# ON THE EXISTENCE OF CLOSED GEODESICS AND UNIQUENESS OF WEAKLY HARMONIC MAPS

by

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A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland

April, 2011

# Abstract

The aim of this dissertation is threefold and it records three distinct results that the author proved, along with his collaborators, during his time as a graduate student at The Johns Hopkins University. Firstly, with L. Wang, in [30] we studied the existence of good sweepouts by curves on closed (i.e., compact and without boundary) Riemannian manifolds via the harmonic map heat flow. This can be thought of as a continuous version of Colding and Minicozzi's width-sweepout construction of closed geodesics in [14] where the discrete local linear (geodesic) replacement was used. Secondly, in [29] the author extended Colding and Minicozzi's width-sweepout construction of closed geodesics on closed manifolds to the case of closed Alexandrov spaces of curvature bounded from above. Finally, together with T. Lamm, in [26] we give an alternate proof, via a new approach that involves Rivière's gauge decomposition technique, to Colding and Minicozzi's energy convexity and uniqueness of weakly harmonic maps with small energy on the two dimensional unit disc $B_1 \subset \mathbf{R}^2$. This is part of a project of generalizing Colding and Minicozzi's energy convexity for harmonic maps to the case of biharmonic maps.

READERS: William P. Minicozzi II (Advisor), Joel Spruck and Chikako Mese.

# Acknowledgments

I would first like to express my deep gratitude to my advisor, Dr. William P. Minicozzi II, for years of consistent support, patient guidance and encouragement. This thesis could not have been completed without his help.

I would also like to express my gratitude to Dr. Joel Spruck and Dr. Chikako Mese for being great teachers and for teaching me PDE and geometry. I would especially like to thank Dr. Tobias Lamm of The Goethe University Frankfurt, Lu Wang of The Massachusetts Institute of Technology and Ling Xiao of The Johns Hopkins University for being such great collaborators and friends. I thank Dr. Jian Kong for his friendship and advice during my years at Hopkins. Many thanks to my fellow graduate students Jin-Cheng Jiang, Qi Zhong, Yifei Chen, Xin Yu, Christine Breiner, Steve Kleene, Caleb Hussey, Jon Dahl, Joel Kramer, Duncan Sinclair, Patrick Zulkowski, Joe Cutrone, Nick Marshburn, Sinan Ariturk, Mike Limarzi and many others for years of company and friendship.

I dedicate this dissertation to my parents Guoying Lin and Meihua Lin, as well as my wife Jie Chen. Thank you all for your constant love and support.

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# Chapter 1

# Introduction

The study of closed geodesics was initiated by Hadamard [21], Poincaré [37] and Birkhoff [4]. Closed geodesics have been investigated mainly in the case of closed (i.e., compact and without boundary) Riemannian manifolds, while various results were obtained for Finsler manifolds and in the more general case of metric spaces with certain special properties (such as Busemann $G$-spaces, see [8]). In the Riemannian case, Hadamard proved that the shortest curves in a nontrivial conjugacy class of the fundamental group $\pi_1$ are closed geodesics. In [19] Grayson showed that there exist simple closed geodesics in each nontrivial $\pi_1$ homotopy class on closed non-simply-connected surfaces (e.g. a torus) by the curve shortening flow. On the 2-sphere whose $\pi_1$ homotopy group is trivial, Birkhoff used the curve shortening map and sweepouts to find non-trivial closed geodesics. Birkhoff's argument works equally well on other closed Riemannian manifolds and it goes back to 1917: one pulls each curve in a sweepout on the manifold as tight as possible, in a continuous way and preserving the sweepout; see [4], [5], [13], [16], [14], [29], [30] and

Part of the material in this thesis was first published in [30] *Existence of good sweepouts on closed manifolds* in Proceedings of the American Mathematical Society, 138 (2010), No. 11, 4081-4088 and in [29] *Closed geodesics in Alexandrov spaces of curvature bounded from above* in Journal of Geometric Analysis, Vol. 21, Issue 2 (2011), 429-454.

section 2 in [17] about Birkhoff's idea.

In [14], Colding and Minicozzi introduced the geometric invariant of closed Riemannian manifolds that they call the *width*. They succeeded in using the local linear (geodesic) replacement as Birkhoff's curve shortening map to explicitly construct good sweepouts by curves that produce at least one closed geodesic, which realizes the width as its energy. In particular, there exist closed geodesics on any closed Riemannian manifold. The argument only produces non-trivial closed geodesics when the width is positive. Their local linear (geodesic) replacement process is a discrete gradient flow (for the length functional of curves), and it depends solely on a local energy convexity for geodesic segments sharing the same end points (see Lemma 4.2 of [14]), which controls the distance of curves in the tightened sweepout from closed geodesics explicitly. Another crucial role the energy convexity plays is that it gives "discrete Palais-Smale condition C" which leads to the convergence to closed geodesics. There are several other applications of the existence of good sweepouts by curves on closed manifolds besides producing close geodesics. For instance, in [14], Colding and Minicozzi showed that the rate of change of the width for a one-parameter family of convex hypersurfaces that flows by mean curvature is bounded from above by a negative constant. The estimate is sharp and leads to a sharp estimate for the extinction time. In [15] a similar bound for the rate of change for the two dimensional width (defined by sweepouts by 2-spheres) is shown for homotopy 3-spheres evolving by Hamilton's Ricci flow, from which the finite time extinction result follows; see also Perelman's work in [36].

However, it requires some work to show the local linear (geodesic) replacement or the discrete shortening process preserves the homotopy class of sweepouts in Colding and Minicozzi's construction of closed geodesics. Therefore in the first part of this thesis, instead of the local linear replacement, we use a continuous method, i.e. the harmonic map heat flow, to tighten sweepouts, which meanwhile provides a natural homotopy of sweepouts. We show that the tightened sweepout has the following good property: *each curve in the tightened sweepout whose energy is close to the maximal energy of curves in the sweepout is itself close to a closed geodesic.* In particular, this implies that the width is the energy of some closed geodesic. This can be thought of as a continuous version (or heat flow version) of Colding and Minicozzi's width-sweepout construction of closed geodesics in closed Riemannian manifolds. As an immediate corollary of the existence of good sweepouts, we have the following theorem. Throughout we let $(\mathcal{N}, g)$ be a general closed Riemannian manifold with metric tensor $g$ which can be isometrically embedded into $(\mathbf{R}^n, \langle , \rangle)$.

**Theorem 1.0.1.** *Suppose that the $k_0$-th homology group $H_{k_0}(\mathcal{N})$ is nonzero for some $k_0 \geq 1$, then $\mathcal{N}$ admits at least one non-trivial closed geodesic.*

One sees that this is a variant of Birkhoff's existence theorem of closed geodesics on simply-connected manifolds.

Recently, Alexandrov spaces re-emerged into prominence and have attracted a lot of attention. Yet the existence of non-trivial closed geodesics in general Alexandrov spaces remained unknown. In the second part of this thesis, we extend Colding and Minicozzi's width-sweepout construction of closed geodesics into the case of closed Alexandrov spaces of curvature bounded from above. We will use the same local linear (geodesic) replacement as Birkhoff's curve shortening map to find good sweepouts by curves in Alexandrov spaces. As mentioned, the local energy convexity is crucial in the width-sweepout construction of closed geodesics. Therefore it is reasonable that if we can find a similar local energy convexity for maps into Alexandrov spaces, we could perhaps extend Colding and Minicozzi's width-sweepout construction to produce closed geodesics in any closed Alexandrov space of curvature bounded from above by $K$, which is locally a $CAT(K)$ space (also known as an $\mathfrak{R}_K$ domain, see [6] or subsection 2.1 below). In theorem 2.2 of [25] (see equation (2.2iv) of [25]), Korevaar and Schoen provided an energy convexity of

$W^{1,2}$ maps from a compact Riemannian domain into a nonpositively curved space (i.e., in the case of $K = 0$) for which the model space* is $\mathbf{R}^2$. Since the model space for the case of $K > 0$ is the standard Euclidean 2-hemisphere $\mathbf{S}_K$, which is locally $\mathbf{R}^2$, it is perhaps not surprising that a similar energy convexity should also hold locally. In fact, we are able to show that, with a small image assumption, the Korevaar and Schoen's energy convexity still holds (up to a constant) for $W^{1,2}$ maps into a $CAT(K)$ space with $K > 0$ .

**Theorem 1.0.2.** *Let $\Sigma$ be a compact Riemannian domain and $(X, d)$ be an Alexandrov space of curvature bounded from above by $K$. In an $\mathfrak{R}_K$ domain of $x \in X$, there exists $\rho = \rho(x, K) > 0$ such that for $u, v \in W^{1,2}(\Sigma, X)$ with images staying in $B_\rho(x) \subset \mathfrak{R}_K$, the following holds:*

$$\frac{1}{4}\int_\Sigma |\nabla d\,(u,v)|^2 \leq E^u + E^v - 2E^w. \tag{1.1}$$

*Here $B_\rho(x)$ is the geodesic ball centered at $x$ with radius $\rho$, $w = \frac{u+v}{2}$ is the mid-point map and $E$ is the 2-energy of maps into metric spaces (see section 2.2).*

We shall remark that Theorem 1.0.2 provides a stronger (quantitative) version of a result of Burago, Burago and Ivanov [6, Proposition 9.1.17]. Theorem 1.0.2 allows us to use Colding and Minicozzi's width-sweepout construction of closed geodesics to produce closed geodesics in another class of general metric spaces, namely, the (closed) Alexandrov spaces of curvature bounded from above. Moreover, as an immediate corollary of the existence of good sweepouts by curves, we obtain the following generalized Birkhoff-Lyusternik theorem on the existence of non-trivial closed geodesics, cf. [32], [31]. This is a generalization of Theorem 1.0.1 if one recalls that any compact smooth Riemannian manifold is an Alexandrov space of curvature bounded from above by some $K$ (see Theorem 2.1.1).

*Also known as the $K$-plane, see footnote †.

**Theorem 1.0.3.** *(Generalized Birkhoff-Lyusternik theorem) Let $(X, d)$ be a closed Alexandrov space of curvature bounded from above by $K$. Suppose that the $k_0$-th homology group $H_{k_0}(X)$ is nonzero for some $k_0 \geq 1$, then $(X, d)$ admits at least one non-trivial closed geodesic.*

Similar to the width-sweepout construction of closed geodesics, in [15] Colding and Minicozzi used the local harmonic map replacement to construct good 2-sweepouts by 2-spheres on a closed 3-manifold and proved the finite extinction time of Hamilton's Ricci flow. Again, an energy convexity for weakly harmonic maps with small energy on $B_1$ (see Theorem 3.1 in [15]) plays a critical role in the construction. This energy convexity can be thought of as the two dimensional analogue of the energy convexity for (one dimensional) geodesic segments in the width-sweepout construction of closed geodesics mentioned above.

The theory of harmonic maps between Riemannian manifolds has been an intensely researched field over the years. This is not only because harmonic maps have rich display of both differential geometric and analytic phenomena, but also their important applications to other research fields such as minimal surfaces (i.e. conformal harmonic maps) and deformations of Riemannian surfaces. The existence, regularity, and uniqueness theories of weakly harmonic maps have been active topics for mathematicians, and various important and interesting results have been obtained under different conditions, such as the dimension of the source manifold, the curvature of the target manifold, its topology and the type of definition chosen for a weak solution, see [24] and the references therein. Above all, harmonic maps in the critical "conformal" dimension, i.e. on surfaces, are of particular interest because of their special features. In [23], Hélein proved the entire interior regularity of weakly harmonic maps on surfaces with the help of the Coulomb moving frame, and Qing showed in [38] continuity up to the boundary in the case of continuous boundary data. More recently, in contrast to Hélein's moving frame technique, in

[40] Rivière provided a new approach to the regularity of a more general 2-dimensional conformally invariant non-linear system of elliptic PDEs (which includes the harmonic maps on surfaces), using gauge decomposition techniques and Wente's lemma.

Regarding the uniqueness of the weakly harmonic maps on surfaces, however, it was not until two years ago that Colding and Minicozzi showed in [15] an energy convexity for $W^{1,2}$ weakly harmonic maps on $B_1 \subset \mathbf{R}^2$ and hence the uniqueness, namely, they proved:

**Theorem 1.0.4.** *([15], [26]) For $u, v \in W^{1,2}(B_1, \mathcal{N})$ with $u$ being weakly harmonic, the Dirichlet energy $E(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \leq \epsilon$ for some small positive constant $\epsilon$ and $u|_{\partial B_1} = v|_{\partial B_1}$, we have*

$$\frac{1}{2}\int_{B_1} |\nabla v - \nabla u|^2 \leq \int_{B_1} |\nabla v|^2 - \int_{B_1} |\nabla u|^2 .$$

An immediate corollary of Theorem 1.0.4 is the uniqueness of solutions to the Dirichlet problem for weakly harmonic maps with small energy on $B_1$.

**Corollary 1.0.5.** *([15]) Let $\epsilon > 0$ be as in Theorem 1.0.4. If $u_1$ and $u_2$ are $W^{1,2}$ weakly harmonic maps from $B_1$ to $\mathcal{N}$, both with Dirichlet energy at most $\epsilon$, and they agree on $\partial B_1$, then $u_1 = u_2$ .*

In the last part of this thesis, by revealing some special Jacobian structures that only hold for the weakly harmonic maps on surfaces and by adapting Rivière's gauge decomposition technique introduced in [40], we will give a new proof of the Colding and Minicozzi's energy convexity and uniqueness results stated above for weakly harmonic maps with small energy on $B_1$, namely, Theorem 1.0.4.

This thesis is organized as follows. In Chapter 2, we collect some needed background material. In Chapter 3, we first derive estimates for the harmonic map heat flow from the unit circle $\mathbf{S}^1$ and then use the flow to construct good sweepouts on closed manifolds. In Chapter 4, we show an energy convexity of $W^{1,2}$ maps into $CAT(K)$ spaces and then use Colding and Minicozzi's width-sweepout construction to show the existence of closed geodesics in closed Alexandrov space of curvature bounded from above. In Chapter 5, we give a new proof to the energy convexity and uniqueness of weakly harmonic maps with small energy on $B^1$.

# Chapter 2

# Background

We begin by providing some necessary background material on Alexandrov space, energy of maps into general metric spaces, the sweepout by curves and the width of closed Riemannian manifolds and closed Alexandrov spaces, and the weakly harmonic maps. We are then in the position where we can formally focus on the two main problems in this work: existence of closed geodesics and uniqueness of 2-dimensional weakly harmonic maps.

## 2.1 Alexandrov space of curvature $\leq K$

In the 1950's, Alexandrov introduced spaces of curvature bounded from above in his papers [1], [2]. The terminology $CAT(K)$ spaces was then coined by Gromov in 1987. The initials are in honor of Cartan, Alexandrov and Toponogov. To make this thesis self-contained, we will recall some basic definitions here.

A metric $d$ of the metric space $(X, d)$ is called *intrinsic* if for every $P, Q \in X$

$$d(P, Q) = \inf_{\mathcal{L}} \{ \text{Length}(\mathcal{L}) \},$$

where the inf is taken over all rectifiable curves $\mathcal{L}$ joining the points $P$ and $Q$, and $\mathrm{Length}(\mathcal{L})$ is the length of $\mathcal{L}$ measured in the metric $d$.

A curve $\mathcal{L}$ in a metric space $(X, d)$ joining a pair of points $A, B$ is called a *shortest arc* if its length is equal to $d(A, B)$.

A metric space is said to be *geodesically connected* or *a length space* if each pair of points in it can be joined by a shortest arc.

An $\mathfrak{R}_K$ domain (also known as a $CAT(K)$ space) of the metric space $(X, d)$ is a metric space with the following properties:

(i) $\mathfrak{R}_K$ is a geodesically connected metric space.

(ii) If $K > 0$, then the perimeter of each triangle in $\mathfrak{R}_K$ is less than $2\pi/\sqrt{K}$.

(iii) $K$-convexity: Each triangle $\triangle ABC \subset \mathfrak{R}_K$ and its comparison triangle $\triangle\overline{ABC}$ in the $K$-plane[†] have the $CAT(K)$-inequality: $d(B, D) \leq d_{K\text{-plane}}(\overline{B}, \overline{D})$, where $\overline{D}$ is the point in the arc $\overline{AC}$ such that $d(A, D) = d_{K\text{-plane}}(\overline{A}, \overline{D})$.

A metric space $(X, d)$ is an Alexandrov space of curvature bounded from above by $K$ if each point of $X$ is contained in some neighborhood that is an $\mathfrak{R}_K$ domain.

To see some examples of such Alexandrov spaces of curvature bounded from above and relate the curvature in the sense of Alexandrov and the sectional curvature of a Riemannian manifold, we have the following theorem due to Alexandrov and Cartan.

**Theorem 2.1.1.** *([1], [9]) A smooth Riemannian manifold $M$ is an Alexandrov space of curvature bounded from above by $K$ if and only if the sectional curvature of $M$ is $\leq K$.*

[†] The $K$-plane is the 2-dimensional model space of constant Gaussian curvature $K$, i.e., $\mathbf{R}^2$ if $K = 0$, the standard Euclidean 2-hemisphere $\mathbf{S}_K$ of radius $1/\sqrt{K}$ if $K > 0$ and the hyperbolic plane of curvature $K$ if $K < 0$. The comparison triangle means $\triangle\overline{ABC}$ has the same length of corresponding side as $\triangle ABC$, measuring in respective metric. See [6].

## 2.2 Energy of maps into metric spaces

Let $(\Sigma, g)$ be a $n$-dimensional compact Riemannian domain, $d_\Sigma$ be the distance function on $\Sigma$ induced by $g$ and $(X, d)$ be any complete metric space. A Borel measurable map $f : \Sigma \to X$ is said to be in $L^2(\Sigma, X)$ if

$$\int_\Sigma d^2(f(x), Q) d\mu < \infty$$

for some $Q \in X$. By the triangle inequality, this definition is independent of the choice of $Q$.

For $\epsilon > 0$, let $\Sigma_\epsilon = \{\eta \in \Sigma : d_\Sigma(\eta, \partial\Sigma) > \epsilon\}$ and $S_\epsilon(\eta) = \{\xi \in \Sigma : d_\Sigma(\eta, \xi) = \epsilon\}$ and $d\sigma_{\eta,\epsilon}(\xi)$ be the $(n-1)$-dimensional surface measure on $S_\epsilon(\eta)$ and $w_n$ be the area form of the unit sphere. For $u \in L^2(\Sigma, X)$, construct an $\epsilon$-approximate energy function $e_\epsilon : \Sigma \to \mathbf{R}$ by setting

$$e_\epsilon(\eta) = \begin{cases} \dfrac{1}{w_n} \displaystyle\int_{S_\epsilon(\eta)} \frac{d^2(u(\eta), u(\xi))}{\epsilon^2} \frac{d\sigma_{\eta,\epsilon}(\xi)}{\epsilon^{n-1}} & \text{for } \eta \in \Sigma_\epsilon, \\ 0 & \text{for } \eta \in \Sigma - \Sigma_\epsilon. \end{cases} \tag{2.1}$$

Define a linear functional $E_\epsilon : C_c(\Sigma) \to \mathbf{R}$ on the set of continuous functions with compact support in $\Sigma$ by setting

$$E_\epsilon(f) = \int_\Sigma f e_\epsilon d\mu.$$

**Definition 2.2.1.** ([25], 1.3ii) The map $u \in L^2(\Sigma, X)$ is said to have finite energy or equivalently $u \in W^{1,2}(\Sigma, X)$ if

$$E^u = \sup_{0 \le f \le 1, f \in C_c(\Sigma)} \limsup_{\epsilon \to 0^+} E_\epsilon(f) < \infty. \tag{2.2}$$

The quantity $E^u$ is defined to be the energy of the map $u$. It is shown in [25] that if $u$ has finite energy, then in fact there exists a function $e(\eta) \in L^1(\Sigma)$ so that $e_\epsilon(\eta) d\mu_g(\eta) \to e(\eta) d\mu_g(\eta)$ as measures. The function $e(\eta)$ is called the energy density of $u$ and we write it as $|\nabla u|^2$ ( $|u'|^2$ when $n = 1$) as an analogue of the Riemannian case. In particular

$$\text{Energy}(u) = E^u = \int_\Sigma |\nabla u|^2 d\mu \ \left(= \int_\Sigma |u'|^2 d\mu \quad \text{when } n = 1\right).$$

*Remark* 2.2.2. By definition, if $u, v \in W^{1,2}(\Sigma, X)$ then the pointwise distance function $d\,(u, v) \in W^{1,2}(\Sigma, \mathbf{R})$ (see also theorem 1.12.2 of [25]). For closed curves $\alpha, \beta \in W^{1,2}(\mathbf{S}^1, X)$, the fact that $d\,(\alpha, \beta) \in W^{1,2}(\mathbf{S}^1, \mathbf{R})$ allows us the define the distance between $\alpha$ and $\beta$ in $W^{1,2}(\mathbf{S}^1, X)$. Note that the Sobolev embedding $C^0(\mathbf{S}^1, \mathbf{R}) \hookrightarrow W^{1,2}(\mathbf{S}^1, \mathbf{R})$ implies two $W^{1,2}$ curves that are $W^{1,2}$ close are also $C^0$ close (cf. (4.11)).

