NEARBY CYCLES, ARCHIMEDEAN COMPLEX AND PERIODICITY IN CYCLIC HOMOLOGY

by
Abhishek Banerjee

A dissertation submitted to Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland
April, 2009

Abstract

This thesis has four parts;

(1) We compare the monodromy on the nearby cycles complex ψ^{**} to the periodicity operator in cyclic homology of a sheaf of differential operators. The nearby cycles complex is quasi-isomorphic to the complex A^{**} of Steenbrink that determines the mixed Hodge structure on the nearby fibre for an algebraic degeneration over a disc. By stacking copies of the nearby cycles complex ψ^{**}, we construct a triple complex $\mathcal{BC}^{***}$ whose "rows" are quasi-isomorphic to the cyclic homology complex of the sheaf of differential operators and show that the periodicity operator on the cyclic homology of the latter is identical to the monodromy on $\mathcal{BC}^{***}$ induced from the ψ^{**}.

(2) We implement the arithmetic analogue of the same construction, using objects $K^{i,j,k}$ constructed by Consani to correspond to direct summands of the E_0-terms of the spectral sequence associated to the Picard-Lefschetz filtration on Steenbrink's complex A^{**}. In this case, we show that the periodicity operator in cyclic cohomology of the ring of differential operators corresponds to the monodromy on Consani's complex.

We assemble summands $K^{i,j,k}$ to form bicomplexes φ^{**} and B^{**}, which correspond to complexes ψ^{**} and A^{**} in part (1) respectively, together with a morphism $\mu : \varphi^{**} \longrightarrow B^{**}$. Both complexes φ^{**} and B^{**} carry monodromy N and we show that $N \circ \mu - \mu \circ N$ is homotopic to zero.

(3) We define algebraic, topological and relative K-theories of the sheaf of differential operators on a locally Stein manifold and demonstrate a long exact sequence connecting them that lifts the periodicity sequence in the cyclic (hyper)homology of the sheaf of differential operators. We also show that similar constructions may be made for formal schemes.

(4) We construct an "enriched archimedean complex" on the modular tower that extends Consani's complex $K^{i,j,k}$ from part (2). We show that the module of functions with values in this complex carries a flat action of the Connes-Moscovici's Hopf Algebra $\mathcal{H}_1$ of codimension 1 foliations and a bimodule structure over a variant $\mathcal{A}_T$ of Connes-Moscovici's modular Hecke algebra $\mathcal{A}$.

Finally, we show that the enriched archimedean complex carries Rankin Cohen brackets of all orders.

Readers: Prof. Caterina Consani (advisor), Prof. Jack Morava.

Acknowledgements

It is with great pleasure that I acknowledge my debt to my advisor, Prof. Caterina Consani, who has helped me along every step of the way. I am also grateful to Prof. Steven Zucker, Prof. Jack Morava and Prof. Vyacheslav Shokurov for their advice with regard to various aspects of this thesis and mathematics in general. A special thank you must needs go out to Prof. Masoud Khalkhali of the University of Western Ontario for a careful reading of the manuscript, thus spotting subtle inaccuracies that had escaped me. It is also a delight to thank Prof. Alain Connes of IHES and Prof. Matilde Marcolli of Caltech for taking interest in my research problem during the early stages of my work and offering important suggestions towards the direction and development of my thesis. In particular, Chapter 5 of this thesis grew out directly from a conversation with Prof. Connes in this regard.

I do also express my deep gratitude to Prof. Ravi A. Rao at Tata Institute of Fundamantal Research, Mumbai, who introduced me to K-theory and Algebraic Geometry, apart from encouraging me to take research interest in it. Besides, I owe much to Prof. Amartya K. Dutta at Indian Statistical Institute, Calcutta and Prof. B. Sury at Indian Statistical Institute, Bangalore for all the support I received from them in studying Algebra and Number Theory during my undergraduate years. Thanks are also due to Prof. K. C. Prasad and Prof. B. B. Bhattacharya at Ranchi University, who kindled my interest in mathematics in the days of high school.

Thus begins a tour de force, one that was meant to gauge the lengths to which a man will go in order to satisfy his future ego, and to fulfil a childhood whim...

This thesis is dedicated to my benefactress Mrs. Uma Saini with respect, love and gratitude.

Contents

0 Introduction **3**

1 Background on Cyclic Homology and Cyclic Cohomology **18**

1.1 Basic Definitions 18

1.2 The case of a topological algebra 27

1.3 The case of filtered algebras 29

1.4 Relations with de Rham Cohomology 32

2 Introduction to Steenbrink's complex and Consani's complex **38**

2.1 The Logarithmic de Rham complex 39

2.2 Cohomology of the universal fibre $\widetilde{X}^*$ 41

2.3 Steenbrink's complex 44

2.4 Cohomology of the fibre at infinity: Consani's complex 49

3 The Steenbrink complex and the Cyclic Homology of the Sheaf of differential operators **52**

3.1 The complex of nearby cycles and de Rham cohomology 55

3.2 The complex $\mathbf{Ker}(\mathbf{N})^*$ and the Hochschild Complex 66

3.3 The Connes periodicity operator and the monodromy operator 69

4 Consani's complex and the Cyclic Cohomology of the ring of differential operators **87**

4.1 Consani's complex 88

4.2 The de Rham complex and the Ring of differential operators 91

4.3 The Connes periodicity operator and the monodromy operator 102

4.4 An analogue of the complex of nearby cycles at archimedean infinity . 106

5 The Connes-Karoubi exact sequence of K-theories 114

5.1 Preliminaries from Algebraic Topology 115

5.2 Definitions and Exact Sequences 118

5.3 Simplicial Sheaves and Generalized Sheaf Cohomology 121

5.4 The exact sequence of K-theories for the sheaf $\mathscr{D}_X$ 124

5.5 Chern characters and Dennis Trace 128

5.6 The long exact sequence for formal schemes 133

6 Modular Hecke Algebras, Rankin Cohen Brackets and an Enriched Archimedean Complex 142

6.1 Hecke Algebras and Hecke Correspondences 144

6.2 The Enriched Archimedean complex with Hopf action 151

6.3 Extension of Rankin Cohen brackets 164

0 Introduction

This thesis is divided into four main parts. In Chapter 3, we start by studying the nearby cycles complex and the monodromy operator on it in the context of cyclic homology and Connes periodicity. Thereafter, in Chapter 4, we obtain a connection between cyclic cohomology and the archimedean complex for the cohomology of the nearby fibre at infinity for an arithmetic variety $\mathfrak{X}$ defined over a discrete valuation ring Λ. The theory of the Steenbrink complex, the nearby cycles complex and the archimedean complex for the fibre at infinity is recalled in Chapter 2.

In Chapter 5, using generalized sheaf cohomology for simplicial sheaves, we obtain a lift of the periodicity sequence in cyclic homology to K-theory. Finally, we deal with the Hopf Algebra $\mathcal{H}_1$ "of codimension 1 foliations" of Connes and Moscovici and define an action of $\mathcal{H}_1$ on an archimedean complex enriched with modular forms. As a consequence, we are able to define Rankin Cohen brackets on the enriched archimedean complex. This is done in Chapter 6.

We will now summarize the main results of the thesis:

Nearby Cycles Complex and Cyclic Homology

The theory of cyclic (co)homology was introduced by A. Connes as a noncommutative analogue of de Rham cohomology within the framework of noncommutative geometry.

Let A be an algebra over a field k and let $HC_q(A)$, *resp.* $HH_q(A)$ denote the cyclic homology, *resp.* the Hochschild homology of A. One of the most interesting results in this theory states the existence of the following periodicity exact sequence (see [14])

$$\dots \longrightarrow HH_q(A) \xrightarrow{I} HC_q(A) \xrightarrow{S} HC_{q-2}(A) \longrightarrow \dots \tag{0.0.1}$$

In this section I shall consider the following geometric situation. Let $j : X^* \hookrightarrow X$ be a compactification of a smooth algebraic variety X^* over the complex numbers. I shall assume that the complement $Y = X - X^*$ is a divisor in X with normal crossings. Then, it is well known (*cfr.* [23]) that the triple

$$(Rj_*\mathbb{Z}_{X^*}, (Rj_*\mathbb{Q}_{X^*}, \tau_{\leq \cdot}), (\Omega^{\cdot}_X(\log Y), W_{\cdot}, F^{\cdot}))$$

and the filtered quasi-isomorphism of complexes

$$(Rj_*\mathbb{Q}_{X^*}, \tau_{\leq \cdot}) \otimes \mathbb{C} \longrightarrow (\Omega^{\cdot}_X(\log Y), W_{\cdot})$$

determine a mixed Hodge structure on the (de Rham) cohomology $H^*(X^*, \mathbb{C})$. In the formulae above, $\tau_{\leq \cdot}$ denotes the canonical filtration, $F^{\cdot}$ is the naive filtration and $W_{\cdot}$ is the weight filtration on the complex of (sheaves of) differential forms on X with logarithmic poles along Y.

The theory of mixed Hodge structures was originally introduced by P. Deligne and has been subsequently generalized by J. Steenbrink (*cf.* [38]) to fibrations of algebraic varieties. In the framework studied by Steenbrink, X is a non-singular complex algebraic variety endowed with a proper map f to the unit disc D, which is smooth everywhere except at the origin $0 \in D$. The fibre $Y = f^{-1}(0)$ is assumed to be a divisor in X with normal crossings. Let $\tilde{X}^* = X^* \times_{D^*} \tilde{D}^*$, where $\tilde{D}^*$ denotes the universal covering of the punctured unit disc $D^* = D \setminus \{0\}$. Under these hypotheses, Steenbrink defines a limiting Hodge structure on the (de-Rham) cohomology of the generic fibre $\tilde{X}^*$. In this construction, the cohomology $H^*(\tilde{X}^*, \mathbb{C})$ gets identified to the hypercohomology of a complex (A^*, d) of sheaves of differential forms on X with logarithmic poles along Y. The complex A^* is also equipped with a monodromy operator N which represents the logarithm of the local monodromy map induced on the cohomology of $\tilde{X}^*$ by the process of looping around the origin $0 \in D$.

In this thesis, we give an interpretation of the above construction in the context of cyclic (co)homology. To obtain this result, we appeal to a well-known theorem of M. Wodzicki (*cfr* [46]) stating that the Hochschild homology $HH_q(\mathscr{D}(U))$ of the ring of differential operators on a Stein manifold U can be described in terms of the de Rham cohomology of U, by means of the following isomorphisms ($n = \dim U$)

$$HH_q(\mathscr{D}(U)) \cong H_{dR}^{2n-q}(U) \qquad \forall\ q \geq 0.$$

Moreover, the cyclic homology $HC_q(\mathscr{D}(U))$ decomposes as a direct sum

$$HC_q(\mathscr{D}(U)) \cong H_{dR}^{2n-q}(U) \oplus H_{dR}^{2n-q+2}(U) \oplus ... \tag{0.0.2}$$

Connes' periodicity operator

$$S : HC_q(\mathscr{D}(U)) \longrightarrow HC_{q-2}(\mathscr{D}(U))$$

fits into a long exact sequence similar to (0.0.1).

To apply these results in the geometric context of algebraic degenerations, we consider the sheaf $\mathscr{D}_{X^*}$ of differential operators on X^*. The cyclic homology $HC_q(\mathscr{D}_{X^*})$ (resp. the Hochschild homology $HH_q(\mathscr{D}_{X^*})$) is defined using hypercohomology techniques. In this way, we prove in Theorem A.3.17 that the cyclic homology $HC_q(\mathscr{D}_{X^*})$ decomposes into a direct sum $H_{dR}^{2n-q}(X^*) \oplus H_{dR}^{2n-q+2}(X^*) \oplus ...$ in a way similar to (0.0.2). Moreover, the periodicity operator $S : HC_q(\mathscr{D}_{X^*}) \longrightarrow HC_{q-2}(\mathscr{D}_{X^*})$ acts by "dropping off" the top summand $H_{dR}^{2n-q}(X^*)$.

