n \leq \int_{\Gamma_u} \det(D^2 u) \leq \int_{\Gamma_u} (\Delta u/n)^n \leq \int_{\Omega} (\Delta u/n)^n = |\Omega|/n^n.$$ 

This implies 

$$|\partial \Omega| \geq n\omega_n^{1/n} |\Omega|^{n-1/n},$$

which is the sharp isoperimetric inequality.
A simplified version of the classical strong maximum principle and boundary point lemma of E. Hopf.

In a bounded domain $\Omega \subset \mathbb{R}^n$, suppose $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta w + c(x)w \geq 0$, $w \leq 0$ where $c(x)$ is bounded. If $w(x_0) = 0$ for an interior point $x_0$, then $w \equiv 0$.

Let $B$ be a ball in $\mathbb{R}^n$ and suppose $w \in C^2(B) \cap C^1(\overline{B})$ satisfies $\Delta w + c(x)w \geq 0$, $w \leq 0$ in $B$, where $c(x)$ is bounded. If $w(x_0) = 0$ for $x_0 \in \partial B$, then $w_\nu(x_0) < 0$ for any direction $\nu$ pointing into $B$. 
The Alexandrov maximum principle for domains of small volume.

Lemma. Assume $\Delta u + c(x)u \geq f$ in $\Omega$, $c(x) \leq 0$, $u \leq 0$ on $\partial \Omega$. Then $M := \sup_{\Omega} u^{+} \leq Cd\|f^{-}\|_{L^{n}(\Omega)}$, $d = \text{diam}(\Omega)$.

Proof. Assume $M > 0$ is achieved at an interior point $x_1 \in \Omega$. Let $v = -u^{+}$. Then $v < 0$ on $\Gamma_v$, the lower contact set of $v$ and $-M = v(x_1) = 0$ on $\partial \Omega$. Hence

$$\Delta v(x) = -\Delta u(x) \leq -f(x) + c(x)u(x) \leq f^{-}(x), \ x \in \Gamma_v.$$
Claim: \( B(0, M/d) \subset \nabla v(\Gamma_v) \).

Let \(|p| < M/d\) and as before translate the plane \( p \cdot x - k \) “up from \(-\infty\)” until there is a first contact with the graph of \( v \) at a point \( x_0 \in \Omega \). Then \( x_0 \not\in \partial \Omega \) for otherwise (since then \( v(x_0) = 0 \))

\[-M = v(x_1) \geq p \cdot (x_1 - x_0) \geq |p|d > -M, \text{ a contradiction.}\]

Therefore \( p = \nabla v(x_1) \) proving the Claim.

Proceeding analogously to the proof of the isoperimetric inequality

\[
\omega_n(M/d)^n \leq \int_{\Gamma_v} \det(D^2v) \leq \int_{\Gamma_v} (\Delta v/n)^n \leq \|f^-\|_{L^n(\Omega)},
\]

and the Lemma follows.
The maximum principle for domains of small volume.

We now formulate the maximum principal for $\Delta u + c(x)u \geq 0$ and drop the assumption $c(x) \leq 0$. To accomplish this, write $c = c^+ - c^-$, $\Delta u - c^- u \geq -c^+ u^+$. Applying the Lemma gives

$$M \leq Cd\|c^+ u^+\|_{L^1(\Omega)} \leq Cd\|c^+\|_{\infty}\Omega^{\frac{1}{n}}M \leq M/2$$

for $|\Omega| \leq (2Cd\|c^+\|_{\infty})^{-n}$.