## 2.3 Sweepout by curves and width

### 2.3.1 For closed Riemannian manifolds

In [14], Colding and Minicozzi introduced the crucial geometric concepts: sweepout (by curves) and width. We will next recall the definitions.

**Definition 2.3.1.** A continuous map $\sigma : \mathbf{S}^1 \times [-1, 1] \longrightarrow \mathcal{N}$ is called a *sweepout* in $\mathcal{N}$, if $\sigma(\cdot, s) \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$ for each $s \in [-1, 1]$, the map $s \longrightarrow \sigma(\cdot, s)$ is continuous from $[-1, 1]$ to $W^{1,2}(\mathbf{S}^1, \mathcal{N})$ and $\sigma$ maps $\mathbf{S}^1 \times \{-1\}\times$ and $\mathbf{S}^1 \times \{1\}$ to points.

Denote by $\Omega$ the set of sweepouts by curves on $\mathcal{N}$. The homotopy class $\Omega_{\hat{\sigma}}$ of $\hat{\sigma} \in \Omega$ is the path connected component of $\hat{\sigma}$ in $\Omega$, where the topology is induced from $C^0([-1, 1], W^{1,2}(\mathbf{S}^1, \mathcal{N}))$.

**Definition 2.3.2.** The *width* $W = W(\Omega_{\hat{\sigma}})$ of the homotopy class $\Omega_{\hat{\sigma}}$ is defined by taking the infimum of the maximum of the energy of each slice. That is, set

$$W = \inf_{\sigma \in \Omega_{\hat{\sigma}}} \max_{s \in [-1,1]} E(\sigma(\cdot, s)), \tag{2.3}$$

where $E(\sigma(\cdot, s))$ is the usual Dirichlet energy for the closed curve $\sigma(\cdot, s) : \mathbf{S}^1 \to \mathcal{N}$ defined by

$$\text{Energy}(\sigma(\cdot, s)) \; = \; E(\sigma(\cdot, s)) \; = \; \frac{1}{2} \int_{\mathbf{S}^1} |\partial_\theta \gamma(\theta, s)|^2 d\theta \tag{2.4}$$

We shall see that the sweepout $\sigma$ induces a map $\tilde{\sigma}$ from the sphere $\mathbf{S}^2$ to $\mathcal{N}$ and the width is always non-negative and is positive if $\hat{\sigma}$ is in a non-trivial homotopy class[‡]. To see this, assume that $W(\Omega_{\hat{\sigma}}) = 0$ and $\sigma \in \Omega_{\hat{\sigma}}$ is such that the energy of each slice of $\sigma$ is sufficiently small. Then each slice, $\sigma(\cdot, s)$, is contained in a strictly convex neighborhood of $\sigma(\theta_0, s)$ and note that $s \longrightarrow \sigma(\theta_0, s)$ is a continuous curve in $\mathcal{N}$. Hence a geodesic homotopy connects $\sigma$ to a path of point curves and thus $\sigma$ is homotopically trivial.

### 2.3.2 For Alexandrov spaces

Throughout the rest of this thesis, we will let $(X, d)$ be a closed Alexandrov space of curvature bounded from above by some $K > 0$. We will next extend the definitions of sweepout and width to a closed Alexandrov space of curvature bounded from above.

**Definition 2.3.3.** A continuous map $\sigma : \mathbf{S}^1 \times [-1, 1] \to X$ is called a *sweepout* in $X$, if for each $s$ the map $\sigma(\cdot, s)$ is in $W^{1,2}(\mathbf{S}^1, X)$, the map $s \to \sigma(\cdot, s)$ is continuous (in the induced topology as in Remark 2.2.2) from $[-1, 1]$ to $W^{1,2}(\mathbf{S}^1, X)$, and finally $\sigma$ maps $\mathbf{S}^1 \times \{-1\}$ and $\mathbf{S}^1 \times \{1\}$ to points.

[‡]A particularly interesting example is when $X$ is a topological 2-sphere with $\pi_1(X) = \{0\}$ and the map induced by a sweepout from $\mathbf{S}^2$ to $X$ has degree one. In this case, the width is positive and realized by a non-trivial closed geodesic with index 1, see footnote 2 of [14].

Let $\Omega$ be the set of sweepouts in $X$. Given a map $\hat{\sigma} \in \Omega$, the homotopy class $\Omega_{\hat{\sigma}}$ is defined to be the set of maps $\sigma \in \Omega$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega$.

**Definition 2.3.4.** The *width* $W = W(\hat{\sigma})$ associated to the homotopy class $\Omega_{\hat{\sigma}}$ is defined by

$$W = \inf_{\sigma \in \Omega_{\hat{\sigma}}} \max_{s \in [-1,1]} \text{Energy}\left(\sigma(\cdot, s)\right), \tag{2.5}$$

where the energy is the energy of maps into metric spaces given in section 2.2, namely,

$$\text{Energy}(\sigma(\cdot, s)) = \sup_{\substack{f \in C_c(\mathbf{S}^1) \\ 0 \le f \le 1}} \limsup_{\epsilon \to 0^+} \int_{\mathbf{S}^1} f \left( \frac{d^2\left(\sigma(\eta - \epsilon, s), \sigma(\eta, s)\right) + d^2\left(\sigma(\eta, s), \sigma(\eta + \epsilon, s)\right)}{2\epsilon^2} \right) d\eta\,. \tag{2.6}$$

We write $\text{Energy}(\sigma(\cdot, s)) = E(\sigma(\cdot, s)) = \int_{\mathbf{S}^1} |\sigma'(x, s)|^2 dx$ (cf. (2.4): these two definitions differ only by a constant $\frac{1}{2}$, but this makes no essential difference). Again, we shall see that a sweepout in $X$ induces a map from the sphere $\mathbf{S}^2$ to $X$ and the width is always non-negative and is positive if $\hat{\sigma}$ is in a non-trivial homotopy class.

*Remark* 2.3.5. The $\epsilon$-approximate length function of $\sigma$ converges to a $L^1$ function, which coincides with the speed function of $\sigma$, as $\epsilon \to 0^+$, namely (see lemma 1.9.3 of [25]),

$$\lim_{\epsilon \to 0^+} \frac{d\left(\sigma(\eta - \epsilon), \sigma(\eta)\right) + d\left(\sigma(\eta), \sigma(\eta + \epsilon)\right)}{2\epsilon} = |\sigma'|(\eta) \quad \text{a.e. } \eta \in \mathbf{S}^1\,.$$

Throughout the rest of this thesis we will use $|\sigma'|$ to denote the speed function of a curve $\sigma$ in $X$.

## 2.4 Weakly harmonic maps

Let $(\mathcal{N}, g)$ be a general closed Riemannian manifold that can be isometrically embedded into $(\mathbf{R}^n, \langle,\rangle)$ as stated above, we define the Dirichlet energy of a sufficiently smooth map $u : B_1 \to \mathcal{N}$ from the unit 2-disc $B_1 \subset \mathbf{R}^2$ to be

$$E(u) \;=\; \frac{1}{2}\int_{B_1} |\nabla u|^2 dx_1 dx_2\,. \tag{2.7}$$

By definition, harmonic maps from $B_1$ to $\mathcal{N}$ are the (regular) stationary points of $E$. In terms of local coordinates $\{y^i\}$ on $\mathcal{N}$, they necessarily satisfy the Euler-Lagrange equations (which is a semilinear second-order elliptic system of partial differential equations):

$$-\Delta u^i \;=\; \Gamma^i_{kl}(u)\nabla u^k \cdot \nabla u^l\,, \tag{2.8}$$

where $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbf{R}^2$ and $\Gamma^i_{kl}$ are the Christoffel symbols of the metric $g$ on $\mathcal{N}$ (with $(g^{ij}) = (g_{ij})^{-1}$):

$$\Gamma^i_{kl} \;=\; \frac{1}{2} g^{ij}\left(g_{kj,l} + g_{lj,k} - g_{kl,j}\right)\,.$$

Here and in the sequel we adopt the *Einstein summation convention* where summation over repeated subscript-superscript pairs is understood.

In the following we will use the short-hand notation

$$-\Delta u \;=\; A(u)(\nabla u, \nabla u)\,, \tag{2.9}$$

where $A(u)$ is the second fundamental form of the fixed isometric embedding $\mathcal{N} \hookrightarrow \mathbf{R}^n$. The weak solutions of (2.9) in the class of maps $W^{1,2}(B_1, \mathcal{N})$ are critical points of the Dirichlet energy $E(u)$ in the distribution sense and they are also called *weakly harmonic* *maps*. That is, $u \in W^{1,2}(B_1, \mathbf{R}^n)$ takes values almost everywhere in $\mathcal{N}$ and solves (2.9) weakly. Moreover, one sees that equation (2.9) is equivalent to the following *intrinsic* harmonic map equation:

$$\Delta u \perp T_u\mathcal{N} \quad \text{or} \quad (\Delta u)^T = 0 \quad (\text{zero tension field})\,, \tag{2.10}$$

where $T_u\mathcal{N}$ is the tangent plane of $\mathcal{N}$ at the point $u$ and the superscript $T$ means the projection onto $T_u\mathcal{N}$.

# Chapter 3

# Closed geodesics in closed manifolds

## 3.1 Harmonic map heat flow from $\mathbf{S}^1$

Throughout this chapter, we use the subscripts $\theta$ and $t$ to denote taking partial derivatives of maps with respect to $\theta$ and $t$; $u$ satisfies the harmonic map heat flow equation, which is defined in (3.1). Given a closed curve $\gamma \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$, define the Dirichlet energy functional $E(\gamma) = \frac{1}{2}\int_{\mathbf{S}^1} |\gamma_\theta|^2 d\theta$. The harmonic map heat flow from $\mathbf{S}^1$ into $\mathcal{N}$ starting with the initial closed curve $u_0 \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$ is the negative $L^2$ gradient flow of the energy functional $E$:

$$
\left\{
\begin{aligned}
u_t &= u_{\theta\theta} - A(u)(u_\theta, u_\theta) \quad \text{on } (0,\infty) \times \mathbf{S}^1\,; \\
\lim_{t \longrightarrow 0^+} u(t,\cdot) &= u_0 \quad \text{in } W^{1,2}(\mathbf{S}^1, \mathcal{N})\,,
\end{aligned}
\right. \tag{3.1}
$$

where $A(u)$ is the second fundamental form of $\mathcal{N}$ in $\mathbf{R}^n$ at point $u$. We study the long time existence and uniqueness of the solution of (3.1), c.f. Struwe's fundamental paper [46] for relevant results of harmonic map heat flow from surfaces. See also Ottarsson's work in [35], where Theorem 3.1.1 was proved under the stronger assumption of $C^1$ initial data (and thus the $C^1$ continuity at $t = 0$)[§].

**Theorem 3.1.1.** *Given $u_0 \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$, there exists a unique solution $u(t,\theta) \in C^\infty((0,\infty) \times \mathbf{S}^1, \mathcal{N})$ of (3.1).*

The following is devoted to the proof of Theorem 3.1.1. First, by theorems 10B and 10A in [18] (or see theorems 6D and 6E in [35]), given any initial data $u_0 \in C^\infty(\mathbf{S}^1, \mathcal{N})$, there exists $T_0 > 0$ and a unique solution $u \in C^\infty([0, T_0) \times \mathbf{S}^1, \mathcal{N})$ of (3.1). We show that the solution $u$ can be extended smoothly beyond $T_0$. First, note that the energy is non-increasing under the harmonic map heat flow:

**Lemma 3.1.2.** *For $0 \le t_1 \le t_2 < T_0$,*

$$E(u(t_1, \cdot)) - E(u(t_2, \cdot)) \;=\; \int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt\,. \tag{3.2}$$

*Proof.* Multiply the first equation of (3.1) by $u_t$ and integrate over $[t_1, t_2] \times \mathbf{S}^1$,

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt &= \int_{t_1}^{t_2} \int_{\mathbf{S}^1} \langle u_{\theta\theta}, u_t \rangle d\theta dt = - \int_{t_1}^{t_2} \int_{\mathbf{S}^1} \langle u_\theta, u_{\theta t} \rangle d\theta dt \\ &= E(u(t_1, \cdot)) - E(u(t_2, \cdot))\,. \end{aligned}$$

This shows (3.2). ∎

We will next derive a global gradient bound of $u$.

**Lemma 3.1.3.** *Suppose that $u$ is a solution of (3.1) in $C^\infty((0,\infty) \times \mathbf{S}^1, \mathcal{N})$. Then*

$$(\partial_t - \partial_\theta^2) |u_\theta|^2 \;\le\; 0\,. \tag{3.3}$$

[§]In our setting, the $C^1$ continuity at $t = 0$ may not be true. For our purpose that the harmonic map heat flow preserves the homotopy class of sweepouts, we use a different argument to show the $W^{1,2}$ continuity at $t = 0$.

*Proof.*

$$\begin{aligned}
\partial_t |u_\theta|^2 &= 2\langle u_\theta, u_{\theta t}\rangle = 2\langle u_\theta, u_{\theta\theta\theta}\rangle - 2\langle u_\theta, (A(u)(u_\theta, u_\theta))_\theta\rangle \\
&= 2\langle u_\theta, u_{\theta\theta\theta}\rangle + 2\langle u_{\theta\theta}, A(u)(u_\theta, u_\theta)\rangle \\
&= \partial_\theta^2 |u_\theta|^2 - 2|u_{\theta\theta}|^2 + 2|A(u)(u_\theta, u_\theta)|^2 \leq \partial_\theta^2 |u_\theta|^2 .
\end{aligned}$$

□

Since $u \in C^\infty((0, T_0) \times \mathbf{S}^1, \mathcal{N})$, it follows from Lemma 3.1.2 and the local maximum principle (see Theorem 2.1 in [20] or Theorem 7.36 in [28]) that for any $\tau > 0$ and $(t, \theta) \in [\tau, T_0) \times \mathbf{S}^1$ we have

$$|u_\theta|^2(t, \theta) \leq C_0 \max\{1, \tau^{-1/2}\} E(u_0) , \tag{3.4}$$

where $C_0$ is a positive constant. Furthermore, by Proposition 7.18 in [28], $|u_{\theta\theta}|$ and $|u_t|$ are bounded on $[2\tau, T_0) \times \mathbf{S}^1$. And by induction, for any $\tau > 0$, the higher order derivatives of $u$ on $[2\tau, T_0) \times \mathbf{S}^1$ are bounded uniformly by constants depending only on $\mathcal{N}$, $E(u_0)$, $\tau$ and $T_0$. Hence $u$ can be extended smoothly to a solution of (3.1) beyond $T_0$. In other words, there exists a unique solution $u \in C^\infty([0, \infty) \times \mathbf{S}^1, \mathcal{N})$ of (3.1), if $u_0 \in C^\infty(\mathbf{S}^1, \mathcal{N})$.

Now given $u_0 \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$, we can find a sequence of $u_0^m \in C^\infty(\mathbf{S}^1, \mathcal{N})$ approaching $u_0$ in the $W^{1,2}$ topology. Let $u^m$ be the solution of the harmonic map heat flow with initial data $u_0^m$. Then by (3.4) and the discussion above, for any $\tau > 0$ and $T_0 > \tau$, $u^m$'s and all their derivatives are bounded uniformly (for all $m$). By the Arzela-Ascoli Theorem and a diagonalization argument, there exists a map $u \in C^\infty((0, \infty) \times \mathbf{S}^1, \mathcal{N})$ solving the harmonic map heat flow with $E(u(t, \cdot)) \leq E(u_0)$. And it follows from the next lemma that $t \longrightarrow u(t, \cdot)$ is a continuous map from $[0, \infty) \longrightarrow W^{1,2}(\mathbf{S}^1, \mathcal{N})$.

**Lemma 3.1.4.** *Given $\epsilon > 0$, there exists $\delta > 0$, depending on $\mathcal{N}$, $u_0$ and $\epsilon$, so that if*

$0 \le t_1 < t_2$ *and* $t_2 - t_1 < \delta$, *then*

$$\|u(t_2,\cdot) - u(t_1,\cdot)\|_{W^{1,2}(\mathbf{S}^1)} \le \epsilon\,. \tag{3.5}$$

*Proof.* First note that

$$\int_{\mathbf{S}^1} |u(t_2,\theta) - u(t_1,\theta)|^2 d\theta \le \int_{\mathbf{S}^1} \left| \int_{t_1}^{t_2} u_t dt \right|^2 d\theta \le (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt\,.$$

And by Lemma 3.1.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_{\mathbf{S}^1} |u_\theta(t_2,\theta) - u_\theta(t_1,\theta)|^2 d\theta \\
&= \int_{\mathbf{S}^1} |u_\theta(t_1,\theta)|^2 d\theta - \int_{\mathbf{S}^1} |u_\theta(t_2,\theta)|^2 d\theta - 2\int_{\mathbf{S}^1} \langle u_\theta(t_2,\theta), u_\theta(t_1,\theta) - u_\theta(t_2,\theta)\rangle d\theta \\
&= 2\int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt + 2\int_{\mathbf{S}^1} \langle u_{\theta\theta}(t_2,\theta), u(t_1,\theta) - u(t_2,\theta)\rangle d\theta \\
&\le 2\int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt + 2\left(\int_{\mathbf{S}^1} |u_{\theta\theta}(t_2,\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_{\mathbf{S}^1} |u(t_2,\theta) - u(t_1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \\
&\le 2\int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt + 2(t_2-t_1)^{\frac{1}{2}} \left(\int_{\mathbf{S}^1} |u_{\theta\theta}(t_2,\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt\right)^{\frac{1}{2}}.
\end{aligned}$$

If $C_0^2\,(t_2 - t_1) < 1$, then by Lemma 3.1.2, (3.1) and (3.4) we get

$$\int_{\mathbf{S}^1} |u_{\theta\theta}(t_2,\theta)|^2 d\theta \le \int_{\mathbf{S}^1} |u_t(t_2,\theta)|^2 d\theta + (t_2 - t_1)^{-1} \sup_{\mathcal{N}} |A|^2 \cdot E(u_0)^2\,. \tag{3.6}$$

We next derive the evolution equation for $|u_t|^2$.

$$\begin{aligned}
\partial_t |u_t|^2 = 2\langle u_t, u_{tt}\rangle &= 2\langle u_t, u_{\theta\theta t}\rangle - 2\langle u_t, (A(u)(u_\theta,u_\theta))_t\rangle \\
&= \partial_\theta^2 |u_t|^2 - 2|u_{t\theta}|^2 + 2\langle u_{tt}, A(u)(u_\theta,u_\theta)\rangle \\
&= \partial_\theta^2 |u_t|^2 - 2|u_{t\theta}|^2 + 2\langle A(u)(u_t,u_t), A(u)(u_\theta,u_\theta)\rangle.
\end{aligned}$$

Thus by (3.4), if $t > t_3 = t_1 + (t_2 - t_1)/2$ there holds

$$(\partial_t - \partial_\theta^2)|u_t|^2 - 4(t_2 - t_1)^{-1} \sup_{\mathcal{N}} |A|^2 \cdot E(u_0)^2 \cdot |u_t|^2 \leq 0\,. \tag{3.7}$$

Hence

$$\begin{aligned}\int_{\mathbf{S}^1} |u_t|^2(t_2,\theta)d\theta &\leq \inf_{t_3 \leq t \leq t_2} \int_{\mathbf{S}^1} |u_t|^2(t,\theta)d\theta + C(t_2 - t_1)^{-1} \int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt \\ &\leq (C+2)(t_2 - t_1)^{-1} \int_{t_1}^{t_2} \int_{\mathbf{S}^1} |u_t|^2 d\theta dt\,,\end{aligned} \tag{3.8}$$

where $C$ depends only on $\mathcal{N}$ and $E(u_0)$. Combining (3.8), (3.6) and Lemma 3.1.2, there exists $\delta > 0$ so that (3.5) holds. $\square$

Given (3.4) and Lemma 3.1.4, one sees that there exists $R_0 > 0$, depending only on $\mathcal{N}$ and $u_0$, so that for $t \geq 0$, $2\pi \cdot \sup_{\mathcal{N}} |A|^2 \cdot \int_{\{t\}\times I_{R_0}} |u_\theta|^2 d\theta < 1/64$ , where $I_{R_0}$ is any segment on the unit circle of length $2R_0$. To prove the uniqueness of the solution of (3.1), we will need the following estimate, c.f. lemma 6.7 on page 225 of [47].