One knows that Steenbrink's complex A^{**} is quasi-isomorphic to a resolution $\psi^{**}(\mathbb{C})$:

$$\psi^{-p,q}(\mathbb{C}) = \mathbb{C}u^p \otimes \Omega_X^{q-p}(\log Y), \quad p, q \in \mathbb{Z}_{\geq 0}$$

of the complex of nearby cycles on X, for the constant sheaf $\mathbb{C}$. I shall denote by $\mu : \psi^{**}(\mathbb{C}) \longrightarrow A^{**}$ the quasi-isomorphism (*cfr.*[28]) connecting these two complexes. The bi-complex $\psi^{**}(\mathbb{C})$ is also endowed with a monodromy operator N. It is important to note that such operator on $\psi^{**}(\mathbb{C})$ does not agree directly with the corresponding N on A^{**}, however one can show that the maps $N \circ \mu$ and $\mu \circ N$ are chain homotopic.

A first link between the theory of nearby cycles and cyclic (co)homology is determined by the result of Proposition 3.8 in which we show that the columns $\psi^{p*}(\mathbb{C})$ of the complex $\psi^{**}(\mathbb{C})$ are quasi-isomorphic to Hochschild complexes of $\mathscr{D}_{X^*}$. More precisely, we show that the spectral sequence associated to the filtration by columns on $\psi^{**}(\mathbb{C})$ converges to the hypercohomology of $\psi^{**}(\mathbb{C})$: the E_1-terms being described by Hochschild homologies. The filtration by columns on $\psi^{**}(\mathbb{C})$ also coincides with the filtration by kernels of powers of N. The result is summarized by the following

Theorem 0.1. *The filtration by columns on $Tot(\psi^{**}(\mathbb{C}))$ coincides with the filtration by kernels of powers of N:*

$$F_k(Tot(\psi^{**}(\mathbb{C}))) = Ker(N^k), \quad k \geq 0.$$

If $C_*^h(\mathscr{D}_{X^*})$ denotes the Hochschild complex of the sheaf of differential operators on X^*, there is a quasi-isomorphism of complexes ($n = dim(X^*)$)

$$C_*^h(\mathscr{D}_{X^*}) \longrightarrow Gr_k^F(Tot(\psi^{**}(\mathbb{C})))[2n], \quad \forall k \geq 0.$$

Thereafter, we construct a triple complex $\mathcal{BC}^{***}$ which consists of copies of the bicomplex $\psi^{**}(\mathbb{C})$ "stacked vertically" in a suitable way. More precisely, we define

$$\mathcal{BC}^{p,q,r} = \psi^{p+r,q-2r}(\mathbb{C}), \qquad p+r \leq 0;\ p,q,r \in \mathbb{Z}.$$

Therefore, a "column" $\mathcal{BC}^{p**}$ of this complex is a bi-complex constructed column-wise by assembling copies of the columns of the complexes $\psi^{**}(\mathbb{C})$: one takes a column from each of these complexes. By construction, the complex $\mathcal{BC}^{***}$ carries also a monodromy operator $N : \mathcal{BC}^{p**} \to \mathcal{BC}^{p+1**}$. By mapping the "column" $\mathcal{BC}^{p**}$ to the next one $\mathcal{BC}^{p+1**}$, N acts by dropping a copy of ψ^{0*}, (i.e. a complex quasi-isomorphic to a Hochschild complex, in view of what I have described before Theorem 0.1). So constructed, the bi-complexes $\mathcal{BC}^{p**}$ are quasi-isomorphic to the cyclic complex of $\mathscr{D}_{X^*}$ on which the periodicity operator S acts. In this set-up, I show that the operator N acting columnwise on $\mathcal{BC}^{***}$ agrees in cohomology with the periodicity operator S on the cyclic theory of $\mathscr{D}_{X^*}$. This result is summarized by the following

Theorem 0.2. *Let us consider the filtration by "columns" on $\mathcal{BC}^{***}$, i.e. the filtration whose p-th graded piece is the bicomplex $\mathcal{BC}^{p**}$. If $CC_*(\mathscr{D}_{X^*})$ denotes the cyclic complex of the sheaf of differential operators on X^*, then there is a quasi-isomorphism of complexes ($n = dim(X^*)$)*

$$Tot(CC_*(\mathscr{D}_{X^*})) \longrightarrow Tot(\mathcal{BC}^{p**})[2n].$$

The bicomplex $\mathcal{BC}^{p**}$ is equipped with an operator N induced by that on the $\psi^{**}(\mathbb{C})$. The periodicity operator S on the complex $CC_*(\mathscr{D}_{X^*})$ corresponds to the monodromy operator N on $\mathcal{BC}^{p**}$, i.e. we have the following commutative diagram of exact se-

quences with vertical isomorphisms; the maps S and N are identical

$$\begin{array}{ccccc}
\ldots \to \mathbb{H}^{2n-k}(Ker(N)^*) & \xrightarrow{I} & \mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p**})) & \xrightarrow{N} & \mathbb{H}^{2n-k+2}(Tot(\mathcal{BC}^{p**})) \to \ldots \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
\ldots \longrightarrow HH_k(\mathscr{D}_{X^*}) & \xrightarrow{I} & HC_k(\mathscr{D}_{X^*}) & \xrightarrow{S} & HC_{k-2}(\mathscr{D}_{X^*}) \longrightarrow \ldots \\
\| & & \| & & \| \\
H^{-k}(C^h_*(\mathscr{D}_{X^*})) & & H^{-k}(CC_*(\mathscr{D}_{X^*})) & & H^{-k+2}(CC_*(\mathscr{D}_{X^*}))
\end{array} \qquad (0.0.3)$$

This construction shows also that the hypercohomology $\mathbb{H}(\mathcal{BC}^{***})$ is the abutment of a spectral sequence whose E_1-terms are cyclic homologies $HC(\mathscr{D}_{X^*})$. On each "column" $\mathcal{BC}^{p**}$ of $\mathcal{BC}^{***}$, both N and S act by dropping a (de-Rham) cohomology group of X^*, which is what is expressed in (0.0.3).

Connes periodicity operator in cyclic homology and monodromy at archimedean infinity

In [2], S. Bloch, H. Gillet and C. Soulé consider an algebraic fibration $f : X \longrightarrow C$ over a curve C, such that the fiber $f^{-1}(P) = Y \subseteq X$ over a closed point $P \in C$, is a divisor with normal crossings. In this set-up, the authors define a cohomological complex on X and show, under certain assumptions, the existence of formal analogues of the Local Invariant Cycle Theorem and the Lefschetz Theorem(s). Motivated in part by their construction, C. Consani defined in [18] a complex K^{***} with monodromy-like operator $N : K^{i,j,k} \longrightarrow K^{i+2,j,k+1}$ on a smooth, projective algebraic variety X over $\mathbb{C}$ or $\mathbb{R}$, with the goal to introduce a theory of nearby cycles at "archimedean infinity". The terms $K^{i,j,k}$ are modules of real differential forms on X twisted by suitable powers of $(2\pi i)$. The cohomologies of the complex K^{***} behave formally as the E_1-terms of the spectral sequence associated to the Picard-Lefschetz filtration on the Steenbrink's complex A^{**}.

In this thesis, we introduce bicomplexes K_t^{**} which are defined by suitably combining the terms $K^{i,j,k}$ in which the same "twist" by $(2\pi i)^t$ appears. This construction (of grouping terms with the same twist) is the analogue of the operation of looking at a fixed "stratum" of Y in the construction of [2]. Then, we show in Proposition 4.8 that there are natural maps from the Hochschild cohomology of the ring of differential operators on X to the cohomology of the graded pieces of the filtration $F^l K_t^{**} := Im(N^l)$ $(l \geq 0)$ on the bicomplex K_t^{**}. This filtration coincides with the filtration by rows on K_t^{**}. More precisely, we obtain the following

Theorem 0.3. *For each fixed $t \in \mathbb{Z}$, the filtration $F^l(Tot(K_t^{**})) = Im(N^l)$ $(l \geq 0)$ coincides with the filtration by rows on $Tot(K_t^{**})$. There are natural maps from the Hochschild cohomology of the ring of differential operators on X to the homology of the graded pieces of the filtration F, i.e. natural maps ($n = \dim X$)*

$$HH^{q+n}(\mathscr{D}(X)) \longrightarrow H^{q-2t}(Gr_F^l Tot(K_t^{**})).$$

The operator N acts on the complexes K_t^{**}; $N : K_t^{**} \longrightarrow K_{t-1}^{*+1,*+1}$ by shifting the index t. Because of that, we consider (in analogy with the construction of the complex $\mathcal{BC}^{***}$ in the Steenbrink case) a triple complex $\mathcal{K}^{***}$, which is again assembled in a precise way as a "vertical stack" of all the bicomplexes K_t^{**} (as t varies in $\mathbb{Z}$). The map N acts between two consecutive vertical levels. $\mathcal{K}^{***}$ carries differentials d', d'' induced by the ones on the K_t^{**}'s. By the q-th "row" of the triple complex $\mathcal{K}^{***}$, we mean the double complex $\mathcal{K}^{*q*}$. Following this technique, we are able to prove in Proposition 4.9 the existence of a natural map from the cyclic cohomology $HC^*(\mathscr{D}(X))$ to the cohomology of the graded pieces of the filtration by "rows" on $\mathcal{K}^{***}$. The "rows" $\mathcal{K}^{*q*}$ carry operators N induced by the complexes K_t^{**} (that have been stacked to form the triple complex $\mathcal{K}^{***}$). The cyclic complex carries the periodicity operator

S which, under the above natural map, corresponds to the operator N on the "row" $\mathcal{K}^{*q*}$. More precisely, we prove the following statement, which is the archimedean counterpart of Theorem 0.2:

Theorem 0.4. *Let us consider the filtration by "rows" on $\mathcal{K}^{***}$, i.e. the filtration whose q-th graded piece is the bicomplex $\mathcal{K}^{*q*}$. There are natural maps connecting the cyclic cohomology of the ring $\mathscr{D}(X)$ to the cohomology of the bicomplex $\mathcal{K}^{*q*}$ ($n = \dim X$)*

$$HC^{j+n}(\mathscr{D}(X)) \to H^j(\mathcal{K}^{*q*}) \qquad \forall\ j \geq 0.$$

The bicomplex $\mathcal{K}^{*q*}$ is equipped with an operator N induced by that on the K_t^{**}. The map N corresponds to the periodicity operator S appearing in the cyclic cohomology of $\mathscr{D}(X)$, i.e. we have the following commutative diagram

$$\begin{array}{ccccc} HC^{j+n-2}(\mathscr{D}(X)) & \xrightarrow{S} & HC^{j+n}(\mathscr{D}(X)) & \longrightarrow & HH^{j+n}(\mathscr{D}(X)) \\ \downarrow & & \downarrow & & \downarrow \\ H^{j-2}(\mathcal{K}^{*q*}) & \xrightarrow{N} & H^j(\mathcal{K}^{*q*}) & \longrightarrow & H^j(Coker(N)^*) \end{array} \qquad (0.0.4)$$

It is important to explicitly remark here that the vertical maps in the diagram (0.0.4), unlike those in (0.0.3), are not in general isomorphisms. There are at least two technical reasons to justify this weaker statement. First of all, the pairing between cyclic homology and cohomology is not in general a perfect pairing *i.e.* the natural map $HC^*(\mathscr{D}(X)) \longrightarrow Hom(HC_*(\mathscr{D}(X)), \mathbb{C})$ is not in general an isomorphism. Hence, the isomorphism (0.0.2) does not have a dual counterpart in cyclic cohomology. The second reason is that in the archimedean case, unlike in the construction over a disc, I work with complexes of modules (*i.e.* global sections of sheaves) rather than with complexes of sheaves. The quasi isomorphism of complexes of sheaves that lead to the vertical isomorphisms in the commutative diagram (0.0.3) is here replaced only by a natural map connecting complexes of global section of the corresponding complexes of sheaves.

I conclude this overview of this part of my thesis by saying that in this archimedean setup I also introduce two further bicomplexes φ^{**} and B^{**} which are linked by a map $\mu : \varphi^{**} \to B^{**}$. These complexes are also endowed with monodromy-type operators N and they are the replacement for the nearby cycles complex $\psi^{**}(\mathbb{C})$ and the Steenbrink complex A^{**} in this framework. In particular, I prove in Proposition 4.11 that the maps $N \circ \mu$ and $\mu \circ N$ are again chain homotopic, in exact analogy with the corresponding result in the theory developed by Steenbrink.

Connes-Karoubi long exact sequence in K-theory

We have already described how the monodromy operator on the nearby cycles complex can be identified with the periodicity operator in cyclic homology. The diagram (0.0.3) shows in particular that the theory of nearby cycles has a very close connection with the cyclic theory of differential operators on X^*. A question that naturally arises, also in view of my result is whether the periodicity exact sequence in cyclic theory can be lifted to a long exact sequence involving a "motivic theory" such as Chow groups or K-theory.

In [13], Connes and Karoubi exhibit a long exact sequence in K-theory that lifts the long exact periodicity sequence (0.0.1).

$$\begin{array}{ccccccccc}
K_{i+1}^{alg}(A) & \longrightarrow & K_{i+1}^{top}(A) & \longrightarrow & K_i^{rel}(A) & \longrightarrow & K_i^{alg}(A) & \longrightarrow & K_i^{top}(A) \\
\downarrow{\scriptstyle D_{i+1}} & & \downarrow{\scriptstyle ch_{i+1}} & & \downarrow{\scriptstyle ch_i^{rel}} & & \downarrow{\scriptstyle D_i} & & \downarrow{\scriptstyle ch_i} \\
H_{i+1}(A,A) & \longrightarrow & HC_{i+1}(A) & \longrightarrow & HC_{i-1}(A) & \longrightarrow & H_i(A,A) & \longrightarrow & HC_i(A)
\end{array} \tag{0.0.5}$$

In the above diagram, $K_i^{alg}(A)$, $K_i^{top}(A)$ and $K_i^{rel}(A)$ denote *resp.* the algebraic, topological and relative K-theories of a locally convex algebra A. The vertical maps D_i is the Dennis trace. The topological and relative Chern maps ch_i^{top} and ch_i^{rel} have been constructed in [13].

To link this construction with the geometric set-up of nearby cycles, we consider once again the sheaf $\mathscr{D}_{X^*}$ of differential operators on X^*, where the variety X^* is as in the Steenbrink setup described earlier (although our results are valid for any locally Stein manifold). For any open set $U \subseteq X^*$, we denote by $\mathscr{D}(U)$ the ring of differential operators on U. Then, by definition one has

$$K_i^{alg}(\mathscr{D}(U)) = \pi_i(BGL(\mathscr{D}(U))^+) \qquad K_i^{top}(\mathscr{D}(U)) = \pi_i(BGL(\mathscr{D}(U))_*)$$

$$K_i^{rel}(\mathscr{D}(U)) = \pi_i((GL(\mathscr{D}(U)_*)/GL(\mathscr{D}(U)))^+),$$

where $\mathscr{D}(U)_*$ denotes the simplicial ring $C^\infty(\Delta^*)\hat{\otimes}\mathscr{D}(U)$ and Δ^* is the standard simplex. Let $\mathbf{BGL}^+$ (resp. $\mathbf{BGL}^{top}$ and $\mathbf{GL}^{rel+}$) to be the simplicial sheaf obtained from the presheaf that associates to an open set $U \subset X^*$ the simplicial set $BGL(\mathscr{D}(U))^+$ (resp. $BGL(\mathscr{D}(U)_*)$ and $(GL(\mathscr{D}(U)_*)/GL(\mathscr{D}(U)))^+$). Using the definition of generalized cohomology for simplicial sheaves given by Brown and Gersten in [5], we introduce the following groups

$$\begin{aligned} K_i^{alg}(\mathscr{D}_X^*) &= H^{-i}(X^*, \mathbf{BGL}^+) := \pi_i\Gamma(X^*, R(\mathbf{BGL}^+)) \\ K_i^{top}(\mathscr{D}_X^*) &= H^{-i}(X^*, \mathbf{BGL}^{top}) := \pi_i\Gamma(X^*, R(\mathbf{BGL}^{top})) \\ K_i^{alg}(\mathscr{D}_X^*) &= H^{-i}(X^*, \mathbf{GL}^{rel+}) := \pi_i\Gamma(X^*, R(\mathbf{GL}^{rel+})) \end{aligned} \tag{0.0.6}$$

where $R(\mathbf{BGL}^+)$ refers to the flasque resolution of the simplicial sheaf $\mathbf{BGL}^+$ (and similarly for $R(\mathbf{BGL}^{top})$ and $R(\mathbf{GL}^{rel+})$). The flasque resolution of a simplicial sheaf $\mathcal{F}$ on X^* is a simplicial sheaf $R(\mathcal{F})$ endowed with a monomorphism $\mathcal{F} \to R(\mathcal{F})$ which is a weak equivalence and a global fibration $R(\mathcal{F}) \longrightarrow *$.

In the thesis, I show that there is a stalkwise fibration (*aka* "local fibration") of simplicial sheaves $\mathbf{BGL}^+ \to \mathbf{BGL}^{top}$ which determines a long exact sequence

$$\ldots \longrightarrow H^{-i}(X^*, \mathbf{GL}^{rel+}) \longrightarrow H^{-i}(X^*, \mathbf{BGL}^+) \longrightarrow H^{-i}(X^*, \mathbf{BGL}^{top}) \longrightarrow$$

Therefore, using the definitions in (0.0.6), we obtain a long exact sequence connecting the groups $K_i^{alg}(\mathscr{D}_{X^*})$, $K_i^{top}(\mathscr{D}_{X^*})$ and $K_i^{rel}(\mathscr{D}_{X^*})$, in exact analogy with the upper sequence in (0.0.5). Moreover, we show that the homotopy sheaves of $R(\mathbf{BGL}^+)$ and the sheaf associated to the presheaf $X^* \supset U \to BGL^+(\mathscr{D}(U))$ are identical. By patching over the Stein open subsets of X^* (which form a basis), we construct the Dennis trace $D_i : K_i^{alg}(\mathscr{D}_{X^*}) \longrightarrow HH_i(\mathscr{D}_{X^*})$. Similarly, we am able to define Chern maps ch_i^{top} and ch_i^{rel} and obtain a commutative diagram analogous to (0.0.5) for the sheaf $\mathscr{D}_{X^*}$. The full result is summarized by the following

Theorem 0.5. *There exists a local fibration of simplicial sheaves* $\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top}$. *This determines a long exact sequence connecting algebraic, topological and relative* K*-theories of the sheaf of differential operators on* X^* *as well as connecting maps* $D_i : \mathscr{K}_i^{alg}(\mathscr{D}_{X^*}) \longrightarrow HH_i(\mathscr{D}_{X^*})$, $ch_i^{top} : \mathscr{K}_i^{top}(\mathscr{D}_{X^*}) \longrightarrow HC_{i+1}(\mathscr{D}_{X^*})$ *and* $ch_i^{rel} : \mathscr{K}_i^{rel}(\mathscr{D}_{X^*}) \longrightarrow HC_{i-1}(\mathscr{D}_{X^*})$ *which fit into the following commutative diagram*

$$\begin{tikzcd}
\ldots \to K_{i+1}^{rel}(\mathscr{D}_{X^*}) \arrow[r] \arrow[d, "ch_i^{rel}"] & K_{i+1}^{alg}(\mathscr{D}_{X^*}) \arrow[r] \arrow[d, "D_i"] & K_{i+1}^{top}(\mathscr{D}_{X^*}) \to \ldots \arrow[d, "ch_i^{top}"] \\
\ldots \to HC_i(\mathscr{D}_{X^*}) \arrow[r] & HH_{i+1}(\mathscr{D}_{X^*}) \arrow[r] & HC_{i+1}(\mathscr{D}_{X^*}) \to \ldots
\end{tikzcd}$$

A similar result holds true also when X^* gets replaced by an algebraizable Noetherian formal scheme $\mathfrak{X}$. In the thesis, I also show that when $\mathfrak{X}$ arises by completing a Noetherian scheme X along an irreducible subscheme Y, the sheaves $\mathbf{GL}^{rel}$, $\mathbf{BGL}$

and $\mathbf{BGL}^{top}$ are flasque, i.e. they coincide with their own flasque resolution.

Modular Hecke Algebras, Rankin-Cohen brackets and an enriched archimedean complex

For any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, Connes and Moscovici have defined in [10] a modular Hecke algebra $\mathcal{A}(\Gamma)$ which is an enlargement of the classical algebra $\mathcal{H}(\Gamma)$ of Hecke operators. The algebra $\mathcal{A}(\Gamma)$ inherits a natural action of the Hopf algebra $\mathcal{H}_1$ of "codimension 1 foliations". The algebra $\mathcal{H}_1$ was originally defined by Connes and Moscovici in [12], in the context of Riemannian geometry, as a member of a larger family $\mathcal{H}_n$ $(n \geq 1)$ of Hopf algebras. It is a remarkable fact that $\mathcal{H}_1$ acts naturally on the specially constructed algebra $\mathcal{A}(\Gamma)$ involving Hecke operators and (elliptic) modular forms. This leads one to naturally introduce the concepts of "Schwarzian derivation" and "Godbillon-Vey cocycle" in the context of modular forms and Hecke operators. The Hopf algebra $\mathcal{H}_1$ captures the symmetries of the modular Hecke algebra $\mathcal{A}(\Gamma)$. $\mathcal{A}(\Gamma)$ is, by definition, an algebra of functions on $\Gamma\backslash G_2^+(\mathbb{Q})$ taking values in the algebra of modular forms, i.e. in the direct limit over all levels of the abelian groups of elliptic modular forms of all weights. The modular forms of level Γ are global sections of the tensor algebra $T(\mathscr{L}(\Gamma))$ of a line bundle $\mathscr{L}(\Gamma)$ over the modular curve $X(\Gamma)$. This suggests that the operators in $\mathcal{H}_1$ should act on a similar module of functions on $\Gamma\backslash G_2^+(\mathbb{Q})$ taking values in a suitably defined "archimedean complex of the modular tower with coefficients in the tensor algebra $\varinjlim T(\mathscr{L}(\Gamma))$".

As a first step of our construction, we shall consider the direct limit $K^{i,j,k} = \varinjlim_N K^{i,j,k}(X(N))$ of the archimedean complexes $K^{i,j,k}(X(N))$ defined by Consani in [18], where the complex algebraic variety is a compactified modular curve $X(N) = \overline{\Gamma(N)\backslash\mathbb{H}}$ and the modules of differential forms are tensored with elliptic modular forms of corresponding level and suitable weights. Elements of $K^{i,j,k}$ are (finite) sums made by terms of the form $f \otimes \omega$, where f is an elliptic modular form and ω is a twisted real differential on the modular curve $X(N)$. Then, we define $\mathbb{B}^{i,j,k}(\Gamma)$ to be the set of functions

$$F : \Gamma\backslash G_2^+(\mathbb{Q}) \longrightarrow K^{i,j,k} \tag{0.0.7}$$

with finite support such that: if $F(\Gamma\alpha) = f \otimes \omega$ for some $\alpha \in G_2^+(\mathbb{Q})$, then $F(\Gamma\alpha\gamma) = f|\gamma \otimes \omega$ for any $\gamma \in \Gamma$. The differentials d' and d'' on the archimedean complex $K^{i,j,k}$ induce differentials $d' : \mathbb{B}^{i,j,k}(\Gamma) \longrightarrow \mathbb{B}^{i+1,j+1,k+1}(\Gamma)$ and $d'' : \mathbb{B}^{i,j,k}(\Gamma) \longrightarrow \mathbb{B}^{i+1,j+1,k}(\Gamma)$.

For technical reasons, we also introduce a slight variant $\mathcal{A}_T(\Gamma)$ of Connes-Moscovici's modular Hecke algebra. We define $\mathcal{A}_T(\Gamma)$ to be the set of functions

$$F : \Gamma\backslash G_2^+(\mathbb{Q}) \longrightarrow \mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T_+] \tag{0.0.8}$$

with finite support, where T_+ denotes the semigroup consisting of all non negative powers of $(2\pi i)^{-1}$. As before, we require that if $F(\Gamma\alpha) = f \otimes \varepsilon$ for any $\alpha \in G_2^+(\mathbb{Q})$, then $F(\Gamma\alpha\gamma) = f|\gamma \otimes \varepsilon$ for any $\gamma \in \Gamma$. Then, $\mathcal{A}_T(\Gamma)$ becomes an algebra under the product

$$(F * G)(\Gamma\alpha) = \sum_{\beta \in \Gamma\backslash G_2^+(\mathbb{Q})} (F(\Gamma\beta) \cdot G(\Gamma\alpha\beta^{-1})|\beta) \qquad F, G \in \mathcal{A}_T(\Gamma). \tag{0.0.9}$$

We then show that $\mathbb{B}^{i,j,k}(\Gamma)$ is a bimodule over $\mathcal{A}_T(\Gamma)$ and moreover that $\mathbb{B}^{i,j,k}(\Gamma)$ carries a flat action of the Hopf algebra $\mathcal{H}_1$ of "codimension 1 foliations". As a Hopf algebra, $\mathcal{H}_1$ is defined by generators X, Y and δ_1 and commutators/coproducts

$$[Y, X] = X \qquad [X, \delta_n] = \delta_{n+1} \qquad [Y, \delta_n] = n\delta_n \qquad [\delta_k, \delta_l] = 0$$

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \quad \Delta(Y) = 1 \otimes Y + Y \otimes 1 \quad \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1. \tag{0.0.10}$$

$\mathcal{H}_1$ acts on both $\mathcal{A}_T(\Gamma)$ and $\mathbb{B}^{i,j,k}(\Gamma)$. The action of $\mathcal{H}_1$ on $\mathbb{B}^{i,j,k}(\Gamma)$ is flat in the sense that

$$h(F * G) = \sum h_{(1)}(F) * h_{(2)}(G) \qquad h \in \mathcal{H}_1$$

if $F \in \mathcal{A}_T(\Gamma)$, $G \in \mathbb{B}^{i,j,k}(\Gamma)$ and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

The generator X of $\mathcal{H}_1$ acts as the Ramanujan derivation on modular forms, Y as the grading operator and δ_1 measures the difference $X(f|\alpha)|\alpha^{-1} - X(f)$ for some $\alpha \in G_2^+(\mathbb{Q})$ and an element f of the modular tower.

The definition of an action of $\mathcal{H}_1$ on the enriched archimedean complex $\mathbb{B}^{i,j,k}(\Gamma)$ is motivated by a formal similarity shared by the operator X and the monodromy operator N on the archimedean complex $K^{i,j,k}$ and by the operator Y and the negative Frobenius operator $-\Phi$ on $K^{i,j,k}$ (see [16]). Note in particular that the relation $[-\Phi, N] = N$ is similar to the relation $[Y, X] = X$.

Thereafter, we also consider a "reduced product" on $\mathcal{A}_T(\Gamma)$, defined as follows. For F, $G \in \mathcal{A}_T(\Gamma)$, we define

$$(F *^r G)(\alpha) = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (F(\beta) \cdot G(\alpha\beta^{-1})|\beta). \tag{0.0.11}$$

Notice that in the above formula one sums over the cosets of $\Gamma \subset SL_2(\mathbb{Z})$ rather than the cosets in $G_2^+(\mathbb{Q})$ as in (0.0.9). We refer to $\mathcal{A}_T^r(\Gamma)$ as to the algebra $\mathcal{A}_T(\Gamma)$ endowed with the product $*^r$. Notice that the operator $\delta_1 \in \mathcal{H}_1$ measures the difference

$X(f|\alpha)|\alpha^{-1} - X(f)$, for $\alpha \in G_2^+(\mathbb{Q})$ and an element f of the modular tower. Because $X(f|\alpha)|\alpha^{-1} - X(f) = 0$ when $\alpha \in SL_2(\mathbb{Z})$, summing over the cosets of Γ in $SL_2(\mathbb{Z})$ as in (0.0.11) allows one to define a "reduced" bimodule action of a smaller Hopf algebra $\mathfrak{h}_1$ on $\mathbb{B}^{i,j,k}(\Gamma)$. The Hopf algebra $\mathfrak{h}_1$ is obtained from $\mathcal{H}_1$ by setting all $\delta_n = 0$ in (0.0.10). When we think of $\mathbb{B}^{i,j,k}$ as a bimodule over $\mathfrak{h}_1$, we denote it by $\mathbb{B}_r^{i,j,k}(\Gamma)$. We then show that $\mathfrak{h}_1$ has a flat action on $\mathbb{B}_r^{i,j,k}(\Gamma)$ which is a bimodule over $\mathcal{A}_T^r(\Gamma)$. These results described in this section can be summarized as follows

Theorem 0.6. *For any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, the algebra $\mathbb{B}^*(\Gamma)$ as in* (0.0.7) *is a bimodule over the algebra $\mathcal{A}_T(\Gamma)$. Moreover, the Hopf algebra $\mathcal{H}_1$ has a flat action on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$, i.e. for any $F \in \mathcal{A}_T(\Gamma)$, $G \in \mathbb{B}^*(\Gamma)$ and any given $h \in \mathcal{H}$, one has*

$$h(F * G) = \sum h_{(1)}(F) * h_{(2)}(G) \qquad \Delta(h) = \sum h_{(1)} \otimes h_{(2)}.$$

Then $\mathbb{B}_r^*(\Gamma)$ is a bimodule over the "reduced" algebra $\mathcal{A}_T^r(\Gamma)$ as in (0.0.11)*, with the action defined by considering cosets of Γ in $SL_2(\mathbb{Z})$, rather than in $G_2^+(\mathbb{Q})$. The Hopf algebra $\mathfrak{h}_1$ ($\delta_n = 0,\ \forall n > 1$) has a flat action on the system $(\mathcal{A}_T^r(\Gamma), \mathbb{B}_r^*(\Gamma))$, i.e. for any $F \in \mathcal{A}_T^r(\Gamma)$ and $G \in \mathbb{B}_r^*(\Gamma)$ and any given $h \in \mathfrak{h}_1$, one has*

$$h(F * G) = \sum h_{(1)}(F) * h_{(2)}(G) \qquad \Delta(h) = \sum h_{(1)} \otimes h_{(2)}.$$

In the final part of the thesis we introduce the definition of Rankin Cohen brackets on $\mathbb{B}_r^{i,j,k}(\Gamma)$ and $\mathbb{B}^{i,j,k}(\Gamma)$. For this construction I use the work of Connes and Moscovici [9] and the definition of a pairing defined on the archimedean complex $K^{i,j,k}$ in [18].

1 Background on Cyclic Homology and Cyclic Cohomology

Cyclic cohomology was introduced by Connes (cf [14]) and B. Tsygan as an analogue for de Rham cohomology in the case of a noncommutative algebra.

1.1 Basic Definitions

Given an algebra A over a commutative ring k, we define its Hochschild homology $HH_q(A)$ to be the q-th homology of the complex

$$C^h_* : \ldots \xrightarrow{b} A \otimes_k A \otimes_k A \xrightarrow{b} A \otimes_k A \xrightarrow{b} A \longrightarrow 0 \tag{1.1.1}$$

We shall denote the $(n+1)$-the tensor power of A over k by $A^{\otimes n+1}$. In the sequence (1.1.1), the differential $b : A^{\otimes n+1} \to A^{\otimes n}$ is defined as follows:

$$\begin{aligned} b(a_0 \otimes a_1 \otimes ... \otimes a_n) &= \sum_{i=0}^{n-1} a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n \\ &+ (-1)^n a_n a_0 \otimes ... \otimes a_{n-1} \end{aligned} \tag{1.1.2}$$

The differential b in (1.1.2) may be described as the alternating sum $b = \sum_{i=0}^{n}(-1)^i d_i$ of the maps $d_i : A^{\otimes n+1} \to A^{\otimes n}$ which are defined as follows

$$\begin{aligned} & d_0(a_0 \otimes a_1 \otimes ... \otimes a_n) = a_0a_1 \otimes ... \otimes a_n \\ & d_i(a_0 \otimes a_1 \otimes ... \otimes a_n) = a_0 \otimes ... \otimes a_ia_{i+1} \otimes ... \otimes a_n \qquad 1 \leq i \leq n-1 \\ & d_n(a_0 \otimes a_1 \otimes ... \otimes a_n) = a_na_0 \otimes a_1 \otimes ... \otimes a_{n-1} \end{aligned} \tag{1.1.3}$$

The maps d_i are usually referred to as the *face maps*. We also set $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ and note that, by construction, $b'^2 = 0$ as well. One also introduces another set of maps $s_i : A^{\otimes n} \to A^{\otimes n+1}$, $0 \leq i \leq n$, which are referred to as *degeneracy operators* and they are defined as follows:

$$\begin{aligned} & s_0(a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) = a_0 \otimes 1 \otimes a_1 \otimes ... \otimes a_{n-1} \\ & s_i(a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) = a_0 \otimes ... \otimes a_{i-1} \otimes 1 \otimes a_i \otimes ... \otimes a_{n-1} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leq i \leq n-2) \\ & s_{n-1}(a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) = a_0 \otimes ... \otimes a_{n-1} \otimes 1 \\ & s_n(a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) = 1 \otimes a_0 \otimes ... \otimes a_{n-1} \end{aligned} \tag{1.1.4}$$

The map $s_n : A^{\otimes n} \to A^{\otimes n+1}$ is called the *extra degeneracy*. In order to define the cyclic bicomplex, we first introduce the *cyclic operator* defined as

$$t_n : A^{\otimes n+1} \to A^{\otimes n+1} \qquad t_n(a_0 \otimes a_1 \otimes ... \otimes a_n) = (-1)^n(a_n \otimes a_0 \otimes ... \otimes a_{n-1}) \tag{1.1.5}$$

as well as

$$N : A^{\otimes n+1} \to A^{\otimes n+1} \qquad N = 1 + t_n + t_n^2 + ... + t_n^n \tag{1.1.6}$$

We notice that the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts on $A^{\otimes n+1}$ through its generator $t = t_n$. From the definition of t_n in (1.1.5), it follows that $t_n^{n+1} = 1$. In what follows, we will often omit the subscript and refer simply to the cyclic operator t when there is no danger of confusion. Sometimes, we will also denote the element $a_0 \otimes a_1 \otimes ... \otimes a_n$ of the tensor product $A^{\otimes n}$ simply by $(a_0, ..., a_n)$. One can easily check that $(1-t)b' = b(1-t)$ and that $b'N = Nb$.

$$\begin{array}{ccccccccc}
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow[N]{} & \dots \\
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow[N]{} & \dots \\
\Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle -b'} & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow[N]{} & \dots
\end{array} \tag{1.1.7}$$

Definition 1.1. *(see [31, 2.1.3]) Define the bicomplex $CC(A)$ of* (1.1.7) *formally as follows: Set*

$$CC(A)_{p,q} = A^{\otimes q+1} \qquad p, q \geq 0 \tag{1.1.8}$$

together with maps (for $k, l \in \mathbb{Z}_{\geq 0}$)

$$\begin{array}{llll}
CC(A)_{2k,q+1} \xrightarrow{b} CC(A)_{2k,q} & & CC(A)_{2k+1,q+1} \xrightarrow{-b'} CC(A)_{2k+1,q} \\
CC(A)_{2k+1,q} \xrightarrow{1-t} CC(A)_{2k,q} & & CC(A)_{2k+2,2l+1} \xrightarrow{N} CC(A)_{2k+1,q}
\end{array} \tag{1.1.9}$$

We define the cyclic homology $HC_q(A)$ to be the q-th total homology of the bicomplex $CC(A)$.

$$HC_q(A) = H_q(Tot\ CC(A)) \tag{1.1.10}$$

We begin by noting that the b'-columns in the bicomplex above are acyclic. If s

denotes the *extra degeneracy* as defined before, it is easy to verify that

$$b's + sb' = id. \tag{1.1.11}$$

and hence the identity map on the b'-complex is chain homotopic to 0. Hence, the b' complex is contractible.

We define the *Connes boundary operator* B as follows:

$$B : A^{\otimes n} \to A^{\otimes n+1} \qquad B = (1-t)sN \tag{1.1.12}$$

where s is, once again, the extra degeneracy as defined above, t denotes the cyclic operator t_n, and N is, as in (1.1.6), defined to be the sum $N = 1+t_{n-1}+t_{n-1}^2+...+t_{n-1}^{n-1}$.

From now onwards, we will assume that A is a unital k-algebra. We define the *mixed complex* $\mathcal{B}_{**}(A)$

$$\mathcal{B}_{p,q}(A) = A^{\otimes q+1-p} \qquad q, p \geq 0 \tag{1.1.13}$$

with the two differentials

$$B : \mathcal{B}_{p+1,q}(A) \to \mathcal{B}_{p,q}(A) \qquad b : \mathcal{B}_{p,q+1}(A) \to \mathcal{B}_{p,q}(A)$$

More explicitly, we have

$$B : A \otimes A^{\otimes n} \longrightarrow A \otimes A^{\otimes n+1} \tag{1.1.14}$$

$$B(a_0, a_1, ..., a_n) = \sum_{i=10}^{n} (-1)^{ni}(1, a_{n-i+1}, ..., a_n, a_0, ..., a_{n-i}) \tag{1.1.15}$$

$$+ \sum_{i=0}^{n} (-1)^{n(i+1)}(a_{n-i}, 1, a_{n-i+1}, ..., a_n, a_0, ..., a_{n-i-1})$$

The following proposition will show that the cyclic homology $HC_q(A)$ of a unital k-algebra A can also be calculated from the mixed complex $\mathcal{B}_{**}(A)$.

Proposition 1.2. *(1) For any associative and unital k-algebra A, the cyclic homology $HC_q(A)$ can be described as the q-th total homology of the complex $\mathcal{B}_{**}$, i.e.*

$$H_q(Tot\ \mathcal{B}(A)) \cong HC_q(A) \tag{1.1.16}$$

(2) Let A be a unital k-algebra and let $\bar{A} = A/k$. If we replace each term $A^{\otimes n+1}$ in $\mathcal{B}_{**}(A)$ by $A \otimes \bar{A}^{\otimes n}$, we obtain a normalized bicomplex $\overline{\mathcal{B}}(A)_{p,q} = A \otimes \bar{A}^{q-p}$ endowed with differentials $\bar{B}$ and $\bar{b}$. Then, there exists a canonical isomorphism

$$H_q(Tot\ \overline{\mathcal{B}}(A)) \cong HC_q(A) \tag{1.1.17}$$

Proof. See [31, 2.1.7-11]. $\square$

From the structure of the bicomplex $\mathcal{B}(A)$ it is clear that we have a short exact sequence

$$0 \longrightarrow C^h_*(A) \longrightarrow Tot(\mathcal{B}(A)) \xrightarrow{\alpha} Tot(\mathcal{B}(A)[-1,-1]) \longrightarrow 0 \tag{1.1.18}$$

The associated long exact sequence of homologies gives rise to the following:

Proposition 1.3. *(Periodicity Sequence) For any associative and unital k-algebra A, the homology long exact sequence associated to* (1.1.18) *determines a long exact sequence*

$$\ldots \longrightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \longrightarrow \ldots \tag{1.1.19}$$

The operator S appearing in (1.1.19) *induced in homology by the map α in* (1.1.18) *is referred to as the Connes periodicity operator.*

The formalism of degeneracies and face maps which we have described above may be conveniently summarized in terms of a *simplicial category* Δ. The objects of the small category Δ are sets $[n]$, for each $n \geq 1$. The morphisms of in Δ are of two kinds; face maps $\delta_i : [n-1] \to [n]$, $0 \leq i \leq n$ and degeneracies $\sigma_j : [n+1] \to [n]$, $0 \leq j \leq n$ for each $n \geq 1$. The relations between the face maps and the degeneracies are as follows:

$$\begin{aligned} &\delta_j \delta_i = \delta_i \delta_{j-1} \\ &\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \\ &\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j \\ id_{[n]} & \text{for } i = j,\ i = j+1 \\ \delta_{i-1} \sigma_j & \text{for } i > j+1 \end{cases} \end{aligned} \tag{1.1.20}$$

The simplicial category Δ is, therefore, isomorphic to a category whose objects are sets $[n] = \{0, 1, 2, ..., n\}$ and whose morphisms from $[m]$ to $[n]$ are non decreasing maps from $\{0, 1, 2, ..., m\}$ to $\{0, 1, 2, ..., n\}$.

Then a *simplicial object* (resp. *cosimplicial object*) in a category $\mathscr{C}$ is a contravariant (resp. covariant) functor $F : \Delta \to \mathscr{C}$. When the functor F is contravariant, the images of the maps δ_i and σ_j are denoted by d_i and s_j respectively. If F is a simplicial object in an abelian category $\mathscr{C}$, we can form a complex $F_h^k = F([-k])$, $k \leq 0$, with the differential $b = \sum_{i=0}^{k} (-1)^i d_i : F([k]) \to F([k-1])$.

The Hochschild homology of the k-algebra A, is therefore obtained as the homology of this complex when we define $F([n]) = A^{\otimes n+1}$ taking values in the categories of k-modules.

The category Δ is subsumed in the larger cyclic category ΔC, which has the same objects as Δ, and which, in addition has a cyclic operator $\tau_n : [n] \to [n]$ that satisfies the following relations:

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \text{ for } 1 \leq i \leq n \qquad \text{and} \qquad \tau_n \delta_0 = \delta_n \qquad \tau_n^{n+1} = id$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \text{ for } 1 \leq i \leq n \qquad \text{and} \qquad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \tag{1.1.21}$$

A *cyclic* (resp. *cocylic*) object in a category $\mathscr{C}$ is a contravariant (resp. covariant) functor $F : \Delta C \to \mathscr{C}$. If F is contravariant, the image of the morphism τ_n is denoted by t_n. When $\mathscr{C}$ is an abelian category, we can form the corresponding cyclic bicomplex (1.1.7) . The cyclic homology of a k-algebra A is, then the total homology of this bicomplex when we define the (contravariant) functor $F([n]) = A^{\otimes n+1}$ taking values in the (abelian)category of k-modules.

Definition 1.4. *(see [31, 2.4.1]) Let A be an associative and unital k-algebra. We denote the by $C^0(A)$ the algebraic dual $Hom(A, k)$ of A. In general, we set $C^n(A) = Hom(A^{\otimes n+1}, k)$. We refer to $C^n(A)$ as the space of n-cochains on A.*

By working with the space of cochains on A, we can dualize the bicomplex $CC_{**}(A)$ to get the bicomplex $CC^{**}(A)$. The bicomplex $CC^{**}(A)$ has vertical differentials b^* and $b'^* : CC^{p,q}(A) \to CC^{p,q+1}(A)$ and horizontal differentials $(1-t)^*$ and $N^* : CC^{p,q}(A) \to CC^{p,q+1}(A)$, which are the duals of the differentials in (1.1.9).

The cyclic cohomology of A is the homology of the cochain complex $Tot(CC^{**}(A))$, i.e.

$$HC^n(A) := H^n(Tot(CC_{**}(A))) \tag{1.1.22}$$

We say that a cochain $f \in C^n(A)$ is cyclic if it satisfies the relation

$$f(a_0, a_1, ..., a_n) = (-1)^n f(a_n, a_0, ..., a_{n-1}) \qquad a_i \in A. \tag{1.1.23}$$

Let us denote the space of cyclic cochains by C^n_λ. It may be checked that the image of a cyclic cochain under the operator b^* is still a cyclic cochain. Therefore, $(C^n_\lambda(A), b^*)$ is a subcomplex of $(C^*(A), b^*)$ and we denote its homology of $H^*_\lambda(A)$.

If k contains $\mathbb{Q}$, it is known (see [31, § 2.4]) the inclusion map $C^*_\lambda(A) \hookrightarrow C^*(A)$ induces an isomorphism

$$H^n_\lambda(A) \xrightarrow{\sim} HC^n(A) \qquad n \geq 0$$

This construction, due to Connes [14], gives an alternative definition of cyclic cohomology. As in the homological framework, we have a periodicity sequence

Proposition 1.5. *[31, S2.4.4] An associative k-algebra gives rise to a long exact sequence*

$$\dots \longrightarrow HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} \dots \tag{1.1.24}$$

Following [31, 1.5.9, 2.4.8], we describe the pairing between cyclic cohomology and cyclic homology of a k-algebra A

$$< .,. >: HC^n(A) \times HC_n(A) \to k \tag{1.1.25}$$

We denote by $A^e = A \otimes_k A^{op}$, the enveloping algebra of A and note that $C^n(A) = Hom(A^{\otimes n+1}, k) = Hom(A^{\otimes n}, A^*)$, where $A^* = Hom(A, k)$. The complex $(C^n(A), b^*)$

may therefore be replaced by an isomorphic complex $(Hom(A^{\otimes n}, A^*), \beta)$ which we denote by $(C^n(A, A^*), \beta)$. We now obtain a pairing

$$C^n(A, A^*) \times C_n(A) \longrightarrow A^* \otimes_{A^e} A \tag{1.1.26}$$

by setting

$$(f, (a_0, a_1, ..., a_n)) \mapsto f(a_1, ..., a_n) \otimes a_0$$

One can easily check that

$$< \beta(f), x > = < f, b(x) > \qquad f \in C^n(A, A^*), x \in C_{n+1}(A) \tag{1.1.27}$$

This induces a pairing on the homologies (called *Kronecker product*)

$$< ., . >: HH^n(A) \times HH_n(A) \longrightarrow A^* \otimes_{A^e} A \tag{1.1.28}$$

This pairing further extends to cyclic homology (see [31, §2.4.8.2]) and cohomology

$$HC^n(A) \times HC_n(A) \longrightarrow A^* \otimes_{A^e} A \tag{1.1.29}$$

When we compose this latter pairing (1.1.29) with the evaluation map $ev : A^* \otimes_{A^e} A \rightarrow k$, $ev(f \otimes a) = f(a)$, we get a pairing

$$HC^n(A) \times HC_n(A) \longrightarrow k \tag{1.1.30}$$

We mention here that this pairing is not usually perfect, but it is non degenerate in certain cases, for instance, if $A = k$ or if k is a field and A is a k-algebra that is finite dimensional as a k-vector space (see [31, §2.4.8]).

1.2 The case of a topological algebra

The definitions of Hochschild and cyclic homology which we have reviewed in the previous section may be extended to a topological algebra over a field k. We shall be particularly interested in the case of the ring of holomorphic (resp. smooth) differential operators over a Stein (resp. compact) manifold. In this regard, we will need to recall some definitions.

If V is a complex vector space, then a seminorm p on V is a map $p : V \to \mathbb{R}$ such that

(1) p is nonnegative, i.e. $p(x) \geq 0$ for all $x \in V$.

(2) p is linear in the sense that $p(\lambda x) = |\lambda| p(x)$ for $\lambda \in \mathbb{C}$. (This implies, in particular, that $p(0) = 0$).

(3) p is subadditive, i.e. $p(x + y) \leq p(x) + p(y)$.

A *locally convex vector space* V is a vector space with a family $\{p_\alpha\}_{\alpha \in I}$ of seminorms defined on it. We make V into a topological space by assigning to it the coarsest topology such that each of the seminorms p_α is continuous. If the family I of seminorms defining the topology on V is countable, we say that V is a *Fréchet space.*

Definition 1.6. *A locally convex algebra A over $\mathbb{R}$ or $\mathbb{C}$ has a family of seminorms $\{p_\alpha\}_{\alpha \in I}$ defining the structure of a locally convex vector space on it. Additionally, each seminorm is assumed to be submultiplicative, i.e.*

$$p_\alpha(fg) \leq p_\alpha(f) p_\alpha(g), \qquad f, g \in A \tag{1.2.1}$$

Finally, we shall need the notion of topological tensor product. If V and W are two topological vector spaces, with norms $|.|_V$ and $|.|_W$ defined on them, a *cross norm* p defined on their algebraic tensor product $V \otimes W$ is a norm satisfying the following two conditions:

(a) $p(x \otimes y) = |x|_V |y|_W$ for all $x \in V$ and $y \in W$.

(b) $p^*(S \otimes T) = |S|_V |T|_W$ for all linear operators $S \in V^*$ and $T \in W^*$, where p^* is the norm on the dual $(V \otimes W)^*$ by p.

It follows that there is a smallest cross norm on $V \otimes W$, called the *injective cross norm* λ, defined to be, for $x \in V \otimes W$,

$$\lambda(x) = \sup\{(S \otimes T)(x) | S \in V^*, T \in W^*, |S|_V \leq 1, |T|_W \leq 1\}. \tag{1.2.2}$$

There is also a largest cross norm, called the *projective cross norm* Λ, defined to be, for $x \in V \otimes W$,

$$\Lambda(x) = \inf\{\sum |a_i|_V |b_i|_W| \text{ over all finite decompositions } x = \sum a_i b_i\}.$$

We also have a notion of algebraic tensor product in either norm, which we denote by $V \otimes_\lambda W$ and $V \otimes_\Lambda W$ respectively.

If the spaces V and W are locally convex, we have a family of seminorms on either of V and W. In that case, we can form an injective cross seminorm and a projective cross seminorm for each pair of seminorms on V and W. We take either all of the injective cross seminorms to define a completion $V \otimes_\epsilon W$ or all of the projective cross seminorms to define a completion $V \otimes_\pi W$. We also have a map $V \otimes_\pi W \to V \otimes_\epsilon W$. When the map $V \otimes_\pi V \longrightarrow V \otimes_\epsilon V$ is an isomorphism, we say that V is a *nuclear space*.

Definition 1.7. *A Fréchet space is a locally convex vector space such that its topology is induced by a countable family $\{p_i\}_{i\in\mathbb{N}}$ of seminorms and the space is complete.*

An algebra A is said to be a Fréchet algebra if it is a Fréchet space as defined above and each of the seminorms is submultiplicative, i.e.

$$p_i(fg) \leq p_i(f)p_i(g), \qquad \forall\ f, g \in A \tag{1.2.3}$$

By replacing the ordinary tensor product $\otimes_k$ by $\otimes_\pi$, we can define a Hochschild complex $C^h_*(A^{top})$ and a cyclic complex $Tot(CC(A^{top}))$. We denote their respective homologies by $HH_*(A^{top})$ and $HC_*(A^{top})$ respectively. There is a morphism of complexes $C^h(A) \to C^h(A^{top})$ and $Tot(CC(A)) \to Tot(CC(A^{top}))$, which induces natural morphisms

$$HH_*(A) \longrightarrow HH_*(A^{top}) \qquad HC_*(A) \longrightarrow HC_*(A^{top}) \tag{1.2.4}$$

We also have a long exact periodicity sequence in the topological context and the morphisms of (1.2.4) induce a morphism of long exact sequences:

$$\begin{array}{ccccccc}
... \to HH_n(A) & \xrightarrow{I} & HC_n(A) & \xrightarrow{S} & HC_{n-2}(A) & \xrightarrow{B} & HH_{n-1}(A) \to ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
... \to HH_n(A^{top}) & \xrightarrow{I} & HC_n(A^{top}) & \xrightarrow{S} & HC_{n-2}(A^{top}) & \xrightarrow{B} & HH_{n-1}(A^{top}) \to ...
\end{array} \tag{1.2.5}$$

1.3 The case of filtered algebras

We will now describe the cyclic homology of filtered algebras. First of all, recall that the ring of differential operators $\mathscr{D}(X)$ on a manifold X has a natural filtration by order of operator (see Appendix 1 to Chapter 3). Along with the cyclic homology of a topological algebra explained above, this allows us to present the description of the Hochschild and cyclic homology of $\mathscr{D}(X)$ due to Wodzicki (cf [46]). In this respect, our principal reference is the paper [3] of Block.

Let A be an algebra over k equipped with an increasing filtration

$$0 = F_{-1}A \subset F_0A \subset F_1A \subset F_2A \subset \dots \tag{1.3.1}$$

that is exhaustive, i.e. $A = \cup F_pA$ and satisfies $F_pA \cdot F_qA \subset F_{p+q}A$. Form the cyclic vector space $C_*(A)$ that computes the cyclic homology of A. Then $C_n(A) = A^{\otimes n+1}$ is equipped with an increasing filtration which is defined as

$$F_pC_n(A) = \sum_{k_0+k_1+\ldots+k_n=p} F_{k_0}A \otimes \ldots \otimes F_{k_n}A \tag{1.3.2}$$

Since the differentials in the cyclic bicomplex preserve the filtration, for each p, $F_pC_n(A)$ is a cyclic vector space.

Definition 1.8. *(cf [3, 3.1]) Let A be an algebra over k. We let $d(A) = inf\{n \in \mathbb{N} | \ HH_i(A) = 0$ for $i > n\}$. Then we refer to $d(A)$ as the Hochschild dimension of A.*

Block (see [3]) proves the following results:

Proposition 1.9. *Let A be an algebra over k endowed with an increasing filtration F_kA as in* (1.3.1). *Then*

(1) $d(A) \leq d(gr(A))$.

(2) There is an isomorphism of associated graded cyclic vector spaces

$$gr(F_*C_*A) \cong C_*gr(A)$$

(3) For $l \geq 1$, the operator S in cyclic homology induces a corresponding map

$$S : HC_{i+2}(F_l C_* A/F_{l-1} C_* A) \longrightarrow HC_i(F_l C_* A/F_{l-1} C_* A)$$

(4) If $d(gr(A)) = n < \infty$, then the natural map

$$HC_i(F_0 A) \longrightarrow HC_i(A)$$

is an ismorphism for $i \geq n$.

Proof. See [3, Theorem 3.4] □

The above result of Block is the analogue in cyclic homology of the following well known result of Quillen (cf. [34]) in K-theory.

Proposition 1.10. *(Quillen) Let A be a ring endowed with an increasing filtration $F_p A$, $p \geq 0$ and such that $F_0 A$ is regular. Suppose that $B = gr(A)$ has finite Tor dimension as a right $F_0 A$-module and that $F_0 A$ has finite Tor dimension as a right B-module. Then, the inclusion $F_0 A \hookrightarrow A$ induces an isomorphism in K-theory;*

$$K_i(F_0 A) \cong K_i(A). \tag{1.3.3}$$

Remark 1.11. The constructions described above are in the context of the algebraic tensor product $\otimes$. It is clear that these results can be suitably reproduced when A is a topological algebra endowed with a filtration.

1.4 Relations with de Rham Cohomology

At the beginning of the chapter we have mentioned that cyclic homology is a noncommutative analogue of de Rham cohomology. In this section, we explain the connection.

We define $\Omega^1_{A|k}$ to be the module of 1-differential forms of A over k. Let $Der_k(A, M)$ denotes the k-module of k-linear derivations on A with values in an A-module M, i.e. the module of k-linear maps $D : A \to M$ such that

$$D(ab) = D(a)b + aD(b) \qquad \forall\ a, b \in A. \tag{1.4.1}$$

Then it is well known that the functor $Der_k(A, \_\_)$ from A-modules to k-modules is represented by $\Omega^1_{A|k}$, i.e.

$$Der_k(A, M) = Hom_k(\Omega^1_{A|k}, A) \tag{1.4.2}$$

Explicitly, $\Omega^1_{A|k}$ is the module generated by all terms of the form adb, for $a, b \in A$ subject to the relations $d(ab) = d(a)b + ad(b)$, $a, b \in A$ and we will denote this module simply by Ω^1_A. We set

$$\Omega^i_A = \wedge^i_A \Omega^1_A \qquad i \geq 0 \tag{1.4.3}$$

and define the following map

$$\pi_n : C_n(A) \longrightarrow \Omega^n_A \qquad\qquad (a_0 \otimes a_1 \otimes ... \otimes a_n) \mapsto a_0 da_1 ... da_n \tag{1.4.4}$$

One may easily check that $\pi_n \circ b = 0$, b being the Hochschild differential, and as such, there is a canonical map

$$\pi_n : HH_n(A) \longrightarrow \Omega^n_A \tag{1.4.5}$$

We denote by S_n the symmetric group acting by permutations on the set $\{1, 2, ..., n\}$. Then $\sigma \in S_n$ acts on $C_n(A)$ on the left as follows:

$$\sigma \cdot (a_0 \otimes a_1 \otimes ... \otimes a_n) = (a_0 \otimes a_{\sigma^{-1}(1)} \otimes ... \otimes a_{\sigma^{-1}(n)}) \tag{1.4.6}$$

We can therefore define the so called *anti-symmetrization map*

$$\varepsilon_n : A \otimes \wedge^n A \longrightarrow C_n(A) \qquad \varepsilon_n(a_0 \otimes a_1 \wedge ... \wedge a_n) = \sum_{\sigma \in S_n} sgn(\sigma)\sigma \cdot (a_0 \otimes a_1 \otimes ... \otimes a_n) \tag{1.4.7}$$

Proposition 1.12. *For any commutative k-algebra A, the anti-symmetrization map induces a canonical map*

$$\varepsilon_n : \Omega^n_A \longrightarrow HH_n(A) \tag{1.4.8}$$

Proof. See [31, 1.3.12]. □

Proposition 1.13. *The composition $\pi_n \circ \varepsilon_n$ coincides with the multiplication by $n!$. Consequently, if k contains $\mathbb{Q}$, then Ω^n_A is a direct summand of $HH_n(A)$.*

Proof. See [31, 1.3.16]. □

The most important application of the above proposition is when A is a smooth algebra. Let k be a Noetherian ring and A a commutative algebra over k which is essentially of finite type. Suppose that $Tor^n_k(A, A) = 0$ for all $n > 0$. Then A is said to be *smooth* over k if A satisfies the following criterion:

For any pair (C, I) where C is a k-algebra and I an ideal in C with $I^2 = 0$, the natural map $Hom_k(A, C) \to Hom_k(A, C/I)$ is a surjection

In particular, the symmetric algebra $S(V) = \oplus_{n \geq 0} V^{\otimes n}$ associated to a k-module V is an example of a smooth algebra over k. It follows that polynomial algebras

$k[x_1, ..., x_m]$, $m \geq 1$, are examples of smooth algebras. In the context of smooth algebras, the following well known result applies.

Theorem 1.14. *(Hochschild-Kostant-Rosenberg) Let A be a smooth algebra over k. Then the anti-symmetrization map*

$$\varepsilon_n : \Omega_A^n \xrightarrow{\sim} HH_n(A) \tag{1.4.9}$$

is an isomorphism for each $n \geq 0$.

Proof. See [31, 3.4.9]. □

In fact, $HH_n(A)$ carries the structure of a graded algebra (with a shuffle product, see [31]) and the isomorphism (1.4.9) is an isomorphism of graded algebras.

Now suppose that A is commutative and unital. The exterior derivative $d : \Omega_A^n \to \Omega_A^{n+1}$, $n \geq 0$, is defined by setting $d(a_0 da_1 da_2 ... da_n) = da_0 da_1 ... da_n$ and the homology groups of the complex

$$0 \xrightarrow{d} \Omega_A^0 \xrightarrow{d} \Omega_A^1 \xrightarrow{d} \Omega_A^2 \xrightarrow{d} ... \tag{1.4.10}$$

are said to be the de Rham cohomology groups $H_{dR}^n(A)$ of A. Then, it may be verified that the following two squares commute.

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A) \\ \pi_n \downarrow & & \pi_{n+1} \downarrow \\ \Omega_A^n & \xrightarrow{(n+1)d} & \Omega_A^{n+1} \end{array} \qquad \begin{array}{ccc} \Omega_A^n & \xrightarrow{d} & \Omega_A^{n+1} \\ \varepsilon_n \downarrow & & \varepsilon_{n+1} \downarrow \\ HH_n(A) & \xrightarrow{B_*} & HH_{n+1}(A) \end{array} \tag{1.4.11}$$

Proposition 1.15. *When the ground ring k contains $\mathbb{Q}$ and A is a unital and commutative k-algebra, the projection map π induces a canonical morphism*

$$HC_n(A) \to \Omega^n_A/d\Omega^{n-1}_A \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \dots \tag{1.4.12}$$

The anti-symmetrization map ε_* induces a map between the bicomplexes

$$\varepsilon_* : (\Omega^*_A, 0, d) \longrightarrow (\mathcal{B}(A), b, B) \tag{1.4.13}$$

When A is a smooth algebra, the morphism ε_* induces an isomorphism on the homologies of the columns of the two complexes. Hence, using the spectral sequences of a double complex, one deduces that

$$\Omega^n_A/d\Omega^{n-1}_A \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \dots \cong HC_n(A) \tag{1.4.14}$$

As a matter of fact, if the ground ring k contains $\mathbb{Q}$ the groups $HC_n(A)$ can be decomposed into summands, each of which admits a canonical map onto one of the groups $H^{n-2i}_{dR}(A)$ (or to $\Omega^n_A/d\Omega^{n-1}_A$). More precisely, if A is a unital commutative ring, the modules $C_n(A)$, $n \geq 0$ may be decomposed into direct sums

$$C_0(A) = C^{(0)}_0(A) \qquad C_n(A) = C^{(1)}_n(A) \oplus ... \oplus C^{(n)}_n(A) \qquad n \geq 1 \tag{1.4.15}$$

by means of Eulerian idempotents (see [31, §4.5-4.6]). The differentials b and B commute with the direct sum decomposition and hence the Hochschild complex $(C_*(A), b)$ (resp. the mixed complex $(\mathcal{B}(A), b, B)$) decomposes as a direct sum $\oplus(C^{(i)}_*(A), b)$ (resp. $\oplus(\mathcal{B}^{(i)}(A), b, B)$). We then obtain a splitting of the corresponding homology groups

$$HH_0(A) = HH_0^{(0)}(A) \qquad HH_n(A) = HH_n^{(1)}(A) \oplus ... \oplus HH_n^{(n)}(A) \text{ when } n \geq 1 \tag{1.4.16}$$

$$HC_0(A) = HC_0^{(0)}(A) \qquad HC_n(A) = HC_n^{(1)}(A) \oplus ... \oplus HC_n^{(n)}(A) \text{ when } n \geq 1 \tag{1.4.17}$$

which is known as the λ-*decomposition*. The λ-decomposition is sometimes referred to as the Hodge filtration on Hochschild/Cyclic homology (see for instance, [41]).

When k contains $\mathbb{Q}$, Ω_A^n, $n \geq 0$, is a direct summand of $HH_n(A)$ for each A. The summand Ω_A^n is actually isomorphic to $HH_n^{(n)}(A)$ via the anti-symmetrization map ε_n. This means that when A is a smooth algebra, $\varepsilon_n : \Omega_A^n \to HH_n(A)$ being an isomorphism, $HH_n^{(i)}(A) = 0$ for each $i < n$.

Theorem 1.16. *When A is a smooth algebra over a ring k and k contains $\mathbb{Q}$, the λ-decomposition coincides with the decomposition in de Rham cohomology, i.e.*

$$\begin{aligned}
&HC_n^{(n)}(A) = \Omega_A^n / d\Omega_A^{n-1} \\
&HC_n^{(i)}(A) = H_{dR}^{2i-n}(A) \qquad \textit{for } [n/2] \leq i \leq n \\
&HC_n^{(i)}(A) = 0 \qquad\qquad\quad \textit{for } i < [n/2]
\end{aligned} \tag{1.4.18}$$

Finally, we recall for future use (see [14]) that the pairing between cyclic homology and cyclic homology may also be understood in terms of differential forms. Define an *abstract cycle of degree n* to be a triple $(\Omega, d, \int)$ where $\Omega = \Omega^0 \oplus \Omega^1 \oplus ... \oplus \Omega^n$ is a graded algebra over k with a differential of degree $+1$ and $\int : \Omega^n \to k$ is a closed graded trace. In other words, the triple $(\Omega, d, \int)$ satisfies the following conditions:

$$\begin{aligned} &(1) \quad d(\omega\omega') = (d\omega)\omega' + (-1)^{\omega}\omega(d\omega') \\ &(2) \quad d^2 = 0 \\ &(3) \quad \int \omega_2\omega_1 = (-1)^{|\omega_1||\omega_2|} \int \omega_1\omega_2 \\ &(4) \quad \int d\omega = 0 \text{ for } \omega \in \Omega^{n-1} \end{aligned} \tag{1.4.19}$$

Given a k-algebra A, we define a cycle over A to be a triple $(\Omega, d, \int)$ as above along with a morphism $\rho : A \to \Omega^0$. The cycle is said to be *reduced* if

(1) The algebra Ω is generated by $\rho(A)$ as a differential graded algebra.

(2) The pairing $(\omega, \omega') \mapsto \int \omega\omega'$ is nondegenerate, i.e. if $\omega \in \Omega$ is such that $\int \omega\omega' = 0$ for all $\omega' \in \Omega$, then $\omega = 0$.

Given any reduced n-cycle over A, we can define its character $\tau : A^{\otimes n+1} \to k$ as

$$\tau(a_0 \otimes a_1 \otimes ... \otimes a_n) = \int a_0 da_1 ... da_n \tag{1.4.20}$$

Connes [14] has shown that the character τ thus defined is always an n-cyclic cocycle and that, conversely, any n-cyclic cocycle over A must necessarily be of this form. This implies that, while cyclic homology is the noncommutative analogue of de Rham cohomology, the cyclic cohomology is a noncommutative analogue of the cohomology of the complex of de Rham currents.

For a general noncommutative algebra A, the reduced Hochschild complex $\bar{C}_n(A)$ itself acts as a substitute for the module of 1-forms over A. Connes and Karoubi have defined the notion of noncommutative de Rham homology which is also directly connected to cyclic homology. We refer to Karoubi [30] for details.

2 Introduction to Steenbrink's complex and Consani's complex

Let X be a complex manifold. We shall denote by $f : X \to D$ a map from X to the unit disc D such that the fibres $f^{-1}(t)$ are smooth for all $t \neq 0$, while the fibre $f^{-1}(0)$ is a divisor Y with normal crossings on X (see §2.1 for the definition of a divisor with normal crossings). We denote the complement $X - Y$ by X^*. If $\tilde{D}^*$ is a universal covering of $D^* = D - \{0\}$, we denote the fibre product $X \times_D \tilde{D}^*$ by $\tilde{X}^*$ ($\tilde{X}^*$ is usually referred to as the nearby fibre or the universal fibre). Under these assumptions, Steenbrink [37] has constructed a complex that calculates the (de Rham) cohomology of the fibre $\tilde{X}^*$. Steenbrink has shown that the cohomology of $\tilde{X}^*$ carries a "limiting mixed hodge structure".

A similar formalism has been introduced to describe the following situation: Let K be a number field and let $\mathcal{O}_K$ denote the ring of integers of K. Let X be an arithmetic variety, i.e. a reduced and irreducible scheme over $Spec(\mathcal{O}_K)$. Then, each of the prime ideals $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ induces a valuation on K with the corresponding valuation ring being the localization $\mathcal{O}_{K,\mathfrak{p}}$. The fibre $X_{\mathfrak{p}}$ of X over each of these primes $\mathfrak{p}$ is a variety over $Spec(k(\mathfrak{p}))$, $k(\mathfrak{p})$ being the residue field of $\mathfrak{p}$. Since each of these valuations is nonarchimedean, we refer to the primes $\mathfrak{p}$ as the *finite primes* of $\mathcal{O}_K$.

We add to this collection of nonarchimedean valuations the archimedean valuations of $\mathcal{O}_K$, given by real embeddings or pairs of conjugate complex embeddings of K and we refer to these archimedean valuations as the *infinite primes* (In general, we may refer to an equivalence class of valuations as a “prime”). As for the nonarchimedean fibres, we would like to describe the fibres of X over these infinite primes, i.e. the vertical divisors of X at infinity. The complex points $X(\mathbb{C})$ of X define a manifold, and Consani [18] has shown that the Deligne cohomology of $X(\mathbb{C})$ can be described by a complex that is analogous to Steenbrink's complex.

In this chapter, we will describe both of these complexes and discuss their main properties.

2.1 The Logarithmic de Rham complex

Let X be a complex manifold of dimension n endowed with a morphism $f : X \to D$ to the unit disc D such that the fibres $f^{-1}(t)$ are smooth for all $t \neq 0$, while the fibre $f^{-1}(0)$ is a divisor Y with normal crossings on X. The fibre Y can therefore be written as a union of smooth irreducible divisors $Y = Y_1 \cup Y_2 \cup ... \cup Y_N$, where, for each $1 \leq k \leq N$, Y_k is a nonsingular subvariety of codimension 1 in X. We denote by $Y^{(p)}$ the union inside X of all the intersections $Y_{i_1} \cap ... \cap Y_{i_p}$, where $1 \leq i_1 < ... < i_p \leq N$ and we let $\tilde{Y}^{(p)}$ denote their disjoint union. Moreover, we shall denote by $a_p : \tilde{Y}^{(p)} \to X$ the canonical map. Note that $\tilde{Y}^{(p)}$ is smooth and equidimensional.

If $dim(X) = n$ and $(z_1, z_2, ..., z_n)$ is a system of local coordinates on X, we may assume that Y is defined locally on X by the equation $z_1^{e_1} z_2^{e_2} ... z_l^{e_l} = 0$, $(l \leq N)$, with the equation $\{z_i = 0\}$ corresponding to the component Y_i, for $1 \leq i \leq N$.

We define the sheaf $\Omega^1_X(\log Y)$ of 1-differentials on X with logarithmic poles along Y to be the free $\mathcal{O}_X$-module generated locally by the sections

$$\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, ..., \frac{dz_l}{z_l}, dz_{l+1}, ..., dz_n$$

We denote the exterior powers $\wedge^p \Omega_X(\log Y)$ by $\Omega^p_X(\log Y)$. The de Rham differential extends to the modules $\Omega^p_X(\log Y)$, $p \geq 0$ with $\Omega^0_X(\log Y) = \mathcal{O}_X$ and the definition of the de Rham complex $\Omega^*_X(\log Y)$ with logarithmic poles along Y follows.

The weight filtration W on $\Omega^*_X(\log Y)$ is defined by

$$W_k \Omega^p_x(\log Y) = \Omega^k_X(\log Y) \wedge \Omega^{p-k}_X \qquad p, k \in \mathbb{Z} \tag{2.1.1}$$

A section of $W_k \Omega^p_X(\log Y)$ can be expressed locally as

$$\alpha = \sum_{1 \leq i_1 < \ldots < i_k \leq l} \omega_{i_1 \ldots i_k} \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k}}, \tag{2.1.2}$$

where $\omega_{i_1 \ldots i_k}$ is a section of Ω^{p-k}_X. Restrictions of the de Rham differential to the strata $\tilde{Y}^k$ determine the morphisms $\rho : \Omega^{p-k}_X \to (a_k)_* \Omega^{p-k}_{\tilde{Y}(k)}$ is defined as follows: If the system of local coordinates is centred at the point P, then, in a neighbourhood of P, the intersection $Y_{i_1} \cap \ldots \cap Y_{i_k}$ is given by the equations $z_{i_1} = \ldots = z_{i_k} = 0$ and hence $\{z_j | j \notin \{i_1, \ldots, i_k\}\}$ form a system of local coordinates on $Y_{i_1} \cap \ldots \cap Y_{i_k}$. Let

$$\omega_{i_1 \ldots i_k} = \sum f_{j_1 \ldots j_{p-k}} dz_{j_1} \wedge \ldots \wedge dz_{j_{p-k}} \tag{2.1.3}$$

be a section of Ω^{p-k}_X where $(j_1, \ldots, j_{p-k})$ runs over the $(p-k)$-tuples of $\{1, 2, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$ and $f_{j_1 \ldots j_{p-k}}$ is a section of $\mathcal{O}_X$. Then $\rho(\omega_{i_1 \ldots i_k})$ is the restriction of the sections $f_{j_1 \ldots j_{p-k}}$ to the intersections $Y_{i_1} \cap \ldots \cap Y_{i_k}$. Set

$$R(\alpha) = \sum_{1 \leq i_1 < \ldots < i_k \leq l} \rho(\omega_{i_1 \ldots i_k}) \tag{2.1.4}$$

The map R is referred to as the Poincaré residue map.

Theorem 2.1. *The Poincaré residue map induces an isomorphism of complexes of sheaves*

$$R : Gr_k^W \Omega_X^*(\log Y) \xrightarrow{\sim} (a_k)_* \Omega^*_{\tilde{Y}^{(k)}}[-k] \tag{2.1.5}$$

Proof. We refer to [27]. □

Since $f : X \to D$ is a holomorphic map which is smooth on $X^* = X - f^{-1}(0)$, one deduces an inclusion $f^*\Omega^1_D(\log 0) \subset \Omega^1_X(\log Y)$: we denote the quotient by $\Omega^1_{X/D}(\log Y)$. Setting $\Omega^p_{X/D}(\log Y) = \wedge^p \Omega^1_{X/D}(\log Y)$, we can form the complex $\Omega^*_{X/D}(\log Y)$. If (t) is the local coordinate on D, suppose that $t \circ f = z_1^{e_1}...z_l^{e_l}$ then $\Omega^p_{X/D}(\log Y)_P$ is the module with generators $\{\frac{dz_1}{z_1}, ..., \frac{dz_l}{z_l}, dz_{l+1}, ..., dz_n\}$ subject to the relation $\sum_{i=1}^l e_i \frac{dz_i}{z_i} = 0$. Let θ denote the form:

$$\theta := f^*(\frac{dt}{t}) = \sum_{i=1}^{l} e_i \frac{dz_i}{z_i} \tag{2.1.6}$$

2.2 Cohomology of the universal fibre $\widetilde{X}^*$

With all the notations as before, we set $X^* = X \backslash Y$ to be the complement of the fibre over 0. With $D^* = D\backslash\{0\}$, we let $\tilde{D}^*$ denote the universal covering of D^* and we let $\tilde{X}^*$ denote the fibre product $X \times_D \tilde{D}^*$. Then, the universal covering map $j : \tilde{D}^* \to D$ is defined by $j(u) = exp(2\pi i u)$. We denote by $k : \tilde{X}^* \to X$ the composition of the natural projection $\tilde{X}^* \to X^*$ with the inclusion $X^* \to X$. The inclusion $f^{-1}(0) = Y \hookrightarrow X$ is denoted by i. One has the following Cartesian diagram:

$$\begin{array}{ccccc} \tilde{X}^* & \xrightarrow{k} & X & \xleftarrow{i} & Y \\ \downarrow & & f\downarrow & & \downarrow \\ \tilde{D}^* & \xrightarrow{j} & D & \longleftarrow & \{0\} \end{array} \tag{2.2.1}$$

Proposition 2.2. *For any point $s \in D^*$, there exists an isomorphism of complexes of sheaves on X_s*

$$\Omega^*_{X/D}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}|X_s \xrightarrow{\sim} \Omega^*_{X_s} \tag{2.2.2}$$

Proof. See [38, 7.5]. □

The following result shows that the de Rham cohomology of $\tilde{X}^*$ can be identified with the hypercohomology of the complex of relative differential forms with logarithmic poles.

Proposition 2.3. *There exist isomorphisms*

$$H^q(\tilde{X}^*, \mathbb{C}) \cong \mathbb{H}^q(Y, i^{\cdot}k_*\Omega^*_{\tilde{X}^*}) \cong \mathbb{H}^q(Y, \Omega^*_{X/D}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \qquad q \in \mathbb{Z} \tag{2.2.3}$$

Proof. See [38, 7.7–7.19] □

The form $\theta = f^*(\frac{dt}{t}) = \sum_{i=1}^{l} e_i \frac{dz_i}{z_i}$ defined in the previous section is, by construction, a section of $W_1\Omega^1_X(\log Y)$. Taking wedge products with the form θ therefore allows us to ascend levels of the weight filtration on $\Omega^*_X(\log Y)$. Let $\mathscr{I}(Y^{red})$ denote the sheaf of ideals associated to the reduced subscheme underlying Y. The following result will be used in the next section.

Proposition 2.4. *(1) The sequence*

$$\Omega^*_X(\log Y) \xrightarrow{\wedge\theta} \Omega^*_X(\log Y)[1] \xrightarrow{\wedge\theta} \Omega^*_X(\log Y)[2] \tag{2.2.4}$$

is an exact sequence of complexes on X.

(2) The following sequence of complexes of sheaves on X is exact:

$$0 \to \mathscr{I}(Y^{red})\Omega_X^*(\log Y) \to W_0\Omega_X^*(\log Y) \xrightarrow{\wedge\theta} Gr_1^W\Omega_X^*(\log Y)[1] \xrightarrow{\wedge\theta} Gr_2^W\Omega_X^*(\log Y)[2] \to \tag{2.2.5}$$

Proof. See [38, 12.2–12.3]. □

In terms of the disjoint unions $\tilde{Y}^{(k)}$ (defined in the previous section), the form θ may be understood as follows: For any $k \geq 1$, there exist $k+1$ distinguished maps

$$\delta_1, ..., \delta_{k+1} : \tilde{Y}^{(k+1)} \longrightarrow \tilde{Y}^{(k)} \tag{2.2.6}$$

where δ_m is induced, for instance, by the inclusion

$$Y_{i_1} \cap ...Y_{i_{m-1}} \cap Y_{i_m} \cap Y_{i_{m+1}} \cap ... \cap Y_{i_{k+1}} \xrightarrow{\delta_m} Y_{i_1} \cap ...Y_{i_{m-1}} \cap Y_{i_{m+1}} \cap ... \cap Y_{i_{k+1}} \tag{2.2.7}$$

which embed a $k+1$-"stratum" into a k-"stratum" by forgetting the corresponding irreducible component Y_{i_m}. Then, we can define

$$d' : (a_k)_*\Omega^p_{\tilde{Y}^{(k)}} \longrightarrow (a_{k+1})_*\Omega^p_{\tilde{Y}^{(k+1)}} \tag{2.2.8}$$

by setting $d' = \sum_{m=1}^{k+1}(-1)^{p+m}\delta_m^*$. It is clear that d' is a differential and it may be shown that, for $\omega \in (a_k)_*\Omega^p_{\tilde{Y}^{(k)}}$, one has

$$d'(\omega) = -\omega \wedge \theta \tag{2.2.9}$$

By applying the Poincaré residue map (2.1.4), we obtain an isomorphism of sheaves $(a_k)_*\Omega^p_{\tilde{Y}^{(k)}} \cong Gr_k^W\Omega_X^{p+k}(\log Y)$.

We define

$$d'' : Gr_k^W \Omega_X^{p+k}(\log Y) \longrightarrow Gr_k^W \Omega_X^{p+k+1}(\log Y) \tag{2.2.10}$$

to be the natural de Rham differential. Notice that, in view of the above isomorphism, d'' can be interpreted as the morphism

$$d'' : (a_k)_* \Omega^p_{\tilde{Y}^{(k)}} \longrightarrow (a_k)_* \Omega^{p+1}_{\tilde{Y}^{(k)}} \tag{2.2.11}$$

2.3 Steenbrink's complex

For nonnegative integers p and q, Steenbrink has defined the bicomplex of sheaves of $\mathcal{O}_X$-modules

$$A^{pq} = \Omega_X^{p+q+1}(\log Y)/W_p \Omega_X^{p+q+1}(\log Y) \tag{2.3.1}$$

The differential $d'' : A^{pq} \to A^{p,q+1}$ is induced by differentiation on the complex $\Omega_X^*(\log Y)$ and the differential $d' : A^{pq} \to A^{p+1,q}$ is defined by $d'(\omega) = (-1)^p \omega \wedge \theta$. Let $A^{\cdot}$ denote the total complex associated to the bicomplex $(A^{\cdot\cdot}, d', d'')$. The projection on the next level of the weight filtration defines an operator

$$N : A^{pq} \longrightarrow A^{p+1,q-1} \qquad N(a) = a \tag{2.3.2}$$

By definition, $\Omega^1_{X/D}(\log Y) = \Omega^1_X(\log Y)/f^*\Omega^1_D(\log 0)$ and hence we have the exact sequence

$$\Omega_X^{q-1}(\log Y) \xrightarrow{\wedge\theta} \Omega_X^q(\log Y) \longrightarrow \Omega^q_{X/D}(\log Y) \longrightarrow 0 \tag{2.3.3}$$

From (2.2.4), we know that the sequence of sheaves

$$\Omega_X^{q-1}(\log Y) \xrightarrow{\wedge\theta} \Omega_X^q(\log Y) \xrightarrow{\wedge\theta} \Omega_X^{q+1}(\log Y) \tag{2.3.4}$$

is exact. Consequently, we deduce an injection

$$\wedge\,\theta : \Omega_{X/D}^q(\log Y) \hookrightarrow \Omega_X^{q+1}(\log Y) \tag{2.3.5}$$

Moreover, from (2) of Proposition 2.4, it follows that, for $\omega \in \Omega_{X/D}^q(\log Y)$ we have $\omega \wedge \theta \in W_0\Omega_X^{q+1}(\log Y)$ if and only if ω is a section of the submodule $\mathscr{I}(Y^{red})\Omega_X^q(\log Y)$. By definition, $A^{0q} = \Omega_X^{q+1}(\log Y)/W_0\Omega_X^{q+1}(\log Y)$ and hence there is an induced morphism of sheaves

$$\phi : \Omega_{X/D}^q(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{red}} \longrightarrow A^{0q} \tag{2.3.6}$$

which induces, in turn, a morphism of complexes

$$\phi : \Omega_{X/D}^*(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{red}} \longrightarrow A^{\cdot} \tag{2.3.7}$$

Further, we define the filtrations L, W and F on $A^{\cdot}$ as follows

$$\begin{aligned} L_nA^{pq} &= W_{2p+n+1}\Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \\ W_nA^{pq} &= W_{p+n+1}\Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \\ F^nA^{pq} &= \begin{cases} A^{pq} & \text{if } p \geq n \\ 0 & \text{if } p < n \end{cases} \end{aligned} \tag{2.3.8}$$

Proposition 2.5. *(1) Let $\sigma_{\geq}$ be the naive filtration on the complex $\Omega_{X/D}^*(\log Y) \