# EXPECTED EULER CHARACTERISTIC OF EXCURSION SETS OF RANDOM HOLOMORPHIC SECTIONS ON COMPLEX MANIFOLDS

by

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# Abstract

We prove a formula for the expected euler characteristic of excursion sets of random sections of powers of an ample bundle $(L, h)$, where $h$ is a Hermitian metric, over a Kähler manifold $(M, \omega)$. We then prove that the critical radius of the Kodaira embedding $\Phi_N : M \to \mathbb{CP}^n$ given by an orthonormal basis of $H^0(M, L^N)$ is bounded below when $N \to \infty$. This result also gives conditions about when the preceding formula is valid.

READERS: Bernard Shiffman(Advisor), Vyacheslav Shokurov.

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# Contents

# Chapter 1

# Introduction

Random complex geometry, as a branch of mathematics, can be considered as a sub-branch of two different fields. On one hand, it is a branch of random real geometry. Because complex structure is much more subtle, in the setting of complex manifolds, we can expect finer or more explicit formulas or conclusions. On the other hand, it is a branch of complex geometry that adopts ideas from statistics. With ideas, we can get new invariants or characterizations of complex manifolds or complex vector bundles.

Let $M$ be a Kähler manifold of dimension $m$. And let $L \to M$ be an ample line bundle with positively curved metric $h$. Take the induced Kähler form $\omega = \frac{i}{2}\Theta_h$ on $M$. We denote by $L^N$ the $N$th tensor power $L^{\otimes N}$ of $L$. Take the induced metric on $L^N$, by abuse of notation, also denoted by $h$. This induces a Hermitian inner product in $H^0(M, L^N)$ which denotes the space of holomorphic sections of $L^N$, given by

$$< \sigma_1, \sigma_2 >= \frac{1}{m!} \int_M h(\sigma_1, \sigma_2)\omega^m$$

In particular, the $L^2$ norm of a section in $s \in H^0(M, L^N)$ is given by

$$|s|_h^2 = \frac{1}{m!} \int_M |s(z)|_h^2 \omega^m$$

We consider random sections in the unit sphere $S_L^N$ in $H^0(M, L^N)$ with probability measure given by the spherical volume normalized so that $Vol(S_L^N) = 1$. For $s \in S_L^N$, the zero locus $Z_s = \{z \in M | s(z) = 0\}$ is very well studied in [12, 13, 14, 16]. It is also interesting to understand the excursion sets $\{z \in M || s(z)|_h > u\}$. In particular, what is $E\chi(|s(z)|_h > u) = \int_{S_L^N} \chi(|s(z)|_h > u) ds$, the expected Euler characteristic of the excursion sets, and what is the probability that the excursion set is non-empty? Here and in the following we denote by $\chi(S)$ the Euler characteristic of a topological space $S$.

It turned out that it is more natural to normalize the excursion sets to be of the form $\{\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > u\}$, where $\Pi_N(z, z)$ is the Szegö kernel of $H^0(M, L^N)$. The Szegö kernel is in general not constant but of the form

$$\Pi_N(z, z) = \frac{N^m}{\pi^m}(1 + O(\frac{1}{N}))$$

([5, 18, 19]), . By the definition of $\Pi_N(z, z)$, we always have $\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} \leq 1$. In fact $\sup_{|s|_h=1} |s(z)|_h^2 = \Pi_N(z, z)$( [4] ). Therefore when $u > 1$, the excursion sets are empty.

In this paper, we will mainly prove two theorems.

The first theorem is interesting in itself. Also it shows that in order to have a nice formula for the expected Euler characteristic we do not need to make $u$ too close to 1.

**Theorem 1.0.1.** *Let $\Phi_N : M \to \mathbb{CP}^n$ be an embedding given by an orthonormal basis of $H^0(M, L^N)$. Let $r_N$ be the critical radius of $\Phi_N(M)$ considered as a submanifold of $\mathbb{CP}^n$. Then there exists a constant $\rho_0(L, h) > 0$ such that $r_N > \rho_0(L, h)$ for all positive integer $N$.*

The proof of this theorem depends mainly on the approximation of the normalized Szegö kernel defined and proved in [12]. The idea is based on the sense that the information of the embedding $\Phi_N : M \to \mathbb{CP}^n$ is totally contained in the normalized Szegö kernel.

The second one is to answer the question about expected Euler characteristic of the normalized excursion set.

**Theorem 1.0.2.** *Let $\Phi_N : M \to \mathbb{CP}^n$ be an embedding given by an orthonormal basis of $H^0(M, L^N)$. Then there exists $\rho_0 > 0$ independent of $N$, such that for $0 \le \rho < \rho_0$, the set $\{z \in M | \frac{|s(z)|_h}{\sqrt{\prod_N(z,z)}} > \cos\rho\}$ is either empty or contractible, therefore*

$$E\chi(\frac{|s(z)|_h}{\sqrt{\prod_N(z,z)}} > \cos\rho) = Prob.\{\sup_{z\in M} \frac{|s(z)|_h}{\sqrt{\prod_N(z,z)}} > \cos\rho\}$$

*Hence the following formula*

$$E\chi(\frac{|s(z)|_h}{\sqrt{\prod_N(z,z)}} > \cos\rho) = \int_M c(M)(1 - Nc_1(L)) \wedge (Nc_1(L)\cos^2\rho + \sin^2\rho)^n \tag{1.1}$$

*Where $c_1(L)$ is the first Chern class of $L$ and $c(M)(1 - Nc_1(L))$ is the Chern polynomial evaluated at $1 - Nc_1(L)$*

When $M$ is a Riemann surface, we have a more explicit formula

**Theorem 1.0.3.** *Let $M$ be a Riemann surface. Then, with the notations above, there exists $\rho_0 > 0$ such that for $u > \cos\rho_N$ and a random section $s(z) \in H^0(M, L^N)$ the expected Euler characteristic*

$$\begin{aligned} E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}} > u) &= (1-u^2)^{(n-1)}[N^2(\deg L)^2u^2 \\ &- N\deg L(gu^2 - 1 + u^2) + (2-2g)(1-u^2)] \end{aligned} \tag{1.2}$$

*for $N\deg(L) > 2g - 2$, where $n = N\deg(L) - g$*

When $M$ is higher dimensional, we get an estimate

**Theorem 1.0.4.** *With the notations above, for $m \ge 1$ and for $N$ big enough*

$$E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}} > u) = (1 + o(1))n^{m+1}(1-u^2)^{n-m}u^{2m}$$

*where*

$$n = \dim H^0(M, L^N) - 1 = \frac{\int_M c_1^m(L)}{m!} N^m + O(N^{m-1})$$

*where the second equality follows from the asymptotic Riemann-Roch formula.*

Our results are complementary to results on excursion probabilities for Gaussian fields (see [17, 11] ) where the probability of large $L^2$ norms plays a role. Here, we consider only sections with $L^2$ norm 1.