**Lemma 3.1.5.** *Suppose that $u$ is a solution of (3.1) in $C^\infty((0,\infty)\times\mathbf{S}^1, \mathcal{N})$. Then*

$$\int_0^T \int_{\mathbf{S}^1} |u_{\theta\theta}|^2 d\theta dt \leq \frac{T}{4R_0^2} E(u_0) + 2\left[E(u_0) - E(u(T,\cdot))\right]\,. \tag{3.9}$$

*Proof.* Fix $(t_1, \theta_1) \in (0,\infty) \times \mathbf{S}^1$. Let $I_R(\theta_1)$ denote the arc segment on the unit circle centered at $\theta_1$ with length $2R$. And let $\phi$ be identically one on $I_{R_0/2}(\theta_1)$ and it cuts off linearly to zero on $I_{R_0}(\theta_1) \setminus I_{R_0/2}(\theta_1)$. Then

$$
\begin{aligned}
&|u_\theta|^4(t_1,\theta_1) = \phi^2 |u_\theta|^4(t_1,\theta_1) \\
&\leq \left( \int_{\mathbf{S}^1} 2|\phi||u_\theta||u_{\theta\theta}|(t_1,\theta)d\theta + \int_{\mathbf{S}^1} |\phi_\theta||u_\theta|^2(t_1,\theta)d\theta \right)^2 \\
&\leq 8 \left( \int_{\mathbf{S}^1} |\phi||u_\theta||u_{\theta\theta}|(t_1,\theta)d\theta \right)^2 + 2 \left( \int_{\mathbf{S}^1} |\phi_\theta||u_\theta|^2(t_1,\theta)d\theta \right)^2 \\
&\leq 8 \int_{I_{R_0}(\theta_1)} |u_\theta|^2(t_1,\theta)d\theta \cdot \int_{\mathbf{S}^1} |u_{\theta\theta}|^2(t_1,\theta)d\theta + \frac{8}{R_0^2} \left( \int_{I_{R_0}(\theta_1)} |u_\theta|^2(t_1,\theta)d\theta \right)^2,
\end{aligned}
$$

where the last inequality follows from Hölder's inequality and that $\phi$ is supported in $I_{R_0}(\theta_1)$ with $|\phi_\theta| \leq 2/R_0$. Hence, for $0 < t_0 \leq T$ we have

$$
\int_{t_0}^T \int_{\mathbf{S}^1} |u_\theta|^4 d\theta dt \leq 16\pi \cdot \epsilon(R_0) \cdot \left( \int_{t_0}^T \int_{\mathbf{S}^1} |u_{\theta\theta}|^2 d\theta dt + R_0^{-2} \int_{t_0}^T \int_{\mathbf{S}^1} |u_\theta|^2 d\theta dt \right),
$$

where

$$
\epsilon(R_0) = \sup_{t\geq 0,\ \theta_1 \in \mathbf{S}^1} \int_{\{t\}\times I_{R_0}(\theta_1)} |u_\theta|^2 d\theta. \tag{3.10}
$$

Then it follows from (3.1) and Lemma 3.1.2 that

$$
\begin{aligned}
\int_{t_0}^T \int_{\mathbf{S}^1} |u_{\theta\theta}|^2 d\theta dt &\leq \int_{t_0}^T \int_{\mathbf{S}^1} |u_t|^2 d\theta dt + \sup_{\mathcal{N}} |A|^2 \cdot \int_{t_0}^T \int_{\mathbf{S}^1} |u_\theta|^4 d\theta dt \\
&\leq [E(u_0) - E(u(T,\cdot))] + \frac{1}{2} \int_{t_0}^T \int_{\mathbf{S}^1} |u_{\theta\theta}|^2 d\theta dt + \frac{T}{8R_0^2} E(u_0).
\end{aligned}
$$

Absorbing the righthand side into the lefthand side and noting that the estimate is independent of $t_0$, we see that (3.9) follows immediately. □

Now we are ready to show the uniqueness of the solution of the harmonic map heat flow (3.1).

**Lemma 3.1.6.** *Given $u_0 \in W^{1,2}(\mathbf{S}^1, \mathcal{N})$, let $u$ and $\tilde{u}$ be solutions of (3.1) in $C^\infty((0,\infty)\times \mathbf{S}^1, \mathcal{N})$. Then $u = \tilde{u}$.*

*Proof.* Define $v = u - \tilde{u}$, we have

$$v_t = v_{\theta\theta} - A_u(u_\theta, u_\theta) + A_{\tilde{u}}(\tilde{u}_\theta, \tilde{u}_\theta)\,. \tag{3.11}$$

Multiplying both sides of (3.11) by $v$ and integrate over $[0, t_0] \times \mathbf{S}^1$, we get

$$\begin{aligned}
&\int_{\{t_0\}\times\mathbf{S}^1} |v|^2 d\theta + 2\int_0^{t_0}\int_{\mathbf{S}^1} |v_\theta|^2 d\theta dt \\
&= 2\int_0^{t_0}\int_{\mathbf{S}^1} \langle A_{\tilde{u}}(\tilde{u}_\theta, \tilde{u}_\theta) - A_u(u_\theta, u_\theta), v\rangle d\theta dt \\
&\leq C(\mathcal{N})\int_0^{t_0}\int_{\mathbf{S}^1} |v|^2(|\tilde{u}_\theta|^2 + |u_\theta|^2) d\theta dt + C(\mathcal{N})\int_0^{t_0}\int_{\mathbf{S}^1} |v||v_\theta|(|\tilde{u}_\theta| + |u_\theta|) d\theta dt \\
&\leq C(\mathcal{N})\int_0^{t_0}\int_{\mathbf{S}^1} |v|^2(|\tilde{u}_\theta|^2 + |u_\theta|^2) d\theta dt + \int_0^{t_0}\int_{\mathbf{S}^1} |v_\theta|^2 d\theta dt \\
&\leq C(\mathcal{N})\int_0^{t_0}\left(\|u_\theta\|^2_{C^0(\{t\}\times\mathbf{S}^1)} + \|\tilde{u}_\theta\|^2_{C^0(\{t\}\times\mathbf{S}^1)}\right)\int_{\mathbf{S}^1} |v|^2 d\theta dt + \int_0^{t_0}\int_{\mathbf{S}^1} |v_\theta|^2 d\theta dt.
\end{aligned}$$

By Lemmas 3.1.2, 3.1.5 and the Sobolev embedding theorem, there exists $\delta > 0$ depending on $\mathcal{N}$ and $u_0$, so that if $t_0 \leq \delta$ then

$$\begin{aligned}
&C(\mathcal{N})\int_0^{t_0} \|\tilde{u}_\theta\|^2_{C^0(\{t\}\times\mathbf{S}^1)} + \|u_\theta\|^2_{C^0(\{t\}\times\mathbf{S}^1)}\, dt \\
&\leq C(\mathcal{N})\int_0^{t_0}\int_{\mathbf{S}^1} |\tilde{u}_{\theta\theta}|^2 + |u_{\theta\theta}|^2 d\theta dt \\
&\leq C(\mathcal{N})\left[E(u_0)\frac{t_0}{2R_0^2} + 4E(u_0) - 2E(u(t_0,\cdot)) - 2E(\tilde{u}(t_0,\cdot))\right] \leq \frac{1}{2}\,.
\end{aligned}$$

Absorbing the righthand side into the lefthand side then gives

$$\sup_{0\leq t\leq\delta}\int_{\{t\}\times\mathbf{S}^1} |v|^2 d\theta + 2\int_0^\delta\int_{\mathbf{S}^1} |v_\theta|^2 d\theta dt \leq 0\,. \tag{3.12}$$

Since $[0, T]$ is compact, the lemma follows by iteration. $\square$

## 3.2 Good sweepouts via harmonic map heat flow

Let $W$ be the width of the closed manifold $\mathcal{N}$ which is defined in Definition 2.3.2. For fixed $\alpha \in (0,1)$, let $\gamma : \mathbf{S}^1 \longrightarrow \mathcal{N}$ be a smooth closed curve and $G$ be the set of closed geodesics in $\mathcal{N}$. Define $\mathrm{dist}_\alpha(\gamma, G) = \inf_{\tilde{\gamma}\in G} \|\gamma - \tilde{\gamma}\|_{C^{1,\alpha}(\mathbf{S}^1)}$. Our aim in this section is to prove the following theorem.

**Theorem 3.2.1.** *Given $0 < \alpha < 1$, $t_0 > 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $j > 1/\delta$ and $s \in [-1,1]$ satisfies $E(\Phi^j(t_0,\cdot,s)) \geq W - \delta$, then $dist_\alpha(\Phi^j(t_0,\cdot,s), G) < \epsilon$.*

*Remark* 3.2.2. Such $s$ exists since $W \leq \max_{s\in[-1,1]} E(\Phi^j(t_0,\cdot,s)) \leq W + 1/j$.

We next prove a useful proposition for the solution of (3.1), which is also the key to the proof of Theorem 3.2.1.

**Proposition 3.2.3.** *Given $0 < \alpha < 1$, $W_0 \geq 0$, $t_0 > 0$ and $\epsilon > 0$, there exists $\delta_0 > 0$ so that if $W_0 - \delta_0 \leq E(u(t_0,\cdot)) \leq E(u_0) \leq W_0 + \delta_0$, then $dist_\alpha(u(t_0,\cdot), G) < \epsilon$.*

*Proof.* Suppose this fails, then there exist $0 < \alpha < 1$, $W_0 \geq 0$, $t_0 > 0$, $\epsilon > 0$ and a sequence of solutions $u^j$ of the harmonic map heat flow satisfying $W_0 - 1/j \leq E(u^j(t_0,\cdot)) \leq E(u_0^j) \leq W_0 + 1/j$ and $\mathrm{dist}_\alpha(u^j(t_0,\cdot), G) \geq \epsilon$ for all $j$. It follows from the evolution equation of $|u_t^j|^2$ (see (3.7)), (3.4), Lemma 3.1.2 and the local maximum principle that

$$\sup_{\theta\in\mathbf{S}^1} |u_t^j|^2(t_0,\theta) \,\leq\, C\left[E(u^j(t_0/2,\cdot)) - E(u^j(t_0,\cdot))\right], \tag{3.13}$$

where $C$ depends on $\mathcal{N}$, $t_0$ and $W_0$. This shows $\sup_{\theta\in\mathbf{S}^1} |u_t^j|(t_0,\theta) \to 0$ as $j \to \infty$. On the other hand, the uniform energy bound and (3.4) imply that $\|u^j(t_0,\cdot)\|_{C^2(\mathbf{S}^1)}$ is uniformly bounded by constants depending only on $\mathcal{N}$, $t_0$ and $W_0$. Therefore by the Arzela-Ascoli's Theorem, there exists a subsequence (relabeled) of $u^j(t_0,\cdot)$ converging to $u^\infty$ in $C^{1,\alpha}(\mathbf{S}^1, \mathcal{N})$ and $u^\infty$ is a closed geodesic in $\mathcal{N}$. This is a contradiction. □

*Proof.* (of Theorem 3.2.1) Let $\sigma$ be a sweepout on $\mathcal{N}$ and $\sigma^j$ be a minimizing sequence of sweepouts in $\Omega_\sigma$, that is

$$W \leq \max_{s\in[-1,1]} E(\sigma^j(\cdot,s)) \leq W + 1/j\,. \tag{3.14}$$

Applying the harmonic map heat flow to each slice of $\sigma^j$, we get a map $\Phi^j : [0,\infty)\times \mathbf{S}^1 \times [-1,1] \longrightarrow \mathcal{N}$ and for each $s \in [-1,1]$ fixed, $\Phi^j(t,\theta,s)$ solves (3.1) with $\Phi^j(0,\theta,s) = \sigma^j(\theta,s)$. It follows from the proof of the long time existence and uniqueness of the solution of (3.1) that for any $t_0 \geq 0$, the map $s \longrightarrow \Phi(t_0,\cdot,s)$ is continuous from $[-1,1]$ to $W^{1,2}(\mathbf{S}^1,\mathcal{N})$ and therefore $\Phi^j(\cdot,t_0,\cdot)$ is still a sweepout on $\mathcal{N}$. Since $[-1,1]$ is compact, for any $\epsilon > 0$, there exists $\delta > 0$ so that if $0 \leq t_1 < t_2 \leq t_0$ and $t_2 - t_1 < \delta$, then $\int_{t_1}^{t_2}\int_{\mathbf{S}^1} |\Phi_t^j(t,\theta,s)|^2 d\theta dt < \epsilon$ for any $s \in [-1,1]$. Hence by Lemma 3.1.4, for any $t_0 > 0$, $\Phi^j(\cdot,t_0,\cdot)$ is homotopic to $\sigma^j$. Then it follows from Proposition 3.2.3 that the $\Phi^j(\cdot,t_0,\cdot)$'s are good sweepouts on $\mathcal{N}$. $\square$

Since $G$ is closed in the $W^{1,2}(\mathbf{S}^1,\mathcal{N})$ topology, we have

**Corollary 3.2.4.** *If $\mathcal{N}$ is a closed Riemannian manifold and $\pi_2(\mathcal{N}) \neq \{0\}$, then there exists at least one non-trivial closed geodesic in $\mathcal{N}$.*

## 3.3 Parameter spaces

Instead of using the interval $[-1,1]$ as parameter space for the circles in the definition of sweepout (see Definition 2.3.1) and assuming that the curves start and end in point curves, one could have use any compact set $\mathcal{P}$ and require that the curves are constant on $\partial\mathcal{P}$ (or that $\partial\mathcal{P} = \emptyset$). Then we let $\Omega^{\mathcal{P}}$ be the set of continuous maps $\sigma : \mathbf{S}^1 \times \mathcal{P} \to \mathcal{N}$ so that for each $s \in \mathcal{P}$ the curve $\sigma(\cdot,s)$ is in $W^{1,2}(\mathbf{S}^1,\mathcal{N})$, the map $s \to \sigma(\cdot,s)$ is continuous from $\mathcal{P}$ to $W^{1,2}(\mathbf{S}^1,\mathcal{N})$ and finally $\sigma$ maps $\partial\mathcal{P}$ to point curves. Given a map $\hat{\sigma} \in \Omega^{\mathcal{P}}$, the homotopy class $\Omega^{\mathcal{P}}_{\hat{\sigma}} \subset \Omega^{\mathcal{P}}$ is defined to be the set of maps $\sigma \in \Omega^{\mathcal{P}}$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega^{\mathcal{P}}$. Finally the *width* $W = W(\hat{\sigma})$ is

$$W = \inf_{\sigma \in \Omega^{\mathcal{P}}_{\hat{\sigma}}} \max_{s \in \mathcal{P}} E(\sigma(\cdot, s)) . \tag{3.15}$$

Theorem 3.2.1 holds for these general parameter spaces and the proof is virtually the same as when $\mathcal{P} = [-1, 1]$.

*Proof.* (of Theorem 1.0.1) We will divide our proof into two cases. In the case of the fundamental group $\pi_1(\mathcal{N}) \neq 0$, we can choose a non-contractible closed curve $\sigma_0 : \mathbf{S}^1 \to \mathcal{N}$. Then by the definition of width we see that $W > 0$. It follows immediately from Theorem 3.2.1 that there exists at lease one non-trivial closed geodesic in $\mathcal{N}$ if we apply the harmonic map heat flow to the minimizing sequence of sweepouts as shown in the proof of Theorem 3.2.1.

In the case of $\pi_1(\mathcal{N}) = 0$, i.e., $\mathcal{N}$ is simply-connected (or 1-connected), then it is well known that $H_1(\mathcal{N}) \cong \pi_1(\mathcal{N})/[\pi_1(\mathcal{N}), \pi_1(\mathcal{N})]$ (see e.g. [22, Theorem 2A.1], page 166) and thus $H_1(\mathcal{N}) = 0$. Then by the assumption of the theorem, there exists the first nonzero $k_1$-th homology group $H_{k_1}(\mathcal{N}) \neq 0$ for some integer $k_1$ with $2 \leq k_1 \leq k_0$. Therefore by the Hurewicz theorem which states that the first nonzero homotopy and homology groups of a simply-connected space occur in the same dimension and are isomorphic (see e.g. [22, Theorem 4.32], page 366), we have

$$\pi_{k_1}(\mathcal{N}) \cong H_{k_1}(\mathcal{N}) \neq 0 .$$

Thus there is a non-contractible map

$$\omega_0 : \mathbf{S}^{k_1} \to \mathcal{N}$$

from the $k_1$-sphere $\mathbf{S}^{k_1}$ to $\mathcal{N}$ for $k_1 \geq 2$. Note that $\mathbf{S}^{k_1}$ is equivalent to $\mathbf{S}^1 \times \bar{B}^{k_1-1}/\sim$, where $\sim$ is the equivalence relation $(\theta_1, y) \sim (\theta_2, y)$ where $\theta_1, \theta_2 \in \mathbf{S}^1$ and $y \in \partial\bar{B}^{k_1-1}$. Here $\bar{B}^{k_1-1}$ is the closed unit ball in $\mathbf{R}^{k_1-1}$. We use this decomposition of $\mathbf{S}^{k_1}$ to define the width of $\mathcal{N}$.

Take $\mathcal{P} = \bar{B}^{k_1-1}$ as the parameter space as in subsection 3.3 and define the width $W$ as in (4.14). We see directly from the fact that $\omega_0$ is non-contractible that $W > 0$. Again, it follows immediately from Theorem 3.2.1 that there exists at lease one non-trivial closed geodesic in $\mathcal{N}$. $\square$

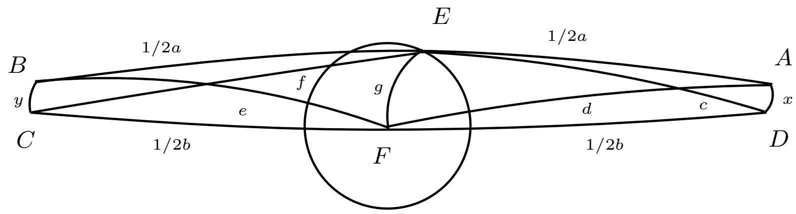
# Chapter 4

# Closed geodesics in closed Alexandrov spaces of curvature bounded from above

## 4.1 Local energy convexity in $CAT(K)$ spaces

This section is devoted to the proof of Theorem 1.0.2. Equation (2.2iv) of [25] already gave the case of $K = 0$ (with any $\rho > 0$). Our idea then follows from [25] to prove the case of $K > 0$. We first provide a local distance convexity in the standard Euclidean 2-hemisphere $\mathbf{S}_K$ and then apply Reshetnyak's majorization theorem to get the local energy convexity for $W^{1,2}$ maps into a $CAT(K)$ space with $K \geq 0$. In [3], Berg and Nikolaev defined the so-called $K$-quadrilateral cosine $cosq_K$ in an Alexandrov space of curvature $\leq K$ which has the property that $|cosq_K| \leq 1$. As we shall see in the following, this quantity is much related to the local distance convexity in $\mathbf{S}_K$.