We have proved with no assumption on the sign of $c(x)$,

**Theorem** Assume $\Delta u + c(x)u \geq 0$ in $\Omega$, $u \leq 0$ on $\partial \Omega$. Then there is a positive constant $\delta$ depending on $n$, diam$(\Omega)$, $\|c^+\|_{\infty}$ so that if $|\Omega| \leq \delta$, then $u \leq 0$ in $\Omega$. 
Motivating PDE questions

Many problems in both Yang-Mills theory, astrophysics and reaction-diffusion systems are modeled by equations of the form \( \Delta u + f(u) = 0 \). For example the power nonlinearity \( f(u) = u^\alpha \) occurs in many applications. The range \( 1 \leq \alpha < \frac{n+2}{n-2} \) is called the subcritical range because of the Sobolev embedding theorem while the range \( \alpha > \frac{n+2}{n-2} \) is called supercritical. The case \( \alpha = \frac{n+2}{n-2} \) is particularly important because the equation becomes conformally invariant. This is the conformally flat case of the famous Yamabe problem.

**Question 1.** Suppose we consider the Dirichlet problem

\[
\Delta u + f(u) = 0, \quad u \geq 0 \text{ in } B = B_1(0), \quad u = 0 \text{ on } \partial B.
\]

Is \( u \) radially symmetric, i.e. \( u = u(|x|) \)?
There is a simple counterexample if we allow $u$ to change sign. Take $f(u) = \lambda_k u$ with $k \geq 2$ where $\lambda_k$ is the $k$th Dirichlet eigenvalue of $B$. Then the eigenfunctions are not radially symmetric.

**Question 2. Global Questions**

Let $\Delta u + f(u) = 0$, $u \geq 0$ in $\mathbb{R}^n$. Assume $u$ decays to 0 at infinity. Is $u$ radially symmetric?

The famous equation $\Delta u + u^{\frac{n+2}{n-2}} = 0$, $u > 0$ in $\mathbb{R}^n$ has the explicit solutions

$$u(x) = \left(\frac{\lambda \sqrt{n(n-2)}}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}} \lambda > 0.$$

On the other hand, $\Delta u + u^\alpha = 0$, $u \geq 0$ in $\mathbb{R}^n$ has only the trivial solution for subcritical $\alpha$. (Gidas-Spruck).
There is also a complicated family of singular solutions of the form
\[ u(x) = r^{-(n-2)/2} \psi(t) \]
where \( r = |x| \), \( t = -\log r \) and \( \psi(t) \) is a periodic translation invariant solution of the ODE:
\[
\psi'' - \left( \frac{n-2}{2} \right)^2 \psi + \psi^{\frac{n+2}{n-2}} = 0.
\]

The simplest singular solution is
\[ u(x) = k / |x|^{\frac{n-2}{2}}, \quad k = \left( \frac{(n-2)}{2} \right)^{\frac{n-2}{2}}. \]

Thus the classification of global singular solutions is quite challenging.
A result of Nirenberg and collaborators.

Nirenberg refined his original method with Gidas and Ni in a paper with Berestycki by strategically using the Alexandrov maximum principle as well as the Hopf maximum principle and boundary point lemma. We present this improved proof. These papers have had a tremendous impact.

**Theorem.** Let $\Omega$ be a bounded domain which is convex in the $e_1$ direction and symmetric with respect to the plane $\{x_1 = 0\}$. Suppose $u \in C^2 \cap C^0(\overline{\Omega})$ satisfies

$$\Delta u + f(u) = 0, \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$

where $f(t)$ is a Lipschitz function. Then $u(x_1, x') = u(-x_1, x')$ and $u_{x_1}(x) < 0$ for any $x$ with $x_1 > 0$. 
The proof by moving planes in the $x_1$ direction.

We will prove that

$$u(x_1, y) < u(x_1^*) \quad \text{for all } x_1 > 0 \text{ and } -x_1 < x_1^* < x_1.$$ 

This implies monotonicity in $x_1$ for $x_1 > 0$. Letting $x_1^* \to -x_1$ implies

$$u(x_1, y) \leq u(-x_1, y) \quad \text{for any } x_1 > 0.$$

Replacing $e_1$ by $-e_1$ gives reflection symmetry

$$u(x_1, y) = u(-x_1, y) \quad \text{for any } (x_1, y) \in \Omega.$$
Moving the plane $T_\lambda := \{x_1 = \lambda\}$ from “infinity”.