Notice that by our estimation, $E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}} > u)$ decays to 0 very rapidly (exponentially) as $N$ goes to $\infty$. It is helpful to compare this observation with the following theorem from [15], which we state using our notations

**Theorem 1.0.5.** *(Theorem 1.1, [15])Let $\nu_N$ denote the measure on $S_L^N$ induced by the metric $ds$. For any integer $k$, there exist constants $C > 0$ depending on $k$, such that*

$$\nu_N\{s_N \in S_L^N : \sup_{z\in M} |s_N(z)|_h > C\sqrt{\log N}\} < O(\frac{1}{N^k})$$

Normalizing the above formula using $\sqrt{\Pi_N(z,z)}$, and by the estimation of $\sqrt{\Pi_N(z,z)}$, we have

$$\nu_N\{s_N \in S_L^N : \sup_{z\in M} \frac{|s_N(z)|_h}{\sqrt{\Pi_N(z,z)}} > \frac{C\sqrt{\log N}}{N^{m/2}}\} < O(\frac{1}{N^k})$$

the term $\frac{C\sqrt{\log N}}{N^m}$ is very small when $N$ is big. But the estimation we made requires $u$ close to 1, so in this sense our estimation is weaker, although it is more explicit.

It should be mentioned here that in the proof of Theorem 1.0.2, we make $\rho_0$ small enough so that the Euler characteristic of an excursion set is either 1 or 0. The proof of Theorem 1.0.2 uses the volume of tubes formula of Gray ([8][9]), which does not hold past the critical radius. For this case, one would need different methods. A related problem is to find the expected Euler characteristic for excursion sets (of any height) for Gaussian random holomorphic sections. Methods of Adler-Taylor ([17]) and [6][7] could be used for this problem, but this requires extensive computations.

This thesis is organized as follows: first in chapter 2 we introduce the definition of Szegö Kernel and state several results from [14],[12],and [19]. In chapter 3 we prove the formula in theorem 1.0.2. In chapter 4 we will analyze the critical radius, by using the results stated in chapter 2 first in the case of Riemann surfaces and then generalizing to the higher dimensional case.

# Chapter 2

# Background

## 2.1 Szegö Kernel

We will follow the notations and arguments in [12] and [14]

Let $L \to M$ be a positive line bundle over a compact Kähler manifold $M$. The associated principle sphere bundle is defined as follows. Let $\pi : L^* \to M$ be the dual bundle to $L$ with dual metric $h^*$. And put $X = \{v \in L^* : \| v \|_{h^*} = 1\}$. Let $r_\theta x = e^{i\theta} x (x \in X)$ denote the $S^1$ action on $X$. Now $X$ is the boundary of a pseudoconvex domain. Anti-holomorphic tangent vectors on $X$ are the anti-holomorphic tangent vectors of $L^{-1}$ that are tangent to $X$. Derivatives by vector fields with anti-holomorphic tangent vectors are called CR-derivatives. A smooth function on $X$ that vanishes under CR-derivatives is called CR-holomorphic.

Now any section $s \in H^0(M, L^N)$ is lifted to an equivariant function $\hat{s}$ on the circle bundle $\pi : X \to M$ with respect to $h$ by the rule

$$\hat{s}(\lambda) = (\lambda^{\otimes N}, s(z)), \lambda \in X_z$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. Notice that $\hat{s}$ satisfies $\hat{s}(e^{i\theta} x) = e^{Ni\theta} \hat{s}(x)$. We denote by $\mathcal{H}^2_N(X)$

the space of CR-holomorphic functions satisfying this homogeneous property. The Szegö projector is the orthogonal projection $L^2(X) \to \mathcal{H}^2_N(X)$. Let $(s_j^N) \subset H^0(M, L^N)$ be an orthonormal basis. The Szegö kernel, which gives the Szegö projector, is given by

$$\Pi_N(x,y) = \sum_{i=0}^{n} \hat{s}_j^N(x)\overline{\hat{s}_j^N(y)} \qquad (x, y \in X)$$

The normalized Szegö kernel is defined as

$$P_N(z,w) := \frac{|\prod_N(z,w)|}{\prod_N(z,z)^{1/2}\prod_N(w,w)^{1/2}}$$

where

$$|\Pi_N(z,w)| := |\Pi_N(x,y)|, \quad z = \pi(x), w = \pi(y) \in M. \tag{2.1}$$

This is independent of the choice of $x$ and $y$, since for different choices of preimages, $\Pi_N(x,y)$ would only be different by some $e^{i\theta}$.

On the diagonal we have

$$\Pi_N(z,z) = \sum_{i=0}^{n} \parallel \mathbf{s}_j^N(z) \parallel_h^2, z \in M$$

The following theorems were proved in [19]

**Theorem 2.1.1** ([19])**.** *Let $M$ be a compact complex manifold of dimension $m$(over $\mathbb{C}$) and let $(L,h) \to M$ be a positive hermitian holomorphic line bundle. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g := \pi Ric(h)$. For each $N \in \mathbb{N}$, $h$ induces a hermitian metric $h_N$ on $L^{\otimes N}$. Let $\{S_0^N, \cdots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^{\otimes N})$, with respect to the inner product $< s_1, s_2 >_{h_N} = \int_M h_N(s_1(z), s_2(z))dV_g$. Here, $dV_g = \frac{1}{m!}\omega_g^m$ is the volume form of $g$. Then there exists a complete asymptotic* *expansion:*

$$\sum_{i=0}^{d_N} \| S_i^N(z) \|_{h_N}^2 \sim a_0 N^m + a_1(z) N^{m-1} + \cdots$$

*for certain smooth coefficients* $a_j(z)$ *with* $a_0 = \frac{1}{\pi^m}$*. More precisely, for any* $k$

$$|\sum_{i=0}^{d_N} \| S_i^N(z) \|_{h_N}^2 - \sum_{j<R} a_j(x) N^{m-j}|_{C^k} \leq C_{R,k} N^{m-R}$$

At a point $z_0 \in M$, we choose a neighborhood $U$ of $z_0$, a local normal coordinate chart $\rho : U, z_0 \to \mathbb{C}^m$, 0 centered at $z_0$, and a preferred local frame at $z_0$, which was defined in [14] to be a local frame $e_L$ such that

$$|e_L(z)|_h = 1 - 1/2|\rho(z)|^2 + \cdots .$$

The following theorem was proved in [12]

**Theorem 2.1.2** ([12], Proposition 2.7)**.** *Let* $(L, h) \to (M, \omega)$ *be a positive Hermitian holomorphic line bundle over a compact m-dimensional Kähler manifold* $M$*. We give* $H^0(M, L^N)$ *the Hermitian Gaussian measure induced by* $h$ *and the Kähler form* $\omega = \frac{i}{2}\Theta_h$*. And let* $P_N(z, w)$ *be the normalized Szegö kernel for* $H^0(M, L^N)$ *and let* $z_0 \in M$ *For* $b, \varepsilon > 0, j \geq 0$*, there is a constant* $C_j = C_j(M, \varepsilon, b)$*, independent of the point* $z_0$*, such that*

$$P_N(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}) = e^{-\frac{1}{2}|u-v|^2}[1 + R_N(u, v)] \tag{2.2}$$

$$| \bigtriangledown^j R_N(u, v)| \leq C_j N^{-1/2+\varepsilon} \quad for \quad |u| + |v| < b\sqrt{logN}$$