**Lemma 4.1.1.** *([3]) Consider a quadruple $\mathcal{Q} = \{A, B, C, D\}$ of order points (see Figure 4.1), $A \neq B$ and $C \neq D$, in $\mathbf{S}_K$ ($\mathbf{R}^2$ if $K = 0$). Let $k = \sqrt{K}$ and $d_{\mathbf{S}_K}$ be the dis-*




$AE = BE = 1/2a, CF = DF = 1/2b, AD = x, BC = y, EF = g, ED = c, AF = d, BF = e, EC = f$

Figure 4.1: Quadruple $\mathcal{Q} = \{A, B, C, D\}$

*tance function in* $\mathbf{S}_K$. *Let* $d_{\mathbf{S}_K}(A, B) = a$, $d_{\mathbf{S}_K}(C, D) = b$, $d_{\mathbf{S}_K}(A, D) = x$, $d_{\mathbf{S}_K}(B, C) = y$, $d_{\mathbf{S}_K}(A, C) = h$ *and* $d_{\mathbf{S}_K}(B, D) = i$. *Then the limit of the* $K$*-quadrilateral cosine equals to the* $0$*-quadrilateral cosine as* $K \to 0$, *i.e.*,

$$\begin{aligned}
&\lim_{K \to 0} cosq_K(\overrightarrow{DA}, \overrightarrow{CB}) \\
&= \lim_{K \to 0} \frac{\cos ka + \cos ky \cos kx \cos kb + \cos ka \cos kb - \cos ky \cos kh - \cos kx \cos ki - \cos kh \cos ki}{(1 + \cos kb) \sin kx \sin ky} \\
&= cosq_0(\overrightarrow{DA}, \overrightarrow{CB}) = \frac{a^2 + b^2 - h^2 - i^2}{2xy}.
\end{aligned}$$

Based on Lemma 4.1.1, the following lemma follows directly from an elementary computation (see Appendix C for the detailed computation).

**Lemma 4.1.2.** *For any* $x \in \mathbf{S}_K$, *there exists* $\tau = \tau(K) > 0$ *such that if* $\{A, D, C, B\} \subset B_\tau(x) \subset \mathbf{S}_K$ *is an ordered sequence and* $E, F$ *are the mid-points of the shortest arcs* $AB$ *and* $CD$ *respectively, we have the following distance convexity:*

$$\frac{1}{4}\left(d_{\mathbf{S}_K}(A, B) - d_{\mathbf{S}_K}(C, D)\right)^2 \leq d^2_{\mathbf{S}_K}(A, D) + d^2_{\mathbf{S}_K}(B, C) - 2d^2_{\mathbf{S}_K}(E, F).$$

We will next recall Reshetnyak's majorization theorem in 1968 for an Alexandrov space of curvature bounded from above, which is a far reaching generalization of the $K$-convexity that was established by Alexandrov.

**Theorem 4.1.3.** *([39]) Let* $(X, d)$ *be an Alexandrov space of curvature* $\leq K$. *In an* $\mathfrak{R}_K$

*domain of $X$, for every rectifiable closed curve $\mathcal{L}$ with length less than $2\pi/\sqrt{K}$ if $K > 0$, there is a convex domain $\mathcal{V}$ in the $K$-plane and a map $\varphi : \mathcal{V} \to \mathfrak{R}_K$ such that $\varphi(\partial\mathcal{V}) = \mathcal{L}$, the lengths of the corresponding arcs coincide, and $d_{K\text{-}plane}(\eta, \xi) \geq d(\varphi(\eta), \varphi(\xi))$, for $\eta, \xi \in \mathcal{V}$.*

*Remark* 4.1.4. In particular, for an ordered sequence of points $\{A, D, C, B\}$ in an Alexandrov space of curvature bounded from above by $K > 0$, let $0 \leq \lambda, \nu \leq 1$ be given. Define $A_\lambda$ to be the point which is the fraction $\lambda$ of the way from $A$ to $B$ (on the geodesic $\gamma_{A,B}$). Let $D_\nu$ be the point which is the fraction $\nu$ of the way from $D$ to $C$ (along the opposite geodesic $\gamma_{D,C}$). By Theorem 4.1.3 (see also [39, lemma 1]), there exists an ordered sequence of points $\{\overline{A}, \overline{D}, \overline{C}, \overline{B}\} \subset \mathbf{S}_K$ which are the consecutive vertices of a quadrilateral. We can construct the corresponding points in $\mathbf{S}_K$ :

$$\overline{A}_\lambda = (1-\lambda)\overline{A} + \lambda\overline{B}, \quad \overline{D}_\nu = (1-\nu)\overline{D} + \nu\overline{C}.$$

Then by Theorem 4.1.3

$$\begin{aligned} &d(A, B) = d_{\mathbf{S}_K}(\overline{A}, \overline{B}), \quad d(C, D) = d_{\mathbf{S}_K}(\overline{C}, \overline{D}), \\ &d(A, D) = d_{\mathbf{S}_K}(\overline{A}, \overline{D}), \quad d(B, C) = d_{\mathbf{S}_K}(\overline{B}, \overline{C}), \\ &d(A_\lambda, D_\nu) \leq d_{\mathbf{S}_K}\left(\overline{A}_\lambda, \overline{D}_\nu\right). \end{aligned}$$

We call $\{\overline{A}, \overline{D}, \overline{C}, \overline{B}\}$ the subembedding of $\{A, D, C, B\}$.

**Lemma 4.1.5.** *Let $(X, d)$ be an Alexandrov space of curvature $\leq K$. In an $\mathfrak{R}_K$ domain of $x \in X$, there exists $\rho = \rho(x, K) > 0$ such that if $\{A, D, C, B\} \subset B_\rho(x) \subset \mathfrak{R}_K$ is an ordered sequence and $E, F$ are the mid-points of the shortest arcs $AB$ and $CD$ respectively, we have*

$$\frac{1}{4}\left(d(A, B) - d(C, D)\right)^2 \leq d^2(A, D) + d^2(B, C) - 2d^2(E, F). \tag{4.1}$$

*Proof.* Equation (2.2iii) of [25] gives the case of $K = 0$ (with any $\rho > 0$). For $K > 0$, let $\varrho = \min\{d(x,y) \,|\, y \in \partial\mathfrak{R}_K\}$ and $\rho = \min\{\tau/4, \varrho\}$ where $\tau$ is from Lemma 4.1.2. Let $\{\overline{A}, \overline{D}, \overline{C}, \overline{B}\} \subset \mathbf{S}_K$ be a subembedding of $\{A, D, C, B\}$. We see first of all that $\overline{A}, \overline{D}, \overline{C}$ and $\overline{B}$ have to be in a geodesic ball of radius at most $4\rho \le \tau$ in $\mathbf{S}_K$ and thus satisfy the condition of Theorem 4.1.3. Then by Theorem 4.1.3 and Remark 4.1.4, letting $\lambda = \nu = \frac{1}{2}$, we obtain

$$
\begin{aligned}
\frac{1}{4}(d(A,B) - d(C,D))^2 &= \frac{1}{4}(d_{\mathbf{S}_K}(\overline{A},\overline{B}) - d_{\mathbf{S}_K}(\overline{C},\overline{D}))^2 \\
&\le d^2_{\mathbf{S}_K}(\overline{A},\overline{D}) + d^2_{\mathbf{S}_K}(\overline{B},\overline{C}) - 2d^2_{\mathbf{S}_K}(\overline{A}_{\frac{1}{2}},\overline{D}_{\frac{1}{2}}) \\
&\le d^2(A,D) + d^2(B,C) - 2d^2(A_{\frac{1}{2}},D_{\frac{1}{2}}), \qquad (4.2)
\end{aligned}
$$

completing the proof. □

*Remark* 4.1.6. For general $\lambda, \nu \in [0,1]$, a similar distance convexity still holds with coefficients in terms of $\lambda$ and $\nu$.

*Proof.* (of **Theorem 1.0.2**) For $u, v \in W^{1,2}(\Sigma, X)$ with images staying in $B_\rho(x)$, set $\{A = u(\xi), B = v(\xi), C = v(\eta), D = u(\eta)\}$ as in Lemma 4.1.5, we have:

$$
\frac{1}{4}(d(u(\eta),v(\eta)) - d(u(\xi),v(\xi)))^2 \le d^2(u(\xi),u(\eta)) + d^2(v(\xi),v(\eta)) - 2d^2(w(\xi),w(\eta)), \qquad (4.3)
$$

where $w(\xi) = \frac{u+v}{2}(\xi)$ is the mid-point of the geodesic connecting $u(\xi)$ and $v(\xi)$.

Multiplying (4.3) by $f(\eta)$ (where $0 \le f \le 1$ and $f \in C_c(\Sigma)$), averaging on the subset $\{|\eta - \xi| < \varepsilon\}$ of $\Sigma \times \Sigma$ and integrating over $\Sigma$ (as in 1.3 of [25] and see (2.1)), then first of all we conclude that $w \in W^{1,2}(\Sigma, X)$. By theorem 1.6.2 and theorem 1.12.2 of [25] we obtain that for any $f \in C_c(\Sigma), 0 \le f \le 1$:

$$
\frac{1}{4}\int_\Sigma f|\nabla d(u,v)|^2 \le \int_\Sigma f|\nabla u|^2 + \int_\Sigma f|\nabla v|^2 - 2\int_\Sigma f|\nabla w|^2.
$$

Hence by definition (2.2), we have an analogue to (2.2iv) of [25]:

$$\frac{1}{4}\int_{\Sigma}|\nabla d(u,v)|^2 \le E^u + E^v - 2E^w. \tag{4.4}$$ □

*Remark* 4.1.7. An immediate corollary of (4.4) is the uniqueness of the solution to the Dirichlet problem into a $CAT(K)$ (with $K>0$) space with small image assumption, cf. [45], [33] and theorem 2.2 of [25].

**Corollary 4.1.8.** *([45]) Let $(\Sigma, g)$ be a Lipschitz Riemannian domain and $(X,d\,)$ be an Alexandrov space of curvature bounded from above by $K>0$. Fix a point $Q\in X$, Let $\phi\in W^{1,2}(\Sigma,X)$ with $\phi(\Sigma)\subset B_\rho(Q)$ where $\rho=\rho(Q,K)$ is given by Theorem 1.0.2. Define*

$$W^{1,2}_\phi = \{u\in W^{1,2}(\Sigma,X) \quad \Big| \quad u(\Sigma)\subset B_\rho(Q) \quad \textit{and} \quad tr(u)=tr(\phi)\}.$$

*Then there exists a unique $u\in W^{1,2}_\phi$ which satisfies*

$$E^u = \int_\Sigma |\nabla u|^2 d\mu = E_0 = \inf_{v\in W^{1,2}_\phi} E^v.$$

## 4.2 Existence of good sweepouts by curves

Throughout the rest of this chapter, we will let $(X,d\,)$ be a closed Alexandrov space of curvature bounded from above by some $K>0$. Using the compactness of $X$, we let

$$\rho = \inf_{x\in X}\{\,\rho(x,K)\,\} > 0\,, \tag{4.5}$$

where $\rho(x,K)$ is as in Theorem 1.0.2. Fix a large positive integer $L$ and let $\Lambda$ denote the space of piecewise linear maps (constant speed geodesics) from $\mathbf{S}^1$ to $X$ with exactly $L^2$ breaks (possibly with unnecessary breaks) such that the length of each geodesic segment is at most $\rho$ defined by (4.5), parametrized by a (constant) multiple of arclength and with Lipschitz bound $L$ (note that a $W^{1,2}$ curve is also a $C^{1/2}$ curve but not necessarily Lipschitz continuous, here the Lipschitz bound denotes the bound of the speed, see (4.11)). Let $G \subset \Lambda$ denote the set of (possibly self-intersecting) closed geodesics in $X$ of length at most $\rho L^2$. (The constant speed of a curve in $\Lambda$ is equal to its length divided by $2\pi$; and its energy is equal to its length squared divided by $2\pi$. In other words, energy and length are essentially equivalent, see (2.6) and (4.11)).

### 4.2.1 Curve shortening map $\Psi$

The curve shortening is a map $\Psi : \Lambda \to \Lambda$ so that (see also section 2 of [17])

(1) $\Psi(\gamma)$ is homotopic to $\gamma$ and $\mathrm{Length}(\Psi(\gamma)) \leq \mathrm{Length}(\gamma)$.

(2) $\Psi(\gamma)$ depends continuously on $\gamma$.

(3) There is a continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ so that

$$\mathrm{dist}^2(\gamma, \Psi(\gamma)) \leq \phi\left(\frac{\mathrm{Length}^2(\gamma) - \mathrm{Length}^2(\Psi(\gamma))}{\mathrm{Length}^2(\Psi(\gamma))}\right). \tag{4.6}$$

(4) Given $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ with $\mathrm{dist}(\gamma, G) \geq \epsilon$, then $\mathrm{Length}\,(\Psi(\gamma)) \leq \mathrm{Length}\,(\gamma) - \delta$.

We will use local linear replacement to define the curve shortening map $\Psi$ which is identical to [14]: fix a partition of $\mathbf{S}^1$ by choosing $2L^2$ consecutive evenly spaced points (note that this is not necessarily where the piecewise linear maps have breaks).

$$x_0, x_1, x_2, \ldots, x_{2L^2} = x_0 \in \mathbf{S}^1\,, \quad \text{so that } |x_j - x_{j+1}| = \frac{\pi}{L^2} \leq \frac{\rho}{2L}\,. \tag{4.7}$$

$\Psi(\gamma)$ is given in the following three steps:

*Step 1*: Replace $\gamma$ on each <u>even</u> interval, i.e., $[x_{2j}, x_{2j+2}]$, by the linear map with the same endpoints to get a piecewise linear curve $\gamma_e : \mathbf{S}^1 \to X$. Namely, for each $j$, we let $\gamma_e\big|_{[x_{2j},x_{2j+2}]}$ be the unique shortest (constant speed) geodesic from $\gamma(x_{2j})$ to $\gamma(x_{2j+2})$.
*Step 2*: Replace $\gamma_e$ on each <u>odd</u> interval by the linear map with the same endpoints to get the piecewise linear curve $\gamma_o : \mathbf{S}^1 \to X$.
*Step 3*: Reparametrize $\gamma_o$ (fixing $\gamma_o(x_0)$) to get the desired constant speed curve $\Psi(\gamma) : \mathbf{S}^1 \to X$.

It is easy to see that $\Psi$ maps $\Lambda$ to $\Lambda$ and has property (1); cf. section 2 of [17]. The proof of properties $(2), (3)$ and (4) for $\Psi$ is virtually the same as [14]. We shall remark that there is a difficulty in the proofs of these properties: the second fundamental form (smoothness) of the manifold is used in the Riemannian case in [14], while we don't have the smoothness in an Alexandrov space of curvature bounded above. But note that the local energy convexity in Theorem 1.0.2 requires only that the two curves both stay in a small region, while the key lemma 4.2 in [14] requires that the two curves have the same endpoints. This fact allows us to get around this difficulty (see (A.6)). For the completeness of this thesis, we include the proofs of properties (3) and (4) in Appendix A and the proof of property (2) in Appendix B. Throughout the rest of this section, we will assume these properties of $\Psi$ and use them to prove the main theorem.

Combining properties (3) and (4) of $\Psi$, we have the following key lemma, which is crucial in producing the desired sequence of good sweepouts.

**Lemma 4.2.1.** *Given $W \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ and*

$$2\pi\left(W-\delta\right) < \mathit{Length}^2\left(\Psi(\gamma)\right) \leq \mathit{Length}^2\left(\gamma\right) < 2\pi\left(W+\delta\right), \tag{4.8}$$

*then $\mathit{dist}(\Psi(\gamma), G) < \epsilon$.*

*Proof.* If $W \leq \epsilon^2/6$, then $\delta = \epsilon^2/6$ gives $\text{Length}\left(\Psi(\gamma)\right) \leq 2\epsilon$. This tells us the bound on distance of $\Psi(\gamma)$ to a point curve (e.g., its mid-point) which is a trivial closed geodesic in $G$.

Assume next that $W > \epsilon^2/6$. The triangle inequality gives

$$\text{dist}(\Psi(\gamma), G) \leq \text{dist}(\Psi(\gamma), \gamma) + \text{dist}(\gamma, G)\,. \tag{4.9}$$

Since $\Psi$ does not decrease the length of $\gamma$ by much by the assumption, property (4) of $\Psi$ bounds $\text{dist}(\gamma, G)$ by $\epsilon/2$ as long as $\delta$ is sufficiently small. Similarly, property (3) of $\Psi$ allows us to bound $\text{dist}(\Psi(\gamma), \gamma)$ by $\epsilon/2$ as long as $\delta$ is sufficiently small. $\square$

### 4.2.2 Defining good sweepouts

Choose a sequence of maps $\hat{\sigma}^j \in \Omega_{\hat{\sigma}}$ with

$$\max_{s \in [-1,1]} \text{Energy}\left(\hat{\sigma}^j(\cdot, s)\right) < W + \frac{1}{j}\,. \tag{4.10}$$

Observe that (4.10) and the Cauchy-Schwarz inequality imply a uniform bound for the length and uniform $C^{1/2}$ continuity for the slices, that are both independent of $j$ and $s$. They follow immediately from the following: for any small $\delta > 0$, $[x, y] \subset [0, 2\pi]$ we pick $f \in C_c([0, 2\pi]), 0 \leq f \leq 1$, with $f = 1$ on $(x, y)$ and *supp* $(f) \subset [x - \delta, y + \delta] \subset [0, 2\pi]$, then

(4.11)

$$\begin{aligned}
&d^2\big(\hat{\sigma}^j(x,s)\,,\hat{\sigma}^j(y,s)\big) \;\leq\; \text{Length}^2\left(\hat{\sigma}^j(\cdot,s)|_{[x,y]}\right) \\
&= \lim_{\delta\to 0^+} \limsup_{\epsilon\to 0^+} \left(\int_{x-\delta}^{y+\delta} f\, \frac{d\left(\hat{\sigma}^j(\eta-\epsilon,s),\hat{\sigma}^j(\eta,s)\right) + d\left(\hat{\sigma}^j(\eta,s),\hat{\sigma}^j(\eta+\epsilon,s)\right)}{2\epsilon}\, d\eta\right)^2 \\
&\leq |y-x| \lim_{\delta\to 0^+} \limsup_{\epsilon\to 0^+} \int_{x-\delta}^{y+\delta} f^2 \left(\frac{d^2\left(\hat{\sigma}^j(\eta-\epsilon,s),\hat{\sigma}^j(\eta,s)\right) + d^2\left(\hat{\sigma}^j(\eta,s),\hat{\sigma}^j(\eta+\epsilon,s)\right)}{2\epsilon^2}\right) d\eta \\
&= |y-x|\, \text{Energy}\left(\hat{\sigma}^j(\cdot,s)|_{[x,y]}\right) \leq\; |y-x|\,(W+1)\,.
\end{aligned}$$

In order to get started and be able to use the properties of $\Psi$, we would like all the initial curves to be in $\Lambda$. We will replace the $\hat{\sigma}^j$'s by sweepouts $\sigma^j$ that, in addition to satisfying (4.10), also satisfy that the slices $\sigma^j(\cdot, s)$ are in $\Lambda$. We will do this by using local linear replacement similar to the construction of $\Psi$. Namely, the uniform $C^{1/2}$ bound for the slices allows us to fix a partition of points $y_0, \ldots, y_N = y_0$ in $\mathbf{S}^1$ so that each interval $[y_i, y_{i+1}]$ is always mapped to a geodesic ball in $X$ of radius at most $\rho$. Next, for each $s$ and each $j$, we replace $\hat{\sigma}^j(\cdot, s)\big|_{[y_i, y_{i+1}]}$ by the linear map (geodesic) with the same endpoints and call the resulting map $\tilde{\sigma}^j(\cdot, s)$. Reparametrize $\tilde{\sigma}^j(\cdot, s)$ to have constant speed to get $\sigma^j(\cdot, s)$. It is easy to see that each $\sigma^j(\cdot, s)$ satisfies (4.10). Furthermore, the length bound for $\sigma^j(\cdot, s)$ also gives a uniform Lipschitz (speed) bound for the linear maps; let $L$ be this bound and $N \leq L^2$.