Let $a = \sup_\Omega x_1$ and for $\lambda < a$, set $\Sigma_\lambda := \{x \in \Omega : x_1 > \lambda\}$.

Given $x = (x_1, y) \in \Omega$, let $x_\lambda = (2\lambda - x_1, y)$ be its reflection in $T_\lambda$.

For $x \in \Sigma_\lambda$, set $u^\lambda(x) = u(x_\lambda)$ and $w(x) = w_\lambda(x) = u(x) - u^\lambda(x)$.

By the mean value theorem,

$$\Delta w = -\frac{f(u) - f(u^\lambda)}{u - u^\lambda} w = -c(x, \lambda) w,$$

where $c := c(x, \lambda)$ is bounded. Note that $\partial \Sigma_\lambda$ consists of a part of $T_\lambda$ where $w = 0$ and a part of $\partial \Omega$ where $w = -u^\lambda < 0$. In summary, we have

$$\Delta w + c(x, \lambda) w = 0 \text{ in } \Sigma_\lambda$$

(6)

$$w \leq 0 \text{ and } w \neq 0 \text{ on } \partial \Sigma_\lambda,$$

with $c(x, \lambda)$ bounded.
Beginning and ending.

For \( a - \delta < \lambda < a \), \( \Sigma_\lambda \) is narrow and so has small volume for \( \delta \) small. Thus we may apply the Alexandrov maximum principle to (6) and conclude that \( w < 0 \) inside \( \Sigma_\lambda \). Note also that by the Hopf boundary point lemma,

\[
\frac{w_{x_1}}{x_1} \bigg|_{x_1=\lambda} = 2u_{x_1} \bigg|_{x_1=\lambda} < 0.
\]

Let

\[
\lambda_0 = \inf \{ 0 < \lambda < a : w < 0 \text{ in } \Sigma_\lambda \}.
\]

If \( \lambda_0 = 0 \) we are done. Assume for contradiction that \( \lambda_0 > 0 \).

By continuity, \( w \leq 0 \) in \( \Sigma_\lambda \) and moreover \( w \not\equiv 0 \) on \( \partial \Sigma_\lambda \).

Therefore the strong maximum principle implies

\[
w < 0 \text{ in } \Sigma_\lambda. \quad (7)
\]
Deriving a contradiction.

Claim. \( w_{\lambda_0 - \varepsilon} < 0 \) in \( \Sigma_{\lambda_0 - \varepsilon} \) for sufficiently small \( 0 < \varepsilon < \varepsilon_0 \).

Choose a simply connected closed set \( K \) in \( \Sigma_{\lambda_0} \) with smooth boundary (which is nearly all of \( \Sigma_{\lambda_0} \) in measure) such that

\[
|\Sigma_{\lambda_0} \setminus K| < \delta/2.
\]

By (7) there exists \( \eta > 0 \) so that

\[
w_{\lambda_0} \leq -\eta \quad \text{for all} \; x \in K,
\]

and so by continuity

\[
w_{\lambda_0 - \varepsilon} \leq -\eta/2 \quad \text{for all} \; x \in K. \tag{8}
\]
Now we are in the situation that
\[ w_{\lambda_0 - \varepsilon} \leq 0 \text{ on } \partial(\Sigma_{\lambda_0 - \varepsilon} \setminus K). \]

However for \( \varepsilon \) sufficiently small we can make \( |\Sigma_{\lambda_0 - \varepsilon} \setminus K| < \delta \).
Thus once again applying the Alexandrov maximum principle and the strong maximum principle gives
\[ w_{\lambda_0 - \varepsilon} < 0 \text{ in } \Sigma_{\lambda_0 - \varepsilon} \setminus K. \]

Combined with (8), we have
\[ w_{\lambda_0 - \varepsilon} < 0 \text{ in } \Sigma_{\lambda_0 - \varepsilon}, \]
a contradiction at last!

**Corollary.** The answer to Question 1 is yes for positive solutions.