As a corollary we have

**Theorem 2.1.3** ([12], Proposition 2.8). *The remainder $R_N$ in the above theorem satisfies*

$$(2.3) \qquad |R_N(u,v)| \leq \frac{C_2}{2}|u-v|^2 N^{-\frac{1}{2}+\varepsilon}, \quad |\bigtriangledown R_N(u,v)| \leq C_2|u-v|N^{-\frac{1}{2}+\varepsilon},$$
$$for \quad |u|+|v| < b\sqrt{logN}.$$

**Theorem 2.1.4** ([12], Proposition 2.6). *With the notations above, for $b > \sqrt{j+2k}$, $j,k \geq 0$, we have*

$$\bigtriangledown^j P_N(z,w) = O(N^{-k}) \quad uniformly \quad for \quad \mathrm{dist}(z,w) \geq b\sqrt{\frac{\log N}{N}}.$$

*Remark:* This theorem implies that the distance between the images of any two points in $M$ that are not very close to each other is close to $\pi/2$. Therefore, when estimating the critical radius of the Kodaira embedding $\Phi_N : M \to \mathbb{CP}^N$ we need only to consider only points that are very close to each other.

## 2.2 Geometry of complex projective spaces

For any integer $n \geq 0$, the complex projective space $\mathbb{CP}^n$ is the space consisting of all complex lines in $\mathbb{C}^{n+1}$ passing through the origin. Points in $\mathbb{CP}^n$ can be described in homogeneous coordinates $[Z_0, Z_1, \cdots, Z_n]$ with $(Z_0, Z_1, \cdots, Z_n) \in \mathbb{C}^{n+1}\backslash\{0\}$. Colinear vectors correspond to a same point, namely for $0 \neq \lambda \in \mathbb{C}$

$$[Z_0, Z_1, \cdots, Z_n] = [\lambda Z_0, \lambda Z_1, \cdots, \lambda Z_n]$$

Consider the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. The unit circle $S^1 \subset \mathbb{C}$ acts naturally as rotations on $S^{2n+1}$ by multiplication. Then $\mathbb{CP}^n$ is identified as the quotient space

$$\mathbb{CP}^n = S^{2n+1}/S^1$$

The well-known Fubini-Study metric $d_{FS}$ on $\mathbb{CP}^n$ is the Riemannian metric $ds^2_{FS}$ induced as the quotient metric of the natural round metric on $S^{2n+1}$ the sphere of radius 1/2.

Considering $CP^n$ as a complex manifold, the Fubini-Study metric $ds^2_{FS}$ is the real part of a Hermitian metric, which in local coordinates has the form

$$h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

where

$$h_{i\bar{j}} = \frac{(1+|z|^2)\delta_{ij} - z_i \bar{z}_j}{(1+|z|^2)^2}$$

The fundamental form $\omega$ of $h$ is a Kähler form with

$$\omega = i\partial\bar{\partial} \log |Z|^2$$

where $|Z|^2 = \sum_{i=0}^{n} |Z_i|^2$ in homogeneous coordinates.

In $\mathbb{CP}^n$ a subset is called a linear subspace if it is the image of a linear subspace of $\mathbb{C}^{n+1}$ passing through the origin under the projection

$$\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n$$

It is obvious that a linear subspace of $\mathbb{CP}^n$ can be naturally identified with $\mathbb{CP}^k$ for some $0 \leq k \leq n$. Now it is clear that for any two points $p, q \in \mathbb{CP}^n$, there exists an unique linear subspace $\mathbb{CP}^1$, denoted by $l_{pq}$ connecting these two points.

**Lemma 2.2.1.** *Given two distinct points $p, q \in \mathbb{CP}^n$ the geodesic connecting both points lies in $l_{pq}$.*

*Proof.* Since the distance of $p$ and $q$ in $l_{pq}$ is the same as that in $\mathbb{CP}^n$. □

*Remark:* With $\mathbb{CP}^1$ identified with the unit sphere $S^2$ with round metric, the pictures of geodesics are very clear. In particular, for each point $a \in \mathbb{CP}^1$ there exists an unique "opposite" point $b$ such that $d_{FS}(a, b) = \pi/2$

Recall a submanifold $M$ in a Riemannian manifold $N$ is called geodesic if for every two points $p, q \in M$ the geodesic in $N$ connecting $p$ and $q$ is contained in $M$

**Corollary 2.2.2.** *All linear subspaces in $\mathbb{CP}^n$ are geodesic.*

**Corollary 2.2.3.** *Let $p, q \in \mathbb{CP}^n$ be two points with homogeneous coordinates*

$$p = [Z_0, Z_1, \cdots, Z_n], \quad q = [Z'_0, Z'_1, \cdots, Z'_n]$$

*with*

$$Z = (Z_0, Z_1, \cdots, Z_n), \quad Z' = (Z'_0, Z'_1, \cdots, Z'_n) \in \mathbb{C}^{n+1} \backslash \{0\}$$

*Then their distance under the Fubini-Study metric has the following form*

$$\cos d_{FS}(p, q) = \frac{|Z \cdot Z'|}{|Z||Z'|}$$

*where $Z \cdot Z'$ denotes the Hermitian inner product in $\mathbb{C}^{n+1}$*

Now fix a point $w = [W_0, W_1, \cdots, W_n] \in \mathbb{CP}^n$, the points whose distances to $w$ are $\pi/2$ form a hyperplane $H(w)$ which is defined by the equation

$$\sum_{i=0}^{n} \bar{W}_i Z_i = 0$$

The projection from $w$ to $H(w)$ defines a holomorphic map $\mathcal{O}_w$ from $\mathbb{CP}^n \backslash w$ to $H(w)$, which geometrically is as follows: For any $z \in \mathbb{CP}^n \backslash w$, the complex line $l_{zw}$ intersects $H(w)$ at one unique point, which is just the image $\mathcal{O}_w(z)$ .

Recall that two submanifolds are said to be orthogonal to each other at their intersections if their tangent spaces, considered as subspace of the tangent space of $\mathbb{CP}^n$ are orthogonal to each other at those points. Since two general linear subspaces of complimentary dimensions intersect at exactly one point, we say that they are orthogonal to each other if they are orthogonal at that intersection.