We can see from the proof of property (2) for $\Psi$ in Appendix B that $\sigma^j$ is continuous in the transversal direction (i.e. with respect to $s$) and homotopic to $\hat{\sigma}$ in $\Omega$, cf. [4], [5], section 2 of [17] and appendix B of [14].

Finally, applying the replacement map $\Psi$ to each $\sigma^j(\cdot, s)$ gives a new sequence of sweepouts $\gamma^j = \Psi(\sigma^j)$. ($\Psi$ depends continuously on $s$ and preserves the homotopy class $\Omega_{\hat{\sigma}}$; it is clear that $\Psi$ fixes the constant maps at $s = \pm 1$.)

### 4.2.3 Almost maximal implies almost critical

We will show that the sequence $\gamma^j = \Psi(\sigma^j)$ of sweepouts is tight in the sense of the Introduction. Namely, we have the following main theorem.

**Theorem 4.2.2.** *Given $W \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $j > 1/\delta$ and for some $s_0$*

$$2\pi \, \mathit{Energy}(\gamma^j(\cdot, s_0)) = \mathit{Length}^2(\gamma^j(\cdot, s_0)) > 2\pi (W - \delta), \tag{4.12}$$

*then for this $j$ we have dist* $(\gamma^j(\cdot, s_0), G) < \epsilon$.

*Proof.* Let $\delta$ be given by Lemma 4.2.1. By (4.12), (4.10), and using that $j > 1/\delta$, we get

$$2\pi(W - \delta) < \text{Length}^2(\gamma^j(\cdot, s_0)) \leq \text{Length}^2(\sigma^j(\cdot, s_0)) < 2\pi(W + \delta)\,. \tag{4.13}$$

Thus, since $\gamma^j(\cdot, s_0) = \Psi(\sigma^j(\cdot, s_0))$, Lemma 4.2.1 gives $\text{dist}(\gamma^j(\cdot, s_0), G) < \epsilon$. □

## 4.3 Generalized Birkhoff-Lyusternik theorem

As in Section 3.3, instead of using the interval $[-1, 1]$ as parameter space for the circles in the definition of sweepout (see Definition 2.3.3) and assuming that the curves start and end in point curves, one could have use any compact set $\mathcal{P}$ and require that the curves are constant on $\partial\mathcal{P}$ (or that $\partial\mathcal{P} = \emptyset$). Then we let $\Omega^{\mathcal{P}}$ be the set of continuous maps $\sigma : \mathbf{S}^1 \times \mathcal{P} \to X$ so that for each $s \in \mathcal{P}$ the curve $\sigma(\cdot, s)$ is in $W^{1,2}(\mathbf{S}^1, X)$, the map $s \to \sigma(\cdot, s)$ is continuous from $\mathcal{P}$ to $W^{1,2}(\mathbf{S}^1, X)$ and finally $\sigma$ maps $\partial\mathcal{P}$ to point curves. Given a map $\hat{\sigma} \in \Omega^{\mathcal{P}}$, the homotopy class $\Omega^{\mathcal{P}}_{\hat{\sigma}} \subset \Omega^{\mathcal{P}}$ is defined to be the set of maps $\sigma \in \Omega^{\mathcal{P}}$ that are homotopic to $\hat{\sigma}$ through maps in $\Omega^{\mathcal{P}}$. Finally the *width* $W = W(\hat{\sigma})$ is

$$W = \inf_{\sigma \in \Omega^{\mathcal{P}}_{\hat{\sigma}}} \max_{s \in \mathcal{P}} \text{Energy}(\sigma(\cdot, s))\,. \tag{4.14}$$

Theorem 4.2.2 holds for these general parameter spaces and the proof is virtually the same.

*Proof.* (of Theorem 1.0.3) The proof is virtually the same as the proof of Theorem 1.0.1 in Section 3.3. □

# Chapter 5

# Uniqueness of weakly harmonic maps from $B_1$

## 5.1 2-dimensional conformally invariant variational probelm

As mentioned in the Introduction, more recently, Rivière [40] succeeded in writing in divergence form of the following 2-dimensional conformally invariant non-linear system of elliptic PDEs (which includes the harmonic map equation (2.9)):

$$-\Delta u^i = \Omega^i_j \cdot \nabla u^j \quad i = 1, 2, ..., n \quad \text{or} \quad -\Delta u = \Omega \cdot \nabla u\,, \tag{5.1}$$

with $\Omega = (\Omega^i_j)_{1 \le i,j \le n} \in L^2(B_1, so(n) \otimes \mathbf{R}^2)$ and $\Omega^i_j = -\Omega^j_i$ and thus provided a conservation law for this system (see (5.11) below). Here and throughout this chapter, the Einstein summation convention is used. Based on this conservation law he proved the (interior) continuity and therefore the regularity of an $W^{1,2}$ solution $u$ to (5.1), cf. [41]. In particular, the harmonic map equation (2.9) can be written in the form of (5.1) if we set

$$\Omega := (\Omega^i_j)_{1\le i,j\le n} \quad \text{where } \Omega^i_j := [A^i(u)_{j,l} - A^j(u)_{i,l}]\nabla u^l \,. \tag{5.2}$$

We shall remark that in a recent paper [27], Lamm and Rivière followed the same approach developed in [40] to show an analogous conservation law for some fourth order system, which includes the extrinsic and intrinsic biharmonic map equations, in 4 dimensions, and thus the continuity of the weak solutions to this system.

Next let us explore a little bit more on Rivière's approach to the regularity problem of the 2-dimensional conformally invariant variational problems (5.1) in [40]. Following the strategy of Uhlenbeck in [48], Rivière used the algebraic feature of $\Omega$, namely $\Omega$ being antisymmetry, to construct $\xi \in W^{1,2}_0(B_1, so(n))$ and a gauge transformation $P \in W^{1,2}(B_1, SO(n))$ which pointwise almost everywhere is an orthogonal matrix in $\mathbf{R}^{n\times n}$ satisfying

$$\|\nabla P\|_{L^2} + \|\nabla P^T\|_{L^2} + \|\xi\|_{W^{1,2}} \,\le\, C\|\Omega\|_{L^2}\,, \tag{5.3}$$

such that in $B_1$ we have

$$\nabla^\perp \xi \,=\, P^T\nabla P + P^T\Omega P\,. \tag{5.4}$$

Here and in what follows $\nabla = (\partial_{x_1}, \partial_{x_2})$ is the gradient and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ denotes the orthogonal gradient. The superscript $T$ denotes the transpose of a matrix.

Equivalently, we have

$$\nabla P \,=\, P\nabla^\perp\xi - \Omega P\,, \tag{5.5}$$

or with indices (and $1 \le m, z \le n$)

$$\nabla P^i_j \;=\; P^i_m \nabla^\perp \xi^m_j - \Omega^i_z\, P^z_j\,, \quad 1 \le i, j \le n\,. \tag{5.6}$$

Here and in what follows, $1 \le m, z \le n$ and the Einstein summation convention is understood.

*Remark* 5.1.1. Besides Uhlenbeck's method to construct the gauge transformation matrix $P$, another nice way to construct $P$ is to minimize the energy functional

$$E(R) \;=\; \int_{B_1} \left| R^T \nabla R + R^T \Omega R \right|^2\,, \quad R \in W^{1,2}(B_1, SO(n))\,. \tag{5.7}$$

See Schikorra's recent results in [43] for this way of construction of $P$, and cf. [10].

*Remark* 5.1.2. The higher dimensional version of this fact was also used in the recent paper of Rivière and Struwe [42] in proving the partial regularity of harmonic maps in dimensions greater than two.

Moreover, by solving the following system of PDEs ([40, equation (II.5)]),

$$\begin{cases} \Delta \widehat{A} \;=\; \nabla^\perp B \cdot \nabla P + \nabla \widehat{A} \cdot \nabla^\perp \xi & \text{in } B_1; \\ \Delta B \;=\; -\nabla^\perp \widehat{A} \cdot \nabla P^T - \mathrm{curl}\,\left( (\widehat{A} + I) \nabla^\perp \xi\, P^T \right) & \text{in } B_1; \\ \dfrac{\partial \widehat{A}}{\partial \nu} \;=\; 0 \quad \text{and} \quad B \;=\; 0 & \text{on } \partial B_1; \\ \text{and } \displaystyle\int_{B_1} \widehat{A} = 0\,, & \end{cases} \tag{5.8}$$

where $I = Id_n$ is the $n \times n$ identity matrix and $\nu$ is the unit normal to $\partial B_1$, Rivière constructed matrices $\widehat{A} \in W^{1,2} \cap C^0(B_1, Gl_n(\mathbf{R})), A \in L^\infty \cap W^{1,2}(B_1, Gl_n(\mathbf{R}))$ and $B \in W^{1,2}_0(B_1, M_n(\mathbf{R}))$ such that

$$\|\widehat{A}\|_{W^{1,2}} + \|\widehat{A}\|_{L^\infty} + \|B\|_{W^{1,2}} \;\le\; C\|\Omega\|_{L^2} \quad \text{and} \quad A \;=\; (\widehat{A} + I)\, P^T\,, \tag{5.9}$$

which gives (see [40, Theorem I.4])

$$\nabla A - A\Omega \;=\; \nabla^{\perp} B \tag{5.10}$$

and the conservation law (using the harmonic map equation (5.1))

$$\text{div}\;(A\nabla u + B\nabla^{\perp} u) = 0\,. \tag{5.11}$$

This conservation law makes a perfect analog to the "almost holomorphic" equation (4.38) of [23] (see Proposition 5.2.4 below which says this conservation law implies an "almost divergence free" structure for weakly harmonic maps). Then by an extension argument and the results of Coifman, Lions, Meyer and Semmes [12], he showed that $u$ lies locally in $W^{2,1}$ which embeds in $C^0$ in two dimensions (see e.g. [7]). In particular, the following theorem holds.

**Theorem 5.1.3.** *([40, theorem 1.2]) Let $\mathcal{N}^k$ be a $C^2$ submanifold of $\mathbf{R}^n$ with $1 \le k \le n$. Let $\omega$ be a $C^1$ 2-form on $\mathcal{N}^k$ such that the $L^\infty$ norm of $d\omega$ is bounded on $\mathcal{N}^k$. Then every critical point in $W^{1,2}(B_1, \mathcal{N}^k)$ of the Lagrangian*

$$F(u) \;=\; \int_{B_1} \left[|\nabla u|^2 + \omega(u)(\partial_{x_1} u, \partial_{x_2} u)\right] dx_1 \wedge dx_2 \tag{5.12}$$

*satisfies an equation of the form (5.1) for some $\Omega$ in $L^2(B_1, so(n) \otimes \mathbf{R}^2)$ and is therefore continuous.*

## 5.2 Energy convexity

We point out that the key ingredient in Colding and Minicozzi's proof of the energy convexity for weakly harmonic maps with small energy on $B_1$, namely, Theorem 1.0.4, is to show a special jacobian structure of the norm square of the holomorphic function that

Hélein constructed in [23] (see page 182 of [23]) where the moving frame was involved and then appeal to the following Wente's lemma.

**Lemma 5.2.1.** *([49], [23, theorem 3.3.8], [12]) If $a, b \in W^{1,2}(B_1)$ and $w$ be the solution of*

$$\begin{cases} \Delta w \,=\, \dfrac{\partial a}{\partial x_2}\dfrac{\partial b}{\partial x_1} - \dfrac{\partial a}{\partial x_1}\dfrac{\partial b}{\partial x_2} \,=\, \nabla a \cdot \nabla^{\perp} b \quad in \;\; B_1\,, \\ \quad w \,=\, 0 \quad or \quad \dfrac{\partial w}{\partial \nu} \,=\, 0 \quad on \;\; \partial B_1\,. \end{cases} \tag{5.13}$$

*Then $w \in C^0(B_1)$ and the following estimates hold*

$$\|w\|_{L^\infty(B_1)} + \|\nabla w\|_{L^2(B_1)} \,\leq\, \|\nabla a\|_{L^2(B_1)}\|\nabla b\|_{L^2(B_1)}\,, \tag{5.14}$$

*where we choose $\int_{B_1} w = 0$ for the Neumann boundary data.*

This special jacobian structure of the norm square of the holomorphic function in particular implied directly that $|\nabla u|^2$ actually lies in the local Hardy space $h^1(B_1)$ (a strict subspace of $L^1(B_1)$), which we will recall next.

**Definition 5.2.2.** ([34], cf. [44, Definition 1.90]) Choose a Schwartz function $\phi \in C_0^\infty(B_1)$ such that $\int_{B_1} \phi \, dx = 1$ and let $\phi_t(x) = t^{-2}\phi\left(\frac{x}{t}\right)$. For a measurable function $f$ defined in $B_1$ we say that $f$ lies in the local Hardy space $h^1(B_1)$ if the radial maximal function of $f$

$$f^*(x) = \sup_{0<t<1-|x|} \left| \int_{B_t(x)} \frac{1}{t^2}\phi\left(\frac{x-y}{t}\right) f(y)dy \right| (x) = \sup_{0<t<1-|x|} \left|\phi_t * f\right|(x) \tag{5.15}$$

belongs to $L^1(B_1)$ and we define

$$\|f\|_{h^1(B_1)} = \|f^*(x)\|_{L^1(B_1)}\,. \tag{5.16}$$

It follows immediately that $h^1(B_1)$ is a strict subspace of $L^1(B_1)$ and $\|f\|_{L^1(B_1)} \leq \|f\|_{h^1(B_1)}$.

In fact, the (local) Hardy space already illustrated its nice distribution features and played important roles in many interesting applications, see e.g. [49],[12],[23] and [40]. Inspired by the results of Rivière [40], in this chapter, we aim to, instead of the use of Hélein's moving frame technique (see the estimate [14, (C.13)] used by Colding-Minicozzi), revisit and understand in more detail Colding and Minicozzi's proof of Theorem 1.0.4, via Rivière's gauge decomposition technique. We will do this by showing that the energy density $|\nabla u|^2$ is in the local Hardy space $h^1(B_1)$ directly using Rivière's approach. And the main idea is to re-exam Rivière's gauge decomposition and the estimates in [40], and to reveal some more special Jacobian structures that hold only for the weakly harmonic maps on surfaces.

The next proposition plays a crucial role in Colding and Minicozzi's proof of Theorem 1.0.4. The rest of the chapter is devoted to a new approach to a proof of this proposition.

**Proposition 5.2.3** ([15]). *There exists a constant $\epsilon_0 > 0$ (depending on $\mathcal{N}$) so that if $u : B_1 \to \mathcal{N}$ is a weakly harmonic map with energy at most $\epsilon_0$, then for any $h \in W_0^{1,2}(B_1)$, we have*

$$\int_{B_1} h^2 |\nabla u|^2 \leq C \left(\int_{B_1} |\nabla h|^2\right) \left(\int_{B_1} |\nabla u|^2\right). \tag{5.17}$$

*for some constant $C > 0$.*

Our observations in this section are the two hidden jacobian structures that $\Delta B$ and $\Delta P$ have. Based on these facts we will be able to obtain an almost jacobian structure for the energy density $|\nabla u|^2$ and show that it is in the local Hardy space $h^1(B_1)$. To see the connection between Rivière's gauge decomposition method and Hélein's moving frame method, we could think of the gauge transformation matrix $P$ as a real analogue of Hélein's gauge transformation matrix in the construction of the adapted moving frame. Most importantly, both transformation matrices enjoy the "compensated compactness" phenomenon which helps to show the improved regularity of the gradient of the weak solution.

To get to the main estimate in Proposition 5.2.3, we first present the two main observations about the matrices $B$ and $P$. Throughout the rest of this chapter, $\epsilon > 0$ is assumed to be sufficiently small and $C$ denotes a universal constant that depends only on $\mathcal{N}$ unless otherwise stated.

**Proposition 5.2.4.** *Suppose $u$ is an $W^{1,2}$ weakly harmonic map satisfying (2.9) with $E(u) \leq \epsilon$, then we have*

$$\|B\|_{L^\infty(B_1)} \leq C\|\nabla u\|^2_{L^2(B_1)} \leq C\epsilon\,.$$

*Proof.* Let us impose the fact that $\Omega = A(u)\nabla u$ if $u$ is a weakly harmonic map, where

$$(A(u))_l = A^i(u)_{j,l} - A^j(u)_{i,l}$$

denotes the second fundamental form as in (5.2). And taking the curl on both sides of equation (5.10) yields

$$\Delta B = -\mathrm{curl}(A\,A(u)\nabla u) = -\nabla^\perp(A\,A(u)) \cdot \nabla u\,. \tag{5.18}$$

Combing the jacobian structure of the right-hand side of (5.18) with the zero boundary condition of $B$ and Lemma 5.2.1 yields the desired estimate. Here we also use the smoothness and compactness of $\mathcal{N}$, (5.3) and (5.9). □

**Lemma 5.2.5.** *Suppose $u$ is an $W^{1,2}$ weakly harmonic map satisfying (2.9) with $E(u) \leq \epsilon$,* *then there exists* $\eta \in W^{1,2}(B_1, \mathbf{R}^n)$ *such that for any* $1 \le i, j \le n$

$$\Delta P^i_j \;=\; \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1})^l_k\right] \cdot \nabla^\perp \eta^k$$

$$+\, \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1}B)^l_k\right] \cdot \nabla^\perp u^k \,. \tag{5.19}$$

*Proof.* By the conservation law (5.11) and Poincaré lemma, there exists $\eta \in W^{1,2}(B_1, \mathbf{R}^n)$ such that

$$\nabla^\perp \eta \;=\; A\nabla u + B\nabla^\perp u \,. \tag{5.20}$$

Moreover, by Proposition 5.2.4, (5.3) and (5.9) we have

$$\|\nabla \eta\|_{L^2(B_1)} \;\le\; C\|\nabla u\|_{L^2(B_1)} \le C\sqrt{\epsilon} \,. \tag{5.21}$$

Multiplying both sides of equation (5.20) by $A^{-1}$ from the left gives (with indices)

$$\nabla u^l \;=\; (A^{-1})^l_k \nabla^\perp \eta^k - (A^{-1}B)^l_k \nabla^\perp u^k \,, \quad l = 1, 2, ..., n \,. \tag{5.22}$$

Taking divergence on both sides of equation (5.6) yields

$$\Delta P^i_j \;=\; \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \operatorname{div}\,(\Omega^i_z\, P^z_j) \,, \quad 1 \le i, j \le n \,. \tag{5.23}$$

Now since $\Omega^i_z = (A^i(u)_{z,l} - A^z(u)_{i,l})\nabla u^l$ (see equation (5.2)), combining (5.22) and (5.23) gives

$$
\begin{aligned}
\Delta P^i_j &= \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \mathrm{div}\,(\Omega^i_z\, P^z_j)\\
&= \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \mathrm{div}\,\left((A^i(u)_{z,l} - A^z(u)_{i,l})\,\left[(A^{-1})^l_k \nabla^\perp \eta^k + (A^{-1}B)^l_k \nabla^\perp u^k\right]\,P^z_j\right)\\
&= \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1})^l_k\right] \cdot \nabla^\perp \eta^k\\
&\qquad + \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1}B)^l_k\right] \cdot \nabla^\perp u^k\,. \qquad \square
\end{aligned}
$$

Next we prove a local estimate on the oscillation of the transformation matrix $P$ based on Lemma 5.2.5.

**Lemma 5.2.6.** *Suppose $u$ is an $W^{1,2}$ weakly harmonic map satisfying (2.9) with $E(u) \leq \epsilon$, then for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have*

$$
|P(y) - P(x)| \leq C\sqrt{\epsilon}\,. \tag{5.24}
$$

*Proof.* Let $\widetilde{P} \in W^{1,2}(B_1, M_n(\mathbf{R}))$ be the weak solution of (for any $1 \leq i, j \leq n$)

$$
\begin{cases}
\Delta \widetilde{P}^i_j = \nabla P^i_m \cdot \nabla^\perp \xi^m_j - \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1})^l_k\right] \cdot \nabla^\perp \eta^k & \\
\qquad\quad + \nabla\left[(A^i(u)_{z,l} - A^z(u)_{i,l}) P^z_j (A^{-1}B)^l_k\right] \cdot \nabla^\perp u^k\,, & \text{in } B_1\\
\widetilde{P}^i_j = 0 & \text{on } \partial B_1\,.
\end{cases} \tag{5.25}
$$

Then by Lemma 5.2.1 we have $\widetilde{P} \in C^0(B_1, M_n(\mathbf{R}))$ and

$$
\|\widetilde{P}\|_{L^\infty(B_1)} + \|\nabla \widetilde{P}\|_{L^2(B_1)} \leq C\epsilon\,. \tag{5.26}
$$

Since $\Delta(P - \widetilde{P}) = 0$ in $B_1$, we know that $V = P - \widetilde{P} \in C^\infty(B_1, M_n(\mathbf{R}))$ is harmonic.

Now for any $x \in B_1$ and any $r > 0$ such that $B_{2r}(x) \subset B_1$, we have

$$
\begin{aligned}
|V(y) - V(x)| \;&\leq\; Cr\|\nabla V\|_{L^\infty(B_r(x))} \\
\leq Cr\frac{1}{\pi r^2}\int_{B_{2r}(x)} |\nabla V| \;&\leq\; \frac{C}{\pi r}\left(\sqrt{\pi}r\|\nabla V\|_{L^2(B_{2r}(x))}\right) \\
\leq C\left(\|\nabla P\|_{L^2(B_{2r}(x))} + \|\nabla \widetilde{P}\|_{L^2(B_{2r}(x))}\right) \;&\leq\; C\sqrt{\epsilon}\,,
\end{aligned}
\tag{5.27}
$$

where we have used the mean value property of $V$ and (5.26), (5.3). Combining (5.26) and (5.27) yields that for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have

$$
|P(y) - P(x)| \;\leq\; C\sqrt{\epsilon}\,, \tag{5.28}
$$

which gives the desired estimate (5.24). $\square$

We will next show

**Lemma 5.2.7.** *Suppose $u$ is a $W^{1,2}$ weakly harmonic map satisfying (2.9) with $E(u) \leq \epsilon$, then we have*

$$
|\nabla u|^2 \in h^1(B_1) \qquad \textit{and} \qquad \||\nabla u|^2\|_{h^1(B_1)} \leq C\int_{B_1} |\nabla u|^2 \leq C\epsilon\,. \tag{5.29}
$$

*Proof.* Using (5.3)-(5.11) and Proposition 5.2.4, for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have (choosing $\epsilon$ sufficiently small)

$$
\begin{aligned}
0 \leq \frac{1}{2}|\nabla u|^2(y) &\leq \left(A\nabla u + B\nabla^\perp u\right)\cdot(P^T\nabla u)(y) \\
&= \left(A\nabla u + B\nabla^\perp u\right)\cdot\left[\left(P^T(x) + \left(P^T - P^T(x)\right)\right)\nabla u\right](y)\,,
\end{aligned}
\tag{5.30}
$$

and therefore by Lemma 5.2.6 and (5.20)

$$\begin{aligned} &\nabla^{\perp}\eta\cdot\big(P^T(x)\nabla u\big)(y) = \big(A\nabla u + B\nabla^{\perp}u\big)\cdot\big(P^T(x)\nabla u\big)(y) \\ &\geq \frac{1}{2}|\nabla u|^2(y) - \big(A\nabla u + B\nabla^{\perp}u\big)\cdot\big[\big(P^T - P^T(x)\big)\nabla u\big](y) \geq \frac{1}{4}|\nabla u|^2(y)\,. \end{aligned} \tag{5.31}$$

Now we choose a function

$$\phi \in C_0^{\infty}(B_1) \text{ with } \phi \geq 0,\ \mathrm{spt}(\phi) \subseteq B_{\frac{1}{2}} \text{ and } \int_{B_1}\phi\,dx = 1\,. \tag{5.32}$$

Moreover, we additionally assume that $\|\nabla\phi\|_{L^{\infty}(B_1)} \leq 100$. Using (5.31), one verifies directly that (using Definition 5.2.2)

$$\begin{aligned} \||\nabla u|^2\|_{h^1(B_1)} &= \int_{B_1}\sup_{0<t<1-|x|}\phi_t * |\nabla u|^2 dx \\ &\leq 4\int_{B_1}\sup_{0<t<1-|x|}\phi_t * \big(\nabla^{\perp}\eta\cdot(P^T(x)\nabla u)\big)\,dx \\ &= 4\int_{B_1}\sup_{0<t<1-|x|}\phi_t * \big[(P^T(x))_{ij}\big(\nabla^{\perp}\eta^i\cdot\nabla u^j\big)\big]\,dx \\ &\leq C\sum_{i,j=1}^{n}\|\nabla^{\perp}\eta^i\cdot\nabla u^j\|_{h^1(B_1)} \\ &\leq C\|\nabla^{\perp}\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)} \leq C\int_{B_1}|\nabla u|^2, \end{aligned}$$

where we have used the fact

$$\nabla^{\perp}\eta^i\cdot\nabla u^j \in h^1(B_1) \quad \text{and} \quad \|\nabla^{\perp}\eta^i\cdot\nabla u^j\|_{h^1(B_1)} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}$$

for all $i,j = 1,2,...,n$. To see this, we first extend $\eta^i - \frac{1}{|B_1|}\int_{B_1}\eta^i$ and $u^j - \frac{1}{|B_1|}\int_{B_1}u^j$ from $B_1$ to $\mathbf{R}^2$ which yields the existence of $\tilde{\eta}^i, \tilde{u}^j \in W_c^{1,2}(\mathbf{R}^2)$ such that

$$\int_{\mathbf{R}^2}|\nabla\tilde{\eta}^i|^2 \leq C\int_{B_1}|\nabla\eta^i|^2 \quad \text{and} \quad \int_{\mathbf{R}^2}|\nabla\tilde{u}^j|^2 \leq C\int_{B_1}|\nabla u^j|^2 \tag{5.33}$$

and

$$\nabla\tilde{\eta}^i = \nabla\eta^i \quad \text{and} \quad \nabla\tilde{u}^j = \nabla u^j \quad \text{a.e. in } B_1\,. \tag{5.34}$$

Then by the results of [12] we know that

$$\begin{aligned}\|\nabla^\perp\tilde{\eta}^i\cdot\nabla\tilde{u}^j\|_{\mathcal{H}^1(\mathbf{R}^2)} :&= \int_{\mathbf{R}^2}\sup_{\phi\in\mathcal{T}}\sup_{t>0}\left|\int_{B_t(x)}\frac{1}{t^2}\phi\left(\frac{x-y}{t}\right)\left(\nabla^\perp\tilde{\eta}^i\cdot\nabla\tilde{u}^j\right)(y)dy\right|dx\\ &\le C\|\nabla\tilde{\eta}^i\|_{L^2(\mathbf{R}^2)}\|\nabla\tilde{u}^j\|_{L^2(\mathbf{R}^2)} \le C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}\,,\end{aligned} \tag{5.35}$$

where $\mathcal{T} = \{\phi\in C^\infty(\mathbf{R}^2) : \mathrm{spt}(\phi)\subset B_1 \text{ and } \|\nabla\phi\|_{L^\infty}\le 100\}$. By (5.34), (5.35) and Definition 5.2.2, it is clear that

$$\begin{aligned}\|\nabla^\perp\eta^i\cdot\nabla u^j\|_{h^1(B_1)} &= \|\nabla^\perp\tilde{\eta}^i\cdot\nabla\tilde{u}^j\|_{h^1(B_1)}\\ &\le \|\nabla^\perp\tilde{\eta}^i\cdot\nabla\tilde{u}^j\|_{\mathcal{H}^1(\mathbf{R}^2)} \le C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}.\end{aligned} \tag{5.36}$$

This completes the proof of the theorem. □

Ware now prepared to present the proof of Proposition 5.2.3. We first recall a regularity result for boundary value problems in the local Hardy space $h^1(B_1)$, which can be thought of as a generalization of Lemma 5.2.1. For a more general version, one can refer [11].

**Theorem 5.2.8** (Theorem A.4 of [26], cf. [11])**.** *Suppose $f\in h^1(B_1)$ and $f\ge 0$, then there exists $\psi\in L^\infty\cap W^{1,2}(B_1)$ which solves the following Dirichlet problem*

$$\begin{cases}\Delta\psi = f & \text{in } B_1\,,\\ \psi = 0 & \text{on } \partial B_1\,.\end{cases} \tag{5.37}$$

*Moreover, there exists constant $C > 0$ such that*

$$\|\psi\|_{L^\infty(B_1)} + \|\nabla\psi\|_{L^2(B_1)} \leq C\,\|f\|_{h^1(B_1)}\,. \tag{5.38}$$

**Lemma 5.2.9.** *Suppose $u$ is a $W^{1,2}$ weakly harmonic map with $E(u) \leq \epsilon$ sufficiently small, then there exists a function $\psi \in L^\infty \cap W_0^{1,2}(B_1)$ such that $\Delta\psi = |\nabla u|^2 \geq 0$ with*

$$\|\psi\|_{L^\infty(B_1)} + \|\nabla\psi\|_{L^2(B_1)} \leq C\|\nabla u\|^2_{L^2(B_1)} \leq C\epsilon\,. \tag{5.39}$$

*Proof.* This is a direct consequence of Lemma 5.2.7 and Theorem 5.2.8. □

*Proof.* (of Proposition 5.2.3) This proof is taken from [14]. Applying Stokes' theorem to $\operatorname{div}(h^2\nabla\psi)$ and using Cauchy-Schwarz inequality and Lemma 5.2.9 gives

$$\int_{B_1} h^2|\nabla u|^2 = \int_{B_1} h^2\Delta\psi \leq \int_{B_1} |\nabla h^2||\nabla\psi| \leq 2\|\nabla h\|_{L^2}\left(\int_{B_1} h^2|\nabla\psi|^2\right)^{1/2}. \tag{5.40}$$

Applying Stokes' theorem to $\operatorname{div}(h^2\psi\nabla\psi)$, using that $\Delta\psi \geq 0$ and (5.40) gives

$$\left(\int_{B_1} h^2|\nabla\psi|^2\right) \leq \int_{B_1} |\psi|(h^2\Delta\psi + |\nabla h^2||\nabla\psi|) \leq 4\|\psi\|_{L^\infty}\|\nabla h\|_{L^2}\left(\int_{B_1} h^2|\nabla\psi|^2\right)^{1/2}, \tag{5.41}$$

so that

$$\left(\int_{B_1} h^2|\nabla\psi|^2\right)^{1/2} \leq 4\|\psi\|_{L^\infty}\|\nabla h\|_{L^2}\,. \tag{5.42}$$

Finally, substituting the bound in (5.42) back into (5.40) and combining with (5.39) gives the proposition. □

*Proof.* (of Theorem 1.0.4) Let $h(x) = v(x) - u(x) \in W_0^{1,2}(B_1)$. Use Stokes' theorem and that $u$ and $v$ are equal on $\partial B_1$ to get

$$\begin{aligned} &\int_{B_1} |\nabla v|^2 - \int_{B_1} |\nabla u|^2 - \int_{B_1} |\nabla (v-u)|^2 \\ &= -2 \int_{B_1} \langle v-u, \Delta u\rangle \;=\; 2\int_{B_1} \langle h, \Omega \cdot \nabla u\rangle \;\equiv\; \Psi\,. \end{aligned} \tag{5.43}$$

Note that there exists a constant $C$ depending only on $\mathcal{N}$ so that

$$|(x-y)^N| \;\leq\; C|x-y|^2\,, \tag{5.44}$$

where the superscript $N$ denotes the normal part of a vector (see (C.17) of [14]). Since $u$ and $v$ both map to $\mathcal{N}$, we can apply (5.44) to get $\left|h^N\right| \;\leq\; C|h|^2$. Therefore, using the fact that the harmonic map equation (2.9) implies

$$\Delta u \perp T_u\mathcal{N} \quad \text{and} \quad |\Delta u| \;\leq\; |\nabla u|^2 \sup_{\mathcal{N}} |A(u)|\,,$$

we have

$$\begin{aligned} |\Psi| = \; &2\int_{B_1} \langle h^N, \Omega\cdot\nabla u\rangle \leq C\int_{B_1} |h|^2|\nabla u|^2 \\ &\leq C\int_{B_1} |\nabla h|^2 \int_{B_1} |\nabla u|^2 \leq C\epsilon \int_{B_1} |\nabla(v-u)|^2, \end{aligned} \tag{5.45}$$

where we used Proposition 5.2.3 already.

Combing (5.43) and (5.45), and taking $\epsilon$ sufficiently small so that $C\epsilon \;\leq\; \frac{1}{2}$, we get the desired energy convexity in Theorem 1.0.4. $\square$

# Appendix A

# Establishing properties (3) and (4) of $\Psi$

To prove property (3) of $\Psi$, we will use the following equivalent way to construct $\Psi(\gamma)$ :

$(A_1)$ Follow Step 1 to get $\gamma_e$.

$(B_1)$ Reparametrize $\gamma_e$ (fixing the image of $x_0$) to get the constant speed curve $\tilde{\gamma}_e$. This reparametrization moves the points $x_j$ to new points $\tilde{x}_j$ (i.e., $\gamma_e(x_j) = \tilde{\gamma}_e(\tilde{x}_j)$).

$(A_2)$ Do linear replacement on the odd $\tilde{x}_j$ intervals to get $\tilde{\gamma}_o$.

$(B_2)$ Reparametrize $\tilde{\gamma}_o$ (fixing the image of $x_0$) to get the constant speed curve $\Psi(\gamma)$.

One sees easily that this gives the same curve since $\tilde{\gamma}_o$ is just a reparametrization of $\gamma_o$. We also see that each of the four steps is energy non-increasing¶. Thus property (3) follows from the triangle inequality once we bound $\mathrm{dist}(\gamma, \gamma_e)$ and $\mathrm{dist}(\gamma_e, \tilde{\gamma}_e)$ in terms of the decrease in length (as well as the analogs for steps $(A_2)$ and $(B_2)$).

The bound on $\mathrm{dist}(\gamma, \gamma_e)$ follows directly from the next corollary of Theorem 1.0.2 .

¶This is obvious for the linear replacements, since linear maps minimize energy. It follows from (4.11) for the reparametrizations, since for a curve $\sigma : \mathbf{S}^1 \to X$ we have $\mathrm{Length}^2(\sigma) \leq 2\pi \, \mathrm{Energy}(\sigma)$ , with equality if and only if its speed is a constant $= \mathrm{Length}(\sigma)/(2\pi)$ almost everywhere.

**Corollary A.0.10.** *There exists $C$ so that if $I$ is an interval of length at most $\rho/L$, $\sigma_1 : I \to X$ is a curve with Lipschitz bound $L$, and $\sigma_2 : I \to X$ is the minimizing geodesic with the same endpoints, then*

$$dist^2(\sigma_1, \sigma_2) \leq C\left(E^{\sigma_1} - E^{\sigma_2}\right).$$

*Proof.* Let $\Sigma = I \subset \mathbf{S}^1$ and note that $w = \frac{\sigma_1+\sigma_2}{2}$ has the same end points as $\sigma_1$ and $\sigma_2$. Since $d(\sigma_1, \sigma_2) \in W_0^{1,2}(I, \mathbf{R})$ (see theorem 1.12.2 of [25]) and from Theorem 1.0.2, the Poincaré inequality and (4.4) imply

$$\text{dist}^2(\sigma_1, \sigma_2) \,\leq\, C(I) \int_I |\nabla d(\sigma_1, \sigma_2)|^2 d\mu \,\leq\, C\left(E^{\sigma_1} - E^{\sigma_2}\right),$$

where we used the minimality of $\sigma_2$. □

Applying Corollary A.0.10 on each of $L^2$ intervals in step $(A_1)$, we get that

$$\text{dist}^2(\gamma, \gamma_e) \,\leq\, C\left(E^{\gamma} - E^{\gamma_e}\right) \,\leq\, \frac{C}{2\pi}\left(\text{Length}^2(\gamma) - \text{Length}^2(\Psi(\gamma))\right). \tag{A.1}$$

This gives the desired bound on $\text{dist}(\gamma, \gamma_e)$ since $\text{Length}(\Psi(\gamma)) \,\leq\, \rho L^2$.

To bound $\text{dist}(\gamma_e, \tilde{\gamma}_e)$, we will use that $\gamma_e$ is just the composition $\tilde{\gamma}_e \circ P$, where $P : \mathbf{S}^1 \to \mathbf{S}^1$ is a monotone piecewise linear map[‖] and let $L$ be its Lipschitz bound as well. Using that the (piecewise constant) speed of $\gamma_e$ is $|\gamma_e'| = |(\tilde{\gamma}_e \circ P)'| = |\tilde{\gamma}_e' \circ P| \cdot |P'| \leq L^2$ (Note: $|\tilde{\gamma}_e' \circ P|(x)$ denotes the speed of $\tilde{\gamma}_e$ at point $P(x)$) and the (constant) speed of $\tilde{\gamma}_e = |\tilde{\gamma}_e'| = \text{Length}(\tilde{\gamma}_e)/(2\pi) \leq L$ (away from the breaks), and also that the integral of

[‖]The map $P$ is Lipschitz, but the inverse map $P^{-1}$ may not be if $\gamma_e$ is constant on an interval.

$P'$ is $2\pi$, we have

$$\int_{\mathbf{S}^1} (P'-1)^2 = \int_{\mathbf{S}^1} (P')^2 - 2\pi = \int_{\mathbf{S}^1} \left( \frac{|\gamma_e'|}{|\tilde{\gamma}_e' \circ P|} \right)^2 - 2\pi = \frac{4\pi^2}{\text{Length}^2(\tilde{\gamma}_e)} \int_{\mathbf{S}^1} |\gamma_e'|^2 - 2\pi$$

$$= 2\pi \, \frac{\text{Energy}(\gamma_e) - \text{Energy}(\tilde{\gamma}_e)}{\text{Energy}(\tilde{\gamma}_e)} \leq 2\pi \, \frac{\text{Energy}(\gamma) - \text{Energy}(\Psi(\gamma))}{\text{Energy}(\Psi(\gamma))} \, . \tag{A.2}$$

Now divide $\mathbf{S}^1$ into two sets, $S_1$ and $S_2$, where $S_1$ is the set of points within distance $(\pi \int_{\mathbf{S}^1} |P'-1|^2)^{1/2}$ of a break point for $\tilde{\gamma}_e$. Since $P(x_0) = x_0$, we have $|P(x) - x| \leq (\pi \int_{\mathbf{S}^1} |P'-1|^2)^{1/2}$. Since $\gamma_e$ and $\tilde{\gamma}_e$ agree at $x_0 = x_{2L^2}$, the Wirtinger inequality** bounds $\text{dist}^2(\gamma_e, \tilde{\gamma}_e)$ in terms of

$$\int_{\mathbf{S}^1} |\nabla d\,(\tilde{\gamma}_e \circ P,\, \tilde{\gamma}_e)|^2 \leq \int_{S_1} \left( |\,(\tilde{\gamma}_e \circ P)'| + |\,\tilde{\gamma}_e'\,| \right)^2 + \int_{S_2} |\nabla d\,(\tilde{\gamma}_e \circ P,\, \tilde{\gamma}_e)|^2 \, , \tag{A.3}$$

where we used that the fact (3) in the proof of Lemma 4.1.2 (see last part of Appendix C) implies

$$\int_{S_1} |\nabla d\,(\tilde{\gamma}_e \circ P,\, \tilde{\gamma}_e)|^2 \, \leq \, \int_{S_1} \left( |\,(\tilde{\gamma}_e \circ P)'| + |\,\tilde{\gamma}_e'\,| \right)^2 . \tag{A.4}$$

We will bound both terms on the right hand side of (A.3) in terms of $\int_{\mathbf{S}^1} |P'-1|^2$ and then appeal to (A.2). To bound the first term, we have

$$\int_{S_1} \left( |\,(\tilde{\gamma}_e \circ P)'| + |\,\tilde{\gamma}_e'\,| \right)^2 \leq \, (L^2 + L)^2 \, \text{Length}(S_1) \leq 8\, L^6 \left( \pi \int_{\mathbf{S}^1} |P'-1|^2 \right)^{1/2} . \tag{A.5}$$

We see that if $(\pi \int_{\mathbf{S}^1} |P'-1|^2)^{1/2} \; \geq \; \frac{\pi}{2L^2} \; = \; \frac{|x_j - x_{j+1}|}{2}$, we are done since in this case $S_2 \, = \, \emptyset$.

On the other hand, suppose $(\pi \int_{\mathbf{S}^1} |P'-1|^2)^{1/2} \; < \; \frac{\pi}{2L^2}$; note that if $x \in S_2$, then $\tilde{\gamma}_e(x)$ and $\tilde{\gamma}_e \circ P(x)$ stay within the $\rho$-neighborhood between two break points (although $\tilde{\gamma}_e$ and

**The Wirtinger inequality is just the usual Poincaré inequality which bounds the $L^2$ norm in terms of the $L^2$ norm of the derivative; i.e., $\int_0^{2\pi} f^2 dt \leq 4 \int_0^{2\pi} (f')^2 dt$ provided $f(0) = f(2\pi) = 0$.

$\tilde{\gamma}_e \circ P$ might not have the same endpoints) and $|\tilde{\gamma}_e' \circ P| = |\tilde{\gamma}_e'| \leq L$ in this neighborhood. Thus, we can bound the second term by applying Theorem 1.0.2. Namely, by summing up the integral over each piece of $S_2$, we have

$$
\begin{aligned}
&\frac{1}{4}\int_{S_2} |\nabla d\,(\tilde{\gamma}_e \circ P,\, \tilde{\gamma}_e)|^2 \\
&\leq \text{Energy}\,((\tilde{\gamma}_e \circ P)|_{S_2}) + \text{Energy}\,(\tilde{\gamma}_e|_{S_2}) - 2\text{Energy}\left(\left(\frac{\tilde{\gamma}_e \circ P + \tilde{\gamma}_e}{2}\right)\Big|_{S_2}\right) \\
&= \int_{S_2} |(\tilde{\gamma}_e' \circ P)P'|^2 + \int_{S_2} |\tilde{\gamma}_e'|^2 - 2\int_{S_2}\left(\frac{|(\tilde{\gamma}_e' \circ P)P'| + |\tilde{\gamma}_e'|}{2}\right)^2 \\
&= \int_{S_2} \frac{(|\tilde{\gamma}_e' \circ P| \cdot |P'| - |\tilde{\gamma}_e'|)^2}{2} \leq \frac{L^2}{2}\int_{\mathbf{S}^1} |P' - 1|^2\,, \qquad \text{(A.6)}
\end{aligned}
$$

completing the proof of property (3).

To prove property (4) of $\Psi$, suppose it is not true, namely, there exist $\epsilon > 0$ and a sequence $\gamma_j \in \Lambda$ with $\text{Energy}(\Psi(\gamma_j)) \geq \text{Energy}(\gamma_j) - 1/j$ and $\text{dist}(\gamma_j, G) \geq \epsilon > 0$; note that the second condition implies a positive lower bound for $\text{Energy}(\gamma_j)$. Observe next that the space $\Lambda$ is compact[††] and, thus, a subsequence of the $\gamma_j$'s must converge to some $\gamma \in \Lambda$. Since property (3) implies that $\text{dist}(\gamma_j, \Psi(\gamma_j)) \to 0$, the $\Psi(\gamma_j)$'s also converge to $\gamma$. The continuity of $\Psi$, i.e., property (2) of $\Psi$, then implies that $\Psi(\gamma) = \gamma$. However, this implies that $\gamma \in G$ since the only fixed points of $\Psi$ are (possibly self-intersected) closed geodesics. This last fact follows immediately from Corollary A.0.10 and (A.2). However, this would contradict that the $\gamma_j$'s remain a fixed distance from any such closed geodesic, completing the proof of (4).

[††] Compactness of $\Lambda$ follows since $\sigma \in \Lambda$ depends continuously on the images of the $L^2$ break points in the compact metric space $X$.

# Appendix B

# Continuity of $\Psi$

**Lemma B.0.11.** *Let $\gamma : \mathbf{S}^1 \to (X, d)$ be a $W^{1,2}$ map with Energy$(\gamma) \leq \rho L$. If $\gamma_e$ and $\tilde{\gamma}_e$ are given by applying $(A_1)$ and $(B_1)$ to $\gamma$, then the map $\gamma \to \tilde{\gamma}_e$ is continuous from $W^{1,2}$ to $\Lambda$ equipped with the $W^{1,2}$ norm as in Remark 2.2.2.*

*Proof.* It follows from (4.11) and the energy bound that $d\,(\gamma(x_{2j}), \gamma(x_{2j+2})) \leq \rho$ for each $j$, and thus we can apply step $(A_1)$. Now suppose that $\gamma^1$ and $\gamma^2$ are non-constant curves in $\Lambda$ (continuity at the constant maps is obvious). For $i = 1, 2$ and $j = 0, 1, 2, ..., L^2 - 1$, let $a^i_j = d\,(\gamma^i(x_{2j}), \gamma^i(x_{2j+2}))$. Let $S^i = \frac{1}{2\pi}\sum_{j=0}^{L^2-1} a^i_j$ be the speed of $\tilde{\gamma}^i_e$, so that $|(\tilde{\gamma}^i_e)'| = S^i$ except at the $L^2$ break points. Since, by Remark 2.2.2, $W^{1,2}$ close curves are also $C^0$ close, it follows that the points $\gamma_e(x_{2j}) = \gamma(x_{2j})$ (identity map) are continuous with respect to the $W^{1,2}$ norm. Thus the $a^i_j$'s are continuous functions of $\gamma^i$, and so is each $S^i$. Moreover, the local energy convexity in Theorem 1.0.2 implies that the $\gamma^i_e$'s are indeed $W^{1,2}$ close on each interval $[x_{2j}, x_{2j+2}]$ if the $\gamma^i$'s are (since $\gamma^i_e\big|_{[x_{2j},x_{2j+2}]}$'s also stay within a $\rho$-neighborhood and the right-hand side of the energy convexity for them is just a continuous function of $S^i$'s). Thus, we have shown $\gamma \to \gamma_e$ is continuous.

To show $\gamma_e \to \tilde{\gamma}_e$ is also continuous, it suffices to show that the $\tilde{\gamma}^i_e$'s are close when the $\gamma^i_e$'s are. Since the point $x_0 = x_{2L^2}$ is fixed under the reparametrization, this will follow from applying Wirtinger's inequality to $d\,(\tilde{\gamma}^1_e, \tilde{\gamma}^2_e) - d\,(\tilde{\gamma}^1_e(x_0), \tilde{\gamma}^2_e(x_0))$ once we show that

$\int_{\mathbf{S}^1} |\nabla d\,(\tilde{\gamma}_e^1, \tilde{\gamma}_e^2)|^2$ can be made small.

The piecewise linear curve $\tilde{\gamma}_e^i$ is linear on the intervals

$$I_j^i = \left[ \frac{1}{S^i} \sum_{\ell < j} a_\ell^i \, , \, \frac{1}{S^i} \sum_{\ell \leq j} a_\ell^i \right] . \tag{B.1}$$

Set $I_j = I_j^1 \cap I_j^2$. Observe first that since the intervals $I_j^i$ in (B.1) depend continuously on $\gamma_e^i$, the measure of the complement $\mathbf{S}^1 \setminus \left[ \cup_{j=0}^{L^2-1} I_j \right]$ can be made small, so that

$$\int_{\mathbf{S}^1 \setminus [\cup I_j]} \left| \nabla d\,(\tilde{\gamma}_e^1, \tilde{\gamma}_e^2) \right|^2 \leq \int_{\mathbf{S}^1 \setminus [\cup I_j]} \left( |(\tilde{\gamma}_e^1)'| + |(\tilde{\gamma}_e^2)'| \right)^2 \leq 4\, L^2 \,\text{Length}\, \left( \mathbf{S}^1 \setminus [\cup I_j] \right) \tag{B.2}$$

can also be made small. We will divide the $I_j$'s into two groups, depending on the size of $a_j^1$. Fix some $\epsilon > 0$ and suppose first that $a_j^1 < \epsilon$; by continuity, we can assume that $a_j^2 < 2\epsilon$. For such $j$, we get

$$\int_{I_j} \left| \nabla d\,(\tilde{\gamma}_e^1, \tilde{\gamma}_e^2) \right|^2 \leq 2 \int_{I_j^1} \left| (\tilde{\gamma}_e^1)' \right|^2 + 2 \int_{I_j^2} \left| (\tilde{\gamma}_e^2)' \right|^2 \leq 2\, L \, \left( a_j^1 + a_j^2 \right) \leq 6\, \epsilon\, L \, . \tag{B.3}$$

Since there are at most $L^2$ breaks, summing over these intervals contributes at most $6\epsilon\, L^3$.

On the other hand, suppose now $a_j^1 \geq \epsilon$; by continuity we can assume that $a_j^2 \geq \epsilon/2$. In this case, $\tilde{\gamma}_e^i$ can be written on $I_j$ as the composition $\gamma_e^i \circ P_j^i$ where $\left| (P_j^i)' \right| = 2\pi\, S^i / (L^2 a_j^i)$. Furthermore, $P_j^1$ and $P_j^2$ both map $I_j$ into $[x_{2j}, x_{2j+2}]$ and arguing as (A.6) we have

$$\begin{aligned} \frac{1}{4} \int_{I_j} \left| \nabla d\,(\tilde{\gamma}_e^1, \tilde{\gamma}_e^2) \right|^2 &= \frac{1}{4} \int_{I_j} \left| \nabla d\,(\gamma_e^1 \circ P_j^1, \gamma_e^2 \circ P_j^2) \right|^2 \\ &\leq \frac{1}{2} \int_{I_j} \left( |(\gamma_e^1)'| \cdot |(P_j^1)'| - |(\gamma_e^2)'| \cdot |(P_j^2)'| \right)^2 . \end{aligned}$$

This can be made small since the speed $\left| (P_j^i)' \right|$ is continuous in $\gamma^i$ and the (piecewise constant) speeds $|(\gamma_e^i)'|$'s are close when $\gamma_e^i$'s are. Therefore, the integral over these intervals can also be made small since there are at most $L^2$ of them. $\square$

# Appendix C

# Proof of Lemma 4.1.2

If $(X, d)$ has curvature bounded from above by $K > 0$ in the sense of Alexandrov, then $(X, \frac{\sqrt{K}d}{\sqrt{\epsilon}})$ has curvature bounded from above by $\epsilon$, so that the local distance convexity in Lemma 4.1.2 is homogenous w.r.t. $K$. Hence, it suffices to assume the metric space has curvature bounded from above by $\epsilon$ which is sufficiently small.

Suppose now $K > 0$ is sufficiently small. Let $d_{\mathbf{S}_K}(A, B) = a$, $d_{\mathbf{S}_K}(C, D) = b$, $d_{\mathbf{S}_K}(A, D) = x$, $d_{\mathbf{S}_K}(B, C) = y$, $d_{\mathbf{S}_K}(E, F) = g$, $d_{\mathbf{S}_K}(E, D) = c$, $d_{\mathbf{S}_K}(A, F) = d$, $d_{\mathbf{S}_K}(B, F) = e$ and $d_{\mathbf{S}_K}(E, C) = f$ (see Figure 4.1). In the rest of this section we aim to prove the following that gives Lemma 4.1.2: for $a, b, x, y$ small enough (i.e., under small region assumption), we have the following inequality:

$$\frac{1}{4}(a-b)^2 \leq x^2 + y^2 - 2g^2 \,. \tag{C.1}$$

Based on Lemma 4.1.1, we first provide three key observations.

**Lemma C.0.12.** *For $x, y$ sufficiently small and some $\alpha, \beta \in \mathbf{R}$ , we have:*

$$W(x^2 + y^2 - 2g^2) = Sx^2 + Ty^2 + U + V - Rg^2 + O(x^2g^2) + O(y^2g^2) + O(g^4 + x^4 + y^4),$$

*where*

(1) $W = \frac{k^4}{6}(2 + \cos\frac{k}{2}a + \cos\frac{k}{2}b)\left(c^2 + d^2 + e^2 + f^2 - \frac{1}{4}(a^2 + b^2) - 2\alpha xg - 2\beta yg\right)$
$+ 2k^2(\cos\frac{k}{2}a + \cos\frac{k}{2}b) - \frac{k^2}{2}(\cos kc + \cos kd + \cos ke + \cos kf)\,,$

(2) $R = \frac{k^4}{2}(2 + \cos\frac{k}{2}a + \cos\frac{k}{2}b)\left(c^2 + d^2 + e^2 + f^2 - \frac{1}{3}(a^2 + b^2) - 2\alpha xg - 2\beta yg\right)$
$+ 6k^2(\cos\frac{k}{2}a + \cos\frac{k}{2}b) - 2k^2(\cos kc + \cos kd + \cos ke + \cos kf)\,,$

(3) $S = \frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(e^2+f^2-2\beta yg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b)+\frac{k^2}{2}(\cos kc+\cos kd-$
$\cos ke - \cos kf)\,,$

(4) $T = \frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2-2\alpha xg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b)-\frac{k^2}{2}(\cos kc+\cos kd-$
$\cos ke - \cos kf)\,,$

(5) $U = k^2(2 + \cos\frac{k}{2}a + \cos\frac{k}{2}b)\left(c^2 + d^2 + e^2 + f^2 - \frac{1}{2}(a^2 + b^2) - 2\alpha xg - 2\beta yg\right)\,,$

(6) $V = 8(\cos\frac{k}{2}a+1)(\cos\frac{k}{2}b+1)-4(\cos kc+1)(\cos kd+1)-4(\cos ke+1)(\cos kf+1)\,.$

*Proof.* Apply Lemma 4.1.1 to $\{A, E, F, D\}$, we can choose $K$ sufficiently small to be determined later so that for some $\alpha$ (with $k = \sqrt{K}$, $|\alpha|$ sufficiently small)

(C.2)

$$\alpha + (\frac{1}{4}a^2 + \frac{1}{4}b^2 - c^2 - d^2)/(2xg) = cosq_K(\overrightarrow{AD}, \overrightarrow{EF}) = cosq_K(\overrightarrow{DA}, \overrightarrow{FE})$$

(C.3)

$$= \frac{\cos\frac{k}{2}a + \cos kg \cos kx \cos\frac{k}{2}b + \cos\frac{k}{2}a\cos\frac{k}{2}b - \cos kg\cos kd - \cos kx \cos kc - \cos kc \cos kd}{(1+\cos\frac{k}{2}b)\sin kx \sin kg}$$

(C.4)

$$= \frac{\cos\frac{k}{2}b + \cos kg \cos kx \cos\frac{k}{2}a + \cos\frac{k}{2}a\cos\frac{k}{2}b - \cos kx\cos kd - \cos kg \cos kc - \cos kc \cos kd}{(1+\cos\frac{k}{2}a)\sin kx \sin kg}\,.$$

By Taylor series expansions for sine and cosine (in $x$ and $g$) and using (C.2)-(C.3), we have

$$\begin{aligned}
&\left[\frac{1}{4}(a^2+b^2) - c^2 - d^2 + 2\alpha xg\right](1+\cos\frac{k}{2}b)(kx - \frac{1}{6}(kx)^3 + O(x^5))(kg - \frac{1}{6}(kg)^3 + O(g^5)) \\
=&2xg\big[(\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b) + (\cos\frac{k}{2}b)(1 - \frac{1}{2}(kx)^2 + \frac{1}{24}(kx)^4 + O(x^6))(1 - \frac{1}{2}(kg)^2 + \frac{1}{24}(kg)^4 \\
&+O(g^6)) - (\cos kd)(1 - \frac{1}{2}(kg)^2 + \frac{1}{24}(kg)^4 + O(g^6)) - (\cos kc)(1 - \frac{1}{2}(kx)^2 + \frac{1}{24}(kx)^4 \\
&+O(x^6)) - \cos kc \cos kd\big]\,.
\end{aligned}$$

Combining the terms in $x^2$ and $g^2$ yields

$$\begin{aligned}
&\left[\frac{k^4}{6}(1+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2) - 2\alpha xg) - k^2\cos kd + k^2\cos\frac{k}{2}b\right]g^2 \\
=&-\left[\frac{k^4}{6}(1+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2) - 2\alpha xg) - k^2\cos kc + k^2\cos\frac{k}{2}b\right]x^2 \\
&+k^2(1+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg) + 2(1+\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b) \\
&-2(1+\cos kc)(1+\cos kd) + O(x^2g^2) + O(x^4+g^4)\,.
\end{aligned}$$

Similarly, using (C.2)-(C.4), we have

$$
\begin{aligned}
&\left[\frac{k^4}{6}(1+\cos\frac{k}{2}a)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)-k^2\cos kc+k^2\cos\frac{k}{2}a\right]g^2\\
=&-\left[\frac{k^4}{6}(1+\cos\frac{k}{2}a)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)-k^2\cos kd+k^2\cos\frac{k}{2}a\right]x^2\\
&+k^2(1+\cos\frac{k}{2}a)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)+2(1+\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b)\\
&-2(1+\cos kc)(1+\cos kd)+O(x^2g^2)+O(x^4+g^4)\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos kc-\cos kd)\big]g^2\\
=&-\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos kc-\cos kd)\big]x^2+k^2(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)\\
&+4(1+\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b)-4(1+\cos kc)(1+\cos kd)+O(x^2g^2)+O(x^4+g^4)\,.
\end{aligned}
$$

Similarly, in $\{B,E,F,C\}$ we have for some $\beta$ (with $|\beta|$ sufficiently small)

$$
\begin{aligned}
&\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(e^2+f^2-\frac{1}{4}(a^2+b^2)-2\beta yg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos ke-\cos kf)\big]g^2\\
=&-\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(e^2+f^2-\frac{1}{4}(a^2+b^2)-2\beta yg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos ke-\cos kf)\big]y^2+k^2(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(e^2+f^2-\frac{1}{4}(a^2+b^2)-2\beta yg)\\
&+4(1+\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b)-4(1+\cos ke)(1+\cos kf)+O(y^2g^2)+O(y^4+g^4)\,.
\end{aligned}
$$

Adding up the above two equations then yields

$$
\begin{aligned}
&\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2+e^2+f^2-\frac{1}{2}(a^2+b^2)-2\alpha xg-2\beta yg)\\
&+k^2(2\cos\frac{k}{2}a+2\cos\frac{k}{2}b-\cos kc-\cos kd-\cos ke-\cos kf)\big]g^2\\
=&-\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2-\frac{1}{4}(a^2+b^2)-2\alpha xg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos kc-\cos kd)\big]x^2\\
&-\big[\frac{k^4}{6}(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(e^2+f^2-\frac{1}{4}(a^2+b^2)-2\beta yg)+k^2(\cos\frac{k}{2}a+\cos\frac{k}{2}b\\
&-\cos ke-\cos kf)\big]y^2\\
&+k^2(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b)(c^2+d^2+e^2+f^2-\frac{1}{2}(a^2+b^2)-2\alpha xg-2\beta yg)\\
&+8(1+\cos\frac{k}{2}a)(1+\cos\frac{k}{2}b)-4(1+\cos kc)(1+\cos kd)-4(1+\cos ke)(1+\cos kf)\\
&+O(x^2g^2)+O(y^2g^2)+O(g^4+x^4+y^4)\,.
\end{aligned}
$$

The lemma follows immediately by rearranging the terms above and using the definitions of $W, R, S, T, U$. $\square$

*Remark* C.0.13. For fixed $a$, as $x, y \to 0$ (and thus $g \to 0, b \to a,$ and $c, d, e, f \to \frac{1}{2}a$), we have $\frac{S}{W} \to 1$ and $\frac{T}{W} \to 1$.

**Lemma C.0.14.** *For $a, b, x, y, k$ small enough, we have*

$$U - Rg^2 \geq \frac{k^2}{2}\,g^2 - 4k^2(2|\alpha|xg + 2|\beta|yg)\,.$$

*Proof.* Let $|\cdot|$ denote the Euclidean distance in $\mathbf{R}^3$ and $\angle EF'D = \theta$ (see Figure C.1). Then

$$|FF'| = \frac{1}{k}(1-\cos\frac{k}{2}b), \quad |FF''| = \frac{1}{k}(1-\cos kg) = \frac{2}{k}\sin^2\left(\frac{k}{2}g\right), \quad |EF''| = \frac{1}{k}\sin kg,$$

and

$$|EF'|^2 = |F'F''|^2 + |EF''|^2 = \frac{1}{k^2}\left(1 - \cos\frac{k}{2}b - 2\sin^2\left(\frac{k}{2}g\right)\right)^2 + \frac{1}{k^2}\sin^2(kg).$$

Thus,

$$\begin{aligned}
&|CE|^2 + |DE|^2 - (|CF|^2 + |DF|^2) \\
&= (|CF'| + |EF'|\cos\theta)^2 + (|EF'|\sin\theta)^2 \\
&\quad + (|CF'| - |EF'|\cos\theta)^2 + (|EF'|\sin\theta)^2 - 2|CF'|^2 - 2|FF'|^2 \\
&= 2(|EF'|^2 - |FF'|^2) \qquad \text{(C.5)} \\
&= \frac{2}{k^2}\left[\left(1 - \cos\frac{k}{2}b - 2\sin^2\left(\frac{k}{2}g\right)\right)^2 + \sin^2(kg) - \left(1 - \cos\frac{k}{2}b\right)^2\right] \\
&= \frac{2}{k^2}\left[\sin^2(kg) - 4\left(1 - \cos\frac{k}{2}b\right)\sin^2\left(\frac{k}{2}g\right) + 4\sin^4\left(\frac{k}{2}g\right)\right] \\
&= 2\left(\cos\frac{k}{2}b\right)g^2 + O(g^4) \;>\; \left(\cos\frac{k}{2}b\right)g^2,
\end{aligned}$$

for $b$, $g$ small, which also implies $|EF'| \geq |FF'|$.

Similarly, for $n \geq 2$,

$$\begin{aligned}
&|CE|^{2n} + |DE|^{2n} - (|CF|^{2n} + |DF|^{2n}) \\
&= \left[(|CF'| + |EF'|\cos\theta)^2 + (|EF'|\sin\theta)^2\right]^n \\
&\quad + \left[(|CF'| - |EF'|\cos\theta)^2 + (|EF'|\sin\theta)^2\right]^n - 2\left[|CF'|^2 + |FF'|^2\right]^n \qquad \text{(C.6)} \\
&\geq 2\left[(|CF'|^2 + |EF'|^2)^n - (|CF'|^2 + |FF'|^2)^n\right] \;\geq\; 0.
\end{aligned}$$

Now, note that

$$c = \frac{2}{k}\arcsin\left(\frac{k|DE|}{2}\right), \; f = \frac{2}{k}\arcsin\left(\frac{k|CE|}{2}\right),$$

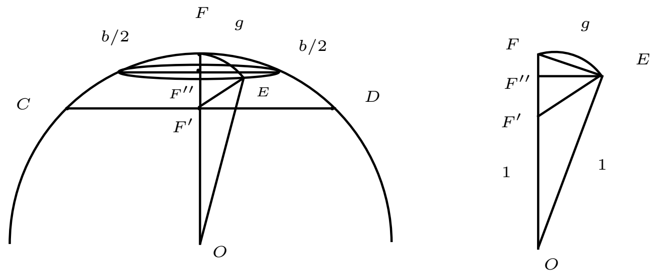



Figure C.1: Angle $\angle EF'D$

and

$$\frac{1}{2}b = \frac{2}{k}\arcsin\left(\frac{k|CF|}{2}\right) = \frac{2}{k}\arcsin\left(\frac{k|DF|}{2}\right).$$

The Taylor series expansion

$$\left(2\arcsin\left(\frac{x}{2}\right)\right)^2 = x^2 + \sum_{n=2} C_n x^{2n} \quad (C_n \geq 0)$$

and (C.5), (C.6) then imply: for $x, y$ small enough (thus $g$ is small enough),

$$\begin{aligned} & c^2 + f^2 - \frac{1}{2}b^2 \\ = & |CE|^2 + |DE|^2 - (|CF|^2 + |DF|^2) + \sum_{n=2} C_n k^{2n-2} \left(|CE|^{2n} + |DE|^{2n} - (|CF|^{2n} + |DF|^{2n})\right) \\ > & \left(\cos\frac{k}{2}b\right) g^2 . \end{aligned}$$

Similarly, $\quad d^2 + e^2 - \frac{1}{2}a^2 > (\cos\frac{k}{2}a)\, g^2$.

Therefore,

$$c^2 + d^2 + e^2 + f^2 - \frac{1}{2}(a^2 + b^2) > \left(\cos\frac{k}{2}a + \cos\frac{k}{2}b\right) g^2. \tag{C.7}$$

Recall the facts that as $x, y \to 0$

(1) $c^2+d^2+e^2+f^2-\frac{1}{3}(a^2+b^2) \;\to\; \frac{1}{6}(a^2+b^2)\,,$

(2) $\cos kc+\cos kd+\cos ke+\cos kf \;\to\; 2(\cos\frac{k}{2}a+\cos\frac{k}{2}b)\,.$

One observes that in a small geodesic ball $B_\tau$ (thus $a,b,x,y$ are small enough), we have:

(1) $\cos\frac{k}{2}a+\cos\frac{k}{2}b \;>\; \frac{15}{8}\,,$

(2) $c^2+d^2+e^2+f^2-\frac{1}{3}(a^2+b^2)-2\alpha xg-2\beta yg \;<\; \frac{1}{2}\,,$

(3) $\cos kc+\cos kd+\cos ke+\cos kf \;>\; \frac{3}{2}\left(\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)\,.$

Therefore, using (C.7) and the definition of $R$, we obtain that for $a,b,x,y$ small and $k\le 1$ :

$$\begin{aligned}
&U-Rg^2\\
=&k^2\left(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)\left(c^2+d^2+e^2+f^2-\frac{1}{2}(a^2+b^2)-2\alpha xg-2\beta yg\right)-Rg^2\\
\ge&\frac{15k^2}{8}\left(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)g^2-\left(\frac{k^4}{4}\left(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)+3k^2\left(\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)\right)g^2\\
&-k^2\left(2+\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)(2\alpha xg+2\beta yg)\\
\ge&k^2\left(\frac{13}{4}-\frac{11}{8}\left(\cos\frac{k}{2}a+\cos\frac{k}{2}b\right)\right)g^2-4k^2(2|\alpha|xg+2|\beta|yg)\\
\ge&\frac{k^2}{2}\,g^2-4k^2(2|\alpha|xg+2|\beta|yg)\,.
\end{aligned}$$

□

**Lemma C.0.15.** *$V\ge 0$ for $a,b,x,y$ small enough.*

*Proof.* By the triangle inequality, we know $\frac{1}{2}(a+b)<c+d<\frac{1}{2}(a+b)+x+g\,,\ \frac{1}{2}(a+b)<e+f<\frac{1}{2}(a+b)+y+g\,,\ c+f>b\,,$ and $d+e>a\,.$ If $c\ge\frac{1}{2}b\,,\ f\ge\frac{1}{2}b$ and $d\ge\frac{1}{2}a\,,\ e\ge\frac{1}{2}a$ then one easily sees $V\ge 0$. Now without loss of generality we suppose that there exists $\varsigma>\sigma>0$ such that

$$c=\frac{1}{2}b-\sigma,\quad d=\frac{1}{2}a+\varsigma,\quad e>\frac{1}{2}a-\varsigma,\quad f>\frac{1}{2}b+\sigma.$$

Assume now $k = 1$, then for $a, b$ small:

$$\begin{aligned}
V =&8(\cos\frac{1}{2}a+1)(\cos\frac{1}{2}b+1)-4(\cos c+1)(\cos d+1)-4(\cos e+1)(\cos f+1)\\
=&8\cos\frac{1}{2}b-4(\cos c+\cos f)+8\cos\frac{1}{2}a-4(\cos d+\cos e)+8(\cos\frac{1}{2}a)(\cos\frac{1}{2}b)\\
&-4(\cos c)(\cos d)-4(\cos e)(\cos f)\\
\geq&8\cos\frac{1}{2}b-4(\cos(\frac{1}{2}b-\sigma)+\cos(\frac{1}{2}b+\sigma))+8\cos\frac{1}{2}a-4(\cos(\frac{1}{2}a+\varsigma)+\cos(\frac{1}{2}a-\varsigma))\\
&+8(\cos\frac{1}{2}a)(\cos\frac{1}{2}b)-4\left[\cos(\frac{1}{2}b-\sigma)\cos(\frac{1}{2}a+\varsigma)+\cos(\frac{1}{2}b+\sigma)\cos(\frac{1}{2}a-\varsigma)\right]\\
=&8(1-\cos\sigma)\cos\frac{1}{2}b+8(1-\cos\varsigma)\cos\frac{1}{2}a+4(\cos\frac{a+b}{2}+\cos\frac{a-b}{2})\\
&-2\left[\cos(\frac{a+b}{2}+\varsigma-\sigma)+\cos(\frac{b-a}{2}+\varsigma+\sigma)+\cos(\frac{a+b}{2}-\varsigma+\sigma)+\cos(\frac{b-a}{2}-\varsigma-\sigma)\right]\\
\geq&4(1-\cos(\varsigma-\sigma))\cos\frac{a+b}{2}+4(1-\cos(\varsigma+\sigma))\cos\frac{b-a}{2}\ \geq\ 0\,.
\end{aligned}$$

One sees that the above argument works for all $k > 0$, completing the proof. □

*Remark* C.0.16. From the proofs of these lemmas we also see that for $a, b, x, y$ small enough, $W, S, T, R > 0$.

*Proof.* (of **Lemma 4.1.2**) Combining previous lemmas we get

$$W(x^2+y^2-2g^2) \geq Sx^2+Ty^2+\frac{k^2}{2}g^2-4k^2(2|\alpha|xg+2|\beta|yg)+O(x^2g^2)+O(y^2g^2)+O(g^4+x^4+y^4).$$

Now using nothing but the facts that

1. $0 < k^2 \leq W \leq 4k^2$ for $a, b, x, y, k$ sufficiently small,

2. $\frac{S}{W} \to 1$ and $\frac{T}{W} \to 1$ uniformly as $x, y \to 0$ (see Remark C.0.13),

3. $|a - b| \leq x + y$,

we obtain for $a, b, x, y$ sufficiently small:

$$x^2 + y^2 - 2g^2 \geq \frac{3}{4}(x^2 + y^2) + \frac{1}{16}g^2 - 4(2|\alpha|xg + 2|\beta|yg)\,. \tag{C.8}$$

Now choose $K$ sufficiently small so that $\max\{|\alpha|, |\beta|\} \leq \frac{1}{128}$ and thus by Cauchy-Schwarz inequality

$$4(2|\alpha|xg + 2|\beta|yg) \leq \frac{1}{16}(x^2 + y^2 + g^2)\,,$$

and therefore

$$x^2 + y^2 - 2g^2 \geq \frac{1}{2}(x^2 + y^2) \geq \frac{1}{4}(x+y)^2 \geq \frac{1}{4}(a-b)^2\,,$$

completing the proof of Lemma 4.1.2. $\square$

# Bibliography

[1] A.D. Alexandrov, A theorem on triangles in a metric space and some of its applications, Trudy Mat.Inst.Steklova 38 (1951), 5–23.

[2] A.D. Alexandrov, Über eine Verallgemeinerung der Riemannschern Geometrie, Schr. Forschungsinst. Math. 1 (1957), 33–84.

[3] I.D. Berg and I.G. Nikolaev, On an extremal property of quadrilaterals in an Alexandrov space of curvature $\leq K$, Contemporary mathematics; 424. Providence, R.I: AMS, c2007.

[4] G.D. Birkhoff, Dynamical systems with two degrees of freedom, Trans. Amer. Math. Soc. 18 (1917), no. 2, 199–300.

[5] G.D. Birkhoff, Dynamical systems, Amer. Math. Soc. Colloq. Publ. vol 9, Providence, RI, 1927.

[6] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, American Mathematical Society, 2001.

[7] V. I. Burenkov and V. A. Gusakov, On exact constants in Sobolev embedding theorems, English translation: Soviet Math. Dokl. 35 (1987), no. 3, 651–655.

[8] H. Busemann, The geometry of geodesics, Acad. Press, 1955.

[9] E. Cartan, Leçons sur la Géométrie des Espaces de Riemann., Gauthier-Villars, Paris, 1928, 2nd edition 1951.

[10] P. Chone, A regularity result for critical points of conformally invariant functionals, Potential Anal., 4 (1995), 269-296.

[11] D. Chang, S. G. Krantz and E.M. Stein, $H^p$ theory on a smooth domain in $R^N$ and elliptic boundary value problems, J. Funct. Anal. 114 (1993), no. 2, 286–347.

[12] R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures Appl. (9) 72 (1993), no. 3, 247–286.

[13] T.H. Colding and C. De Lellis, The min-max construction of minimal surfaces, Surveys in differential geometry, Vol. 8, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3–5, 2002, Sponsored by JDG, (2003) 75–107.

[14] T.H. Colding and W.P. Minicozzi II, Width and mean curvature flow, Geom. Topol. 12 (2008), no. 5, 2517–2535.

[15] T.H. Colding and W.P. Minicozzi II, Width and finite extinction time of Ricci flow, Geom. Topol. 12 (2008), no. 5, 2537–2586.

[16] T.H. Colding and W.P. Minicozzi II, Estimates for the extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman, J. Amer. Math. Soc. 18 (2005), no. 3, 561–569.

[17] C.B. Croke, Area and the length of the shortest closed geodesic, J. Differential Geom. 27 (1988), no. 1, 1–21.

[18] J. Eells and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. no. 86 (1964) 109-160.

[19] M.A. Grayson, Shortening embedded curves, Ann. of Math. (2) 129 (1989), no. 1, 71–111.

[20] M. Gruber, Harnack inequalities for solutions of general second order parabolic equations and estimates of their Hölder constants, Math. Z. 185 (1984), no. 1, 23–43.

[21] J. Hadamard, Les surfaces à courbures opposées et leurs lignes géodésique, J. Math. Pure Appl., 4 (1898) pp. 27-75.

[22] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. xii+544 pp. ISBN: 0-521-79160-X; 0-521-79540-0.

[23] F. Hélein, Harmonic maps, conservation laws, and moving frames, Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[24] F. Hélein and J. Wood, *Harmonic maps*, Handbook of global analysis, 417–491, 1213, Elsevier Sci. B. V., Amsterdam, 2008.

[25] N. Korevaar and R. Schoen, Sobolev Spaces and Harmonic Maps for Metric Space Targets, Communications in Analysis and Geometry, 1 (1993), 561–659.

[26] T. Lamm and L. Lin, Estimates on the energy density for critical points of a class of conformally invariant variational problems, preprint 2012.

[27] T. Lamm and T. Rivière, Conservation laws for fourth order systems in four dimensions, Communications in Partial Differential Equations, 33 (2008), no.2, 245-262.

[28] G.M. Lieberman, Second order parabolic differential equations, World Scientific Publishing Co., Inc., River Edge, NJ, 1996 (revised edition 2005).

[29] L. Lin, Closed geodesics in Alexandrov spaces of curvature bounded from above, Journal of Geometric Analysis, Volume 21, Issue 2 (2011), 429–454.

[30] L. Lin and L. Wang, Existence of good sweepouts on closed manifolds, Proc. Amer. Math. Soc. 138 (2010), No. 11, 4081-4088.

[31] L.A. Lyusternik, Topology of functional spaces and calculus of variations in the large (Russian), Trudy Inst. Steklov. 19, Izdat. Akad. Nauk SSSR, Moscow, 1947; The topology of the calculus of variations in the large, translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 16 American Mathematical Society, Providence, R.I. 1966 vii+96 pp.

[32] L.A. Lyusternik and L. Šnirel'man, Topological methods in variational problems and their application to the differential geometry of surfaces, (Russian) Uspehi Matem. Nauk (N.S.) 2, (1947). no. 1(17), 166–217.

[33] C. Mese, Uniqueness theorems for harmonic maps into metric spaces, Commun. Contemp. Math. 4 (2002), no. 4, 725–750.

[34] A. Miyachi, $H^p$ spaces over open subsets of $\mathbf{R}^n$, Studia Math. 95 (1990), no. 3, 205–228.

[35] S.K. Ottarsson, Closed geodesics on Riemannian manifolds via the heat flow, J. Geom. Phys. 2 (1985), no. 1, 49–72.

[36] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three manifolds, math.DG/0307245.

[37] H. Poincaré, Sur les lignes géodésiques des surfaces convexes, Trans. Amer. Math. Soc., 6 (1904), pp. 237-274.

[38] J. Qing, Boundary regularity of weakly harmonic maps from surfaces, J. Funct. Anal. 114 (1993), no. 2, 458–466.

[39] Y.G. Reshetnyak, Non-expanding mappings in a space of curvature not greater than K, Sibirsk.Mat. Zh., s9, 918C927 (1968) (in Russian); English translation: Sib.Math. J. 9, 683–689 (1968).

[40] T. Rivière, Conservation laws for conformally invariant variational problems, Inventiones Mathematicae, v.168 (2007), no.1, 1-22.

[41] T. Rivière, The role of Integrability by Compensation in Conformal Geometric Analysis, to appear in Analytic aspects of problems from Riemannian Geometry S.M.F. (2008).

[42] T. Rivière and M. Struwe, Partial regularity for harmonic maps and related problems, Comm. Pure Appl. Math. 61 (2008), no. 4, 451–463.

[43] A. Schikorra, A remark on Gauge transformations and the moving frame method, Annales de l'Institut Henri Poincaré - Analyse non linéaire (2009), doi:10.1016/j.anihpc.2009.09.004.

[44] S. Semmes. A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller, Commun. Partial Differ. Equations, 19(1-2), 277C319, 1994.

[45] T. Serbinowski, Harmonic maps into metric spaces with curvature bounded above, Ph.D. thesis, University of Utah, 1995.

[46] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv. 60 (1985), no. 4, 558–581.

[47] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 34. Springer-Verlag, Berlin, 1996.

[48] K. Uhlenbeck, Connections with $L^p$ bounds on curvature, Comm. Math. Phys. v. 83 (1982), no. 1, 31-42.

[49] H. Wente, An existence theorem for surfaces of constant mean curvature, J. Math. Anal. Appl. 26 (1969), 318-344.

# Vitae

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