**Proposition 2.2.4.** *For any point $z \in \mathbb{CP}^n \backslash w$, the line $l_{zw}$ is orthogonal to $H(w)$. Conversely, any line that is orthogonal to $H(w)$ passes through $w$*

*Proof.* One just needs to consider the special case when $w = [1, 0, \cdots, 0$ and $z = [0, 1, 0, \cdots, 0]$, since the actions of $U(n+1)$ fixing $w$ is transitive on $H(w)$ □

Now we consider a general submanifold. By a germ of a submanifold at a point $z_0$ we will mean a complex submanifold in a neighborhood of $z_0$ in $\mathbb{CP}^n$. Let $(M, z_0)$ be a germ of submanifold of dimension $m$, let $F : (\mathbb{C}^m, 0) \to (M, z_0)$ be a local parameter. In homogeneous coordinates,

$$F(x) = [F_0(x), F_1(x), \cdots, F_n(x)]$$

A first observation is that there exists an unique linear subspace $T_{z_0}$ of $\mathbb{CP}^n$ of dimension $m$ such that the tangent spaces at $z_0$ of $M$ and $T_{z_0}$ are identical considered as subspace of the tangent space of $\mathbb{CP}^n$ at $z_0$. We call $T_{z_0}$ the tangent subspace of $(M, z_0)$.

Recall that the span of a set of finite points in $\mathbb{CP}^n$ is the smallest linear subspace of $\mathbb{CP}^n$ that contains all these points.

**Proposition 2.2.5.** *$T_{z_0}$ is spanned by $z_0$ and*

$$[\frac{dF_0}{dx_i}(0), \frac{dF_1}{dx_i}(0), \cdots, \frac{dF_n}{dx_i}(0)]$$

*for $1 \leq i \leq m$, where $x_i$ is the $i-th$ coordinate in $\mathbb{C}^m$*

*Proof.* By considering one coordinate at a time, it is clear that we need only to show the 1-dimensional case.

Without lose of generality, we assume $F(0) \neq 0$. Then in the affine open set $U_0 = \{Z_0 \neq 0\}$, $F$ is

$$(\frac{F_1}{F_0}, \frac{F_2}{F_0}, \cdots, \frac{F_n}{F_0})$$

Then the complex line in $U_0$ that is tangent to $M$ at $z_0 = (\frac{F_1}{F_0}(0), \frac{F_2}{F_0}(0), \cdots, \frac{F_n}{F_0}(0))$ is of the form

$$(\frac{F_1}{F_0}(0), \frac{F_2}{F_0}(0), \cdots, \frac{F_n}{F_0}(0)) + t(\frac{d}{dx}\frac{F_1}{F_0}(0), \frac{d}{dx}\frac{F_2}{F_0}(0), \cdots, \frac{d}{dx}\frac{F_n}{F_0}(0))$$

for $t \in \mathbb{C}$

Since $\frac{d}{dx}\frac{F_i}{F_0}(0) = \frac{F_i'(0)}{F_0(0)} - \frac{F_i(0)F_0'(0)}{F_0^2(0)}$, we get that $(\frac{F_1'(0)}{F_0'(0)}, \frac{F_2'(0)}{F_0'(0)}, \cdots, \frac{F_1'(0)}{F_0'(0)})$ is on this complex line by letting $t = \frac{F_0(0)}{F_0'(0)}$. Switching to homogeneous coordinates, we get the conclusion. $\square$

# Chapter 3

# Expected Euler Characteristic

Let $s_N^j \in H^0(M, L^N), 0 \leq j \leq n$, where $n+1 = \dim(H^0(M, L^N))$ be an orthonormal basis. By the Kodaira embedding theorem, for $N$ big enough this gives an embedding $\Phi_N : M \to \mathbb{CP}^N$, locally given by $\Phi_N(x) = [f_N^0(x), f_N^1(x), \cdots, f_N^n(x)]$, where $s_N^j = f_N^j e_L^N$ with $e_L$ a local frame of $L$. For a random section with norm 1,

$$s = \sum_{i=0}^{N} c_i s_n^i, \quad \sum_{i=0}^{N} \parallel c_i \parallel^2 = 1$$

Let $|s(z)|_h$ denote the norm of $s$ at $z \in M$ under the metric induced by $h$. Let $C = (c_i)$ and $f_N(z) = (f_N^i(z))$.Then $\sum_{i=0}^{N} |s_N^i(z)|_h^2 = |f_N(z)|^2 |e_L(z)|_h^{2N}$

$$|s(z)|_h = |C \cdot f_N(z)||e_L|^N = \frac{C \cdot f_N(z)}{|C||f_N(z)|} \sqrt{\sum_{i=0}^{N} |s_N^i(z)|_h^2}$$

By definition $\Pi_N(z,z) = \sum_{i=0}^{n} |s_N^i(z)|_h^2$ is just the Szego kernel for $H^0(M, L^N)$ on the diagonal.

Therefore we have

$$\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} = \frac{|C \cdot \Phi_n(z)|}{|C||\Phi_n(z)|}$$

By identifying $\mathbb{CP}^1$ with the sphere in $\mathbb{R}^3$ of radius $1/2$, one sees that with the Fubini-

Study metric on $\mathbb{CP}^n$,

$$\cos d_{FS}(C, \Phi_N(Z)) = \frac{|C \cdot \Phi_N(Z)|}{|C||\Phi_n(Z)|}$$

So we get

$$\frac{|s(Z)|_h}{\sqrt{\Pi_N(z,z)}} = \cos d_{FS}(C, \Phi_N(Z))$$

**Lemma 3.0.6.** *Let $M$ be a compact submanifold of $(A, g)$, where $(A, g)$ is a $C^\infty$ Remannian manifold, let $B_\rho(P)$denote the ball centered at $P \in A$ with radius $\rho$, then for $\rho > 0$ small enough, $\mathbb{B}_\rho(P) \cap M$ is contractible if not empty.*

*Proof.* Consider the normal bundle of $\pi : N \to M$ in $A$ and the exponential map $\exp : N \to A$. Since $M$ is compact, there exists $\rho_1 > 0$ such that restricted to the open neighborhood $O_M(\rho_1) = \{(p, v)| \parallel v \parallel < \rho_1\} \subset N$, the exponential map is injective. Now we claim that any $\rho < \rho_1$ satisfies the requirement of the lemma.

Now suppose $\mathbb{B}_\rho(P) \cap M$ is not empty, then $P \in \exp(O_M(\rho_1))$ and $P = \exp(p, v)$ with $(p, v) \in O_M(\rho_1)$. Consider $d(P, -)$ as a smooth function on $\mathbb{B}_\rho(P) \cap M$, then by assumption $p$ is the only critical point of $d(P, -)$, since the geodesic that connects $p$ and a critical point of $\mathbb{B}_\rho(P) \cap M$ is orthogonal to $M$. Let $r = d(P, p)$, then $\mathbb{B}_r(P) \cap M = p$. So by Morse theory $\mathbb{B}_\rho(P) \cap M$ is contractible.

□

*Remark:* The proof of the lemma above implies that above a high level a section can have at most one critical point.

**Corollary 3.0.7.** *Let $r_N$ be the critical radius of the embedding $\Phi_N(M) \subset \mathbb{CP}^n$, then for $\rho < r_N$, the excursion set $\{z \in M | \frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos\rho\}$ is either contractible or empty.*

By theorem 1.0.1(which will be proved in the next section), $r_N$ is bounded below by $\rho_0 > 0$. Therefore as a corollary, taking into account that the Fubini-Study metric on $\mathbb{CP}^n$

is the quotient of the "round metric" under the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ we have

$$E\chi(\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos\rho) = Prob.\{\sup_{z\in M} \frac{|s(z)|_h}{\sqrt{\prod_N(z,z)}} > \cos\rho\} = \frac{Vol(T(\Phi_N(M),\rho))}{Vol(\mathbb{CP}^n)}$$

for $\rho < \rho_0$

First we calculate the volume $V(T(\Phi_N(M),\rho))$. We use theorems and formulas from [9](Theorem 7.20)(see also [8]).

**Theorem 3.0.8.** *Let $M^m$ be an embedded complex submanifold of $(\mathbb{CP}^n, \omega_{FS})$, and let $N$ be the normal bundle of $M$ in $\mathbb{CP}^n$ suppose that* $\exp : \{(p,v) \in N | \parallel v \parallel < r\} \to T(M,r)$ *is a diffeomorphism. Then*

$$V_M(r) = \frac{1}{n!}\int_M \prod_{a=1}^{m}(1 - \frac{\omega_{FS}}{\pi} + x_a) \wedge (\pi\sin^2(r) + \cos^2(r)\omega_{FS})^n$$

*Where $x_a$ is defined formally in the factorization of the Chern polynomial* $c(M)(t) = \prod_{a=1}^{m}(t + x_a)$

As a corollary of this theorem and by plugging in $\Phi_N^*(\omega_{FS}) = N\pi c_1(L)$,and dividing by the Fubini-Study volume $\pi^n/n!$ of $\mathbb{CP}^n$, we get theorem 1.0.2

When $M$ is a Riemann surface, $m = 1$, so $x_1 = c_1(M)$, the first Chern class of $M$. So

$$V_M(r) = \frac{1}{n!}\int_M (1 - \frac{\omega_{FS}}{\pi} + c_1(M)) \wedge [(\pi\sin^2(r))^n + n(\pi\sin^2(r))^{n-1}\cos^2(r)\omega_{FS}]$$

therefore

$$V_M(r) = \frac{1}{n!}\int_M [(\pi\sin^2(r))^n(c_1(M) - \frac{\omega_{FS}}{\pi}) + n(\pi\sin^2(r))^{n-1}\cos^2(r)\omega_{FS}]$$

We know by the Gauss-Bonnet formula $\int_M c_1(M) = \chi(M) = 2 - 2g$ and since $\Phi_N(N)$ is of degree $N\deg(L)$ in $\mathbb{CP}^n$, $\int_M \omega_{FS} = N\deg(L)\pi$. Now we can write out the explicit formula for $V(T(\Phi_N(M),\rho))$, that is

$$\begin{aligned}
V(T(\Phi_N(M),\rho)) &= \frac{1}{n!}[(\pi\sin^2(\rho))^n(\chi(M)-N\deg(L))+nN\deg(L)\pi(\pi\sin^2(\rho))^{n-1}\cos^2(\rho)] \\
&= \frac{\pi^n}{n!}(\sin^{2(n-1)}\rho)[N^2(\deg L)^2\cos^2\rho - N\deg L(g\cos^2\rho-\sin^2\rho)+(2-2g)\sin^2\rho]
\end{aligned}$$

where $\chi(M)=2-2g$ and by the Riemann-Roch formula $n=N\deg(L)-g$ for $N\deg(L)>2g-2$

To summarize, we have the following theorem

**Theorem 3.0.9.** *Let $M$ be a Riemann surface. Then, with the notations above, there exists $\rho_0>0$ such that for $\rho<\rho_0, NdegL>2g-2$ and a random section $s(z)\in H^0(M,L^N)$, the expected Euler characteristic is given by*

$$\begin{aligned}
&E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}}>\cos\rho) \\
&= (\sin^{2(n-1)}\rho)[N^2(\deg L)^2\cos^2\rho - N\deg L(g\cos^2\rho-\sin^2\rho)+(2-2g)\sin^2\rho]
\end{aligned}$$

If we write $u=\cos\rho$ and plug in $\sin^2\rho=1-u^2$, we get theorem 1.0.3

Note that when $m>1$, the expansion of $E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}}>\cos\rho)$ is complicated and the author can not get a more intuitive formula. We can calculate the leading term to have an estimation of formula 1.1.

Observe that the leading term in the expansion should be

$$\frac{1}{\pi^n}\int_M\binom{n}{m}(\pi\sin^2\rho)^{n-m}(\cos^2\rho\,\omega_{FS})^m$$

Let $O_n(1)$ denote the hyperplane bundle on $\mathbb{CP}^n$. Then $\omega_{FS}$ is a multiple of the first Chern class of $O_n(1)$, that is $\omega_{FS}=\pi c_1(O_n(1))$. Also the pull back $\Phi_N^*(c_1(O_n(1)))=$

$Nc_1(L)$. Therefore we have

$$\int_M \omega_{FS}^m = \pi^m N^m \int_M c_1^m(L)$$

which is independent of the metric on $L$.

So formula 3 becomes

$$\begin{pmatrix} n \\ m \end{pmatrix} (\sin^2 \rho)^{n-m} (\cos^2 \rho)^m N^m \int_M c_1^m(L)$$

By the asymptotic Riemann-Roch formula (ref. Theorem 1.1.22[10])for $N$ big enough

$$n = \frac{\int_M c_1^m(L)}{m!} N^m + O(N^{m-1})$$

So the leading term is

$$n^{m+1} (\sin^2 \rho)^{n-m} (\cos^2 \rho)^m$$

Therefore we have the following theorem

**Theorem 3.0.10.** *With the notations above, for $m \geq 1$ and for $N$ big enough*

$$E\chi(\frac{s(z)}{\sqrt{\Pi_N(z,z)}} > \cos \rho) = (1 + o(1)) n^{m+1} (\sin^2 \rho)^{n-m} (\cos^2 \rho)^m$$

Again, plugging in $u = \cos \rho$, we get theorem 1.0.4

Let $r_n = \sup\{\rho_n$ that satisfies the requirement of theorem 1.0.3$\}$. we are going to show that although $r_n$ might get smaller as $n$ grows, there is a positive lower bound.

# Chapter 4

# Critical Radius

We will first analyze the case of Riemann surfaces, then generalize the results to that of higher dimensional smooth projective variety.

## 4.1 Riemann Surfaces

Let $X$ be a compact Riemann Surface, and let $(L, h)$ be a positive Hermitian holomorphic line bundle over $X$. The curvature of $(L, h)$ induces a Kähler metric on $X$ with Kähler form $\omega = \frac{i}{2}\Theta_h$. Let $s_0, s_1, \cdots, s_n$ be an orthonormal basis of $H^0(X, L^N)$. Here we write $n$ instead of $n(N)$ for short. This gives an embedding $\Phi_N : X \to \mathbb{CP}^n$ for $N$ big enough by Kodaira. If we choose a holomorphic local frame $e_L$of $L$, then $s_i = f_i e_L^N$ with $f_i$ holomorphic functions. So $\Phi_N$ is locally given by $\Phi_N(z) = [f_0(z), f_1(z), \cdots, f_n(z)]$. We denote the vector $(f_0(z), f_1(z), \cdots, f_n(z)) \in \mathbb{C}^{n+1}$ by $F(z)$, and the vector $(f_0'(z), f_1'(z), \cdots, f_n'(z))$ by $F'(z)$. At each point $\Phi_N(z)$, the holomorphic tangent line is given by $[F(z) + tF'(z)], t \in \mathbb{C} \cup \{\infty\}$. By $[v]$ for $v \in \mathbb{C}^{n+1}$, we mean the image under the projection $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$. Consider the normal bundle $N \subset T\mathbb{CP}^n|_{\Phi_N(X)}$. At any point $p \in \Phi_N(X)$, $\exp(N_p)$ is the hyperplane $H_p$ passing $p$ which is orthogonal to $T_p\Phi_N(X)$ at $p$. We define $T_\infty(z)$ as the only point on the tangent line through $\Phi_N(z)$ with distance $\pi/2$ to $\Phi_N(z)$ in $\mathbb{CP}^N$. Then

$$[T_\infty(z)] = [F'(z) - \frac{< F'(z), F(z) >}{< F(z), F(z) >} F(z)]$$

We denote by $O_z()$ the projection of $\mathbb{CP}^n$ from $T_\infty(z)$ to its orthogonal hyperplane, which is just $H_z$. In particular we have

$$[O_z([v])] = [v - \frac{< v, T_\infty(z) >}{|T_\infty(z)|^2} T_\infty(z)]$$

Also by $d_N(,)$ we mean the distance in $\mathbb{CP}^n$ induced by the Fubini-Study metric.

**Lemma 4.1.1.** *Let $H_{z\cap w}$ denote the intersection of the normal hyperplanes $H_z, H_w$ of $\Phi_N(X)$ through $\Phi_N(z)$ and $\Phi_N(w)$ respectively, then*

$$\sin^2(d_N(\Phi_N(z), H_{z\cap w})) = \cos^2(d_N(\Phi_N(z), O_z(T_\infty(w))))$$

*Proof.* By unitary change of coordinates, we can assume that $\Phi_N(z) = [0, \cdots, 0, 1]$,and that $T_\infty(z) = [0, \cdots, 0, 1, 0]$. For any $q \in H_{z\cap w}$, let $q = [v_0, v_1, \cdots, v_n]$ with $\sum_{i=0}^n |v_i|^2 = 1$. Then $\cos(d_N(\Phi_N(z), q)) = |v_n|$. So $\cos^2(d_N(\Phi_N(z), H_{z\cap w})) = max|v_n|^2$. Let $T_\infty(w) = [c_0, c_1, \cdots, c_n]$. So the $v_i$'s satisfies the following equations

$$v_{n-1} = 0, \quad \sum_{i=0}^n c_i \bar{v}_i = 0$$

So the maximum $|v_n|$ is

$$|v_n|^2 = 1 - \frac{|c_n|^2}{\sum_{i\neq n-1} |c_i|^2}$$

On the other hand it is clear that

$$\cos^2(d_N(\Phi_N(z), O_z(T_\infty(w)))) = \frac{|c_n|^2}{\sum_{i\neq n-1} |c_i|^2}$$

Combining the equations, we get the conclusion. $\square$

Therefore, by switching $z$ and $w$, we have the following equation

$$\sin^2(d_N(\Phi_N(w), H_{z\cap w})) = \frac{|< F(w), O_w(T_\infty(z)) >|^2}{|F(w)|^2|O_w(T_\infty(z))|^2}$$

where by abuse of notation, we consider the homogeneous coordinate of a point in $\mathbb{CP}^n$ as a vector in $\mathbb{C}^{n+1}$.

Before we go on calculating the right side of the equation, we recall the normalized Szegö kernels in [12] is defined as

$$P_N(z, w) := \frac{|\Pi_N(z, w)|}{\Pi_N(z, z)^{1/2}\Pi_N(w, w)^{1/2}}$$

Since $|s_i|_h^2 = |f_i|^2 h^N$, we have

$$P_N(z, w) = \frac{|< F(z), F(w) >|}{< F(z), F(z) >^{1/2}< F(w), F(w) >^{1/2}}$$

Now we let $E(z, w) = P_N^2(z, w)$, then

$$E(z, w) = \frac{< F(z), F(w) >< F(w), F(z) >}{< F(z), F(z) >< F(w), F(w) >}$$

Therefore

$$\frac{\partial}{\partial z}E(z, w) = \frac{< F(w), F(z) >}{|F(z)|^2|F(w)|^2}[< F'(z), F(w) > - \frac{< F(z), F(w) >}{|F(z)|^2} < F'(z), F(z) >]$$

and

From now on we use the following convention, by $Df(*, *)$, where $D$ is some differential, we always mean the the value of $Df$ at $(*, *)$

$$\begin{aligned}\frac{\partial^2}{\partial z \partial \bar{w}} E(z,w) &= \frac{< F(w), F(z) >}{|F(z)|^2 |F(w)|^2} [< F'(z), F'(w) > \\ &- \frac{< F(w), F'(w)}{|F(w)|^2} < F'(z), F(w) > - \frac{< F(z), F'(w)}{|F(z)|^2} < F'(z), F(z) > \\ &+ \frac{< F(z), F(w) >< F(w), F'(w) >}{|F(z)|^2 |F(w)|^2} < F'(z), F(z) >]\end{aligned}$$

We denote $\frac{\partial}{\partial z} E(z,w)$ and $\frac{\partial^2}{\partial z \partial \bar{w}} E(z,w)$ considered as functions of $(z,w)$ by $E_z(z,w)$ and $E_{z\bar{w}}(z,w)$ respectively. So in particular

$$E_{z\bar{w}}(z,z) = \frac{1}{|F(z)|^2} [< F'(z), F'(z) > - \frac{| < F'(z), F(z) > |^2}{|F(z)|^2}]$$

Now we calculate $\sin^2(d_N(\Phi_N(w), H_{z\cap w}))$.

First we have

$$O_w(T_\infty(z)) = T_\infty(z) - \frac{< T_\infty(z), T_\infty(w) >}{|T_\infty(w)|^2} T_\infty(w)$$

since $O_w(T_\infty(z))$ is orthogonal to $T_\infty(w)$ we have

$$|O_w(T_\infty(z))|^2 = |T_\infty(z)|^2 - \frac{| < T_\infty(z), T_\infty(w) > |^2}{|T_\infty(w)|^2}$$

and since $F(w)$ is also orthogonal to $T_\infty(w)$ we also have

$$| < F(w), O_w(T_\infty(z)) > |^2 = | < F(w), T_\infty(z) > |^2$$

Since

$$< T_\infty(z), F(w) > = < F'(z), F(w) > - \frac{< F'(z), F(z) >}{|F(z)|^2} < F(z), F(w) >$$

We get the following equation

**Lemma 4.1.2.** *With the notations above*

$$E_z(z,w) = \frac{< F(w), F(z) >}{|F(z)|^2 |F(w)|^2} < T_\infty(z), F(w) >$$

Moreover,

$$|T_\infty(z)|^2 =< F'(z), F'(z) > - \frac{| < F'(z), F(z) > |^2}{|F(z)|^2}$$

and

$$\begin{aligned}
(4.1) < T_\infty(z), T_\infty(w) > &= < F'(z), F'(w) > - \frac{< F(w), F'(w) >}{|F(w)|^2} < F'(z), F(w) > \\
(4.2) \qquad &- \frac{< F(z), F'(w) >}{|F(z)|^2} < F'(z), F(z) > \\
(4.3) \qquad &+ \frac{< F(z), F(w) >< F(w), F'(w) >}{|F(z)|^2 |F(w)|^2} < F'(z), F(z) >
\end{aligned}$$

Therefore we have

$$E_{z\bar{w}}(z,z) = \frac{1}{|F(z)|^2} |T_\infty(z)|^2$$

and

$$E_{z\bar{w}}(z,w) = \frac{< F(w), F(z) >}{|F(z)|^2 |F(w)|^2} < T_\infty(z), T_\infty(w) >$$

Combining these equations we have

$$\begin{aligned}
&\sin^2(d_N(\Phi_N(w), H_{z\cap w})) \\
&= \frac{(|F(z)|^2 |F(w)|^2)^2 |E_z(z,w)|^2}{| < F(w), F(z) > |^2 |F(w)|^2 \{ |F(z)|^2 E_{z\bar{w}}(z,z) - \frac{[|F(z)|^2|F(w)|^2]^2 |E_{z\bar{w}}(z,w)|^2}{|F(w)|^2 E_{z\bar{w}}(w,w) |<F(w),F(z)>|^2} \}} \\
&= \frac{|E_z(z,w)|^2}{E(z,w)[E_{z\bar{w}}(z,z) - \frac{|E_{z\bar{w}}(z,w)|^2}{E(z,w)E_{z\bar{w}}(w,w)}]}
\end{aligned}$$

So we have the following theorem

**Theorem 4.1.3.** *With the notations above we have the equation*

$$\sin^2(d_N(\Phi_N(w), H_{z\cap w})) = \frac{|E_z(z,w)|^2}{E(z,w)E_{z\bar{w}}(z,z)[1 - \frac{|E_{z\bar{w}}(z,w)|^2}{E_{z\bar{w}}(z,z)E_{z\bar{w}}(w,w)}\frac{1}{E(z,w)}]}$$

As in the last section, we choose local coordinates such $z_0 = 0$. Then

$$E(z,w) = P_N^2(z,w) = e^{-|u-v|^2}[1 + R_N(u,v)]^2,$$

where $u = \sqrt{N}z, v = \sqrt{N}w$. So

$$\begin{aligned} \frac{\partial}{\partial z}E(z,w) &= \sqrt{N}\frac{\partial}{\partial u}e^{-|u-v|^2}[1 + R_N(u,v)]^2 \\ &= \sqrt{N}[e^{-|u-v|^2}(\bar{v} - \bar{u})[1 + R_N(u,v)]^2 \\ &+ e^{-|u-v|^2}2(1 + R_N(u,v))\frac{\partial}{\partial u}R_N(u,v)] \end{aligned} \tag{4.4}$$

When $z = 0$

$$E(0,w) = e^{-N|w|^2}[1 + O(N^{1/2+\varepsilon}|w|^2)]^2$$

$$\begin{aligned} \frac{\partial}{\partial z}E(0,w) &= \sqrt{N}e^{-N|w|^2}(1 + O(N^{-1/2+\varepsilon})\sqrt{N}\bar{w} + O(N^{\varepsilon}|w|)) \\ &= (1 + o(1))Ne^{-N|w|^2}\bar{w} \end{aligned} \tag{4.5}$$

where we make $\varepsilon < 1/2$ and use the estimation that

$$|\frac{\partial}{\partial u}R_N(u,v)_{u=0}| \le C_2|v|N^{-1/2+\varepsilon}$$

Furthermore we can calculate

$$
\begin{aligned}
\frac{\partial^2}{\partial z \partial \bar{w}} E(z,w) &= N\frac{\partial}{\partial \bar{v}}\{\sqrt{N}[e^{-|u-v|^2}(\bar{v}-\bar{u})][1+R_N(u,v)]^2 \\
&+ e^{-|u-v|^2}2(1+R_N(u,v))\frac{\partial}{\partial u}R_N(u,v)\} \\
&= Ne^{-|u-v|^2}(u-v)[(\bar{v}-\bar{u})(1+R_N)^2)+2(1+R_N)\frac{\partial}{\partial u}R_N] \\
&+ Ne^{-|u-v|^2}\{(1+R_N)^2+(\bar{v}-\bar{u})\frac{\partial}{\partial \bar{v}}(1+R_N)^2 \\
&+ 2\frac{\partial}{\partial \bar{v}}R_N\frac{\partial}{\partial u}R_N+2(1+R_N)\frac{\partial^2}{\partial u \partial \bar{v}}R_N\}
\end{aligned} \tag{4.6}
$$

Therefore

$$
\begin{aligned}
E_{z\bar{w}}(w,w) &= N[(1+R_N)^2+2\frac{\partial}{\partial \bar{v}}R_N\frac{\partial}{\partial u}R_N](v,v) \\
&= N(1+2\frac{\partial^2}{\partial u \partial \bar{v}}R_N(v,v))
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
E_{z\bar{w}}(0,w) &= Ne^{-N|w|^2}\{(1+O(N^{1/2+\varepsilon}|w|^2))^2(1-N|w|^2) \\
&+ 2(1+O(N^{1/2+\varepsilon}|w|^2))O(N^{1/2+\varepsilon}|w|^2) \\
&+ O(N^{1/2+\varepsilon}|w|^2)(1+O(N^{1/2+\varepsilon}|w|^2)) \\
&+ O(N^{2\varepsilon}|w|^2)+2(1+O(N^{1/2+\varepsilon}|w|^2))\partial u\partial\bar{v}R_N(0,v))\} \\
&= Ne^{-N|w|^2}[1-(1+o(1))N|w|^2+2\frac{\partial^2}{\partial u \partial \bar{v}}R_N(0,v)]
\end{aligned}
$$

With the notations above we have the following theorem

**Theorem 4.1.4.** *There exist $r(L,h)>0$, which is independent of $N$, such that*

$$\sin^2(d_N(\Phi_N(w),H_{0\cap w}))>r(L,h)$$

*for* $|w| < \frac{1}{\sqrt{2N}}$

*Proof.* Before applying theorem 4.1.3, we need the following estimations

$$\frac{\partial^2}{\partial u \partial \bar{v}} R_N(0,v) - \frac{\partial^2}{\partial u \partial \bar{v}} R_N(0,0) = A(v) + O(N^{-1/2+\varepsilon}|v|^2)$$
$$\frac{\partial^2}{\partial u \partial \bar{v}} R_N(v,v) - \frac{\partial}{\partial u \partial \bar{v}} R_N(0,0) = A(v) + \overline{A(v)} + O(N^{-1/2+\varepsilon}|v|^2)$$

where $A(v) = O(N^{-1/2+\varepsilon}|v|)$ .

Note first that $R_N(u,v)$ is a real analytic function, so we can write $R_N(u,v)$ as power series in $(u, \bar{u}, v, \bar{v})$. So the first equation follows directly from the theorem in [12]. Now we prove the second equation.

We denote by $g(u,v)$ the homogeneous part of degree 3 in the power series, since this is the part that contribute terms of degree 1 in the second derivatives. Notice that $g_{u\bar{v}}(0,0) = 0$,we need to show that

$$g_{u\bar{v}}(0,x) + \overline{g_{u\bar{v}}(0,x)} = g_{u\bar{v}}(x,x)$$

for all $x \in \mathbb{C}$.

Since $g(u,v)$ is real, $\overline{g_{u\bar{v}}(0,x)} = g_{v\bar{u}}(0,x)$. We write $p(x) = g_{u\bar{v}}(x,x) - g_{u\bar{v}}(0,x) - g_{v\bar{u}}(0,x)$. So $p(x)$ is linear in $(x, \bar{x})$ and $p(0) = 0$. To show that $p(x) \equiv 0$, we just need to show that $\frac{\partial}{\partial x} p(x) = 0$ and $\frac{\partial}{\partial \bar{x}} p(x) = 0$. But

$$\frac{\partial}{\partial x} p(x) = g_{uu\bar{v}}(x,x) + g_{vu\bar{v}}(x,x) - g_{vu\bar{v}}(0,x) - g_{vv\bar{u}}(0,x) = 0$$

where the second equation follows from the fact that all terms in the middle are constant and that $g$ is symmetic with respect to $u$ and $v$. So we have proved the second equation.

We let $a = \frac{\partial^2}{\partial u \partial \bar{v}} R_N(0,0)$, then $a = O(N^{-1/2+\varepsilon})$ and $a$ is real. So

$$
\begin{aligned}
(4.8)\quad E_{z\bar{w}}(0,0) &= N(1+2a) \\
(4.9)\quad E_{z\bar{w}}(0,w) &= Ne^{-N|w|^2}[1-(1+o(1))N|w|^2+2a+2A(v)+O(N^{-1/2+\varepsilon}|v|^2)] \\
(4.10)\,E_{z\bar{w}}(w,w) &= N[1+2a+2A(v)+2\overline{A(v)}+O(N^{-1/2+\varepsilon}|v|^2)]
\end{aligned}
$$

Plug in these estimations together with the ones about $E_z(0,w)$ and $E(0,w)$ to the expression of $\sin^2(d_N(\Phi_N(w), H_{0\cap w}))$ in the last theorem, and use the Taylor series of $e^{-N|w|^2}$, both the numerator and denominator is bounded by positive multiples of $N|w|^2$, then it is easy to see that there is a constant $r>0$ independent of $N$ and $w$ for $w<\frac{1}{\sqrt{2N}}$ such that $\sin^2(d_N(\Phi_N(w), H_{0\cap w}))>r$. Also since the constant in the approximation of the normalized Szegö kernel is independent of the point $z$, $r$ can be chosen independent of $z$. $\square$

As a corollary, we have the following theorem

**Theorem 4.1.5.** *Let $r_N$ be the critical radius of $\Phi_N(X)$ considered as a submanifold of $\mathbb{CP}^n$. There exists a constant $c(X,h)>0$, such that $r_N>c(X,h)$*

*Proof.* We still use the preferred coordinates chosen centered at $z$

Let $N_x(b)=\{v\in N_x(\Phi_N(X)), \parallel v\parallel\leq b\}$ Notice that by theorem 2.1.2 and 2.1.3, for $w\geq\frac{1}{\sqrt{2N}}$, and for $N$ big enough, $d_N(\Phi_N(z),\Phi_N(z+w))\geq\cos^{-1}[(1+o(1))e^{-1/4}]$.

Combining this fact and theorem 4.1.4, there exists a constant $c>0$, which is independent of $z$ such that for any point $q\in\Phi_N(X)$,

$$\exp_{\Phi_N(z)}(N_{\Phi_N(z)}(c))\cap\exp_q(N_q(c))=\emptyset$$

This implies that the critical radius is bounded below, namely $r_N>c(X,h)$ $\square$

## 4.2 Higher Dimension

Actually the argument for Riemann surfaces carries directly to high dimensional Kähler manifolds. Now use the notations in section 3, we have the following theorem

**Theorem 4.2.1.** *Let $r_N$ be the critical radius of $\Phi_N(M)$ considered as a submanifold of $\mathbb{CP}^n$. There exists a constant $c(M,h) > 0$, such that $r_N > c(M,h)$*

*Proof.* We just need a high dimensional version of theorem 4.1.4.

We still choose a preferred coordinates centered at $z$, and let $w < \frac{1}{\sqrt{2N}}$.

In order to apply theorem 4.1.3, we let $X$ be the complex line in the coordinates chart connecting 0 and $w$, and restrict $\Phi_N$ to an open set $V \subset X$. Then all the estimations we used in proving theorem 4.1.4 hold for $V$. So theorem 4.1.4 can be applied to $V$. Notice that the lower bounds we can get come from the approximation of the normalized Szegö kernel of $(L,h) \to M$, hence is independent of $w$

Notice that the normal space of $M$ at $\Phi_N(z)$ is contained in the normal hyperplane of $\Phi_N(V)$ at $\Phi_N(z)$, the same is true for $\Phi_N(z+w)$. Therefore the intersection of the two normal spaces $N_{0\cap w} = N_{\Phi_N(z)}(M) \cap N_{\Phi_N(z+w)}(M)$ of $M$ is contained in the intersection of the two normal hyperplanes. Therefore the distance from $\Phi_N(z)$ to the intersection $N_{0\cap w}$ is also bounded below independent of $z$, $w$ and $N$.

Now use the same argument as in theorem 4.1.5, we get the expected conclusion. □

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## Vitae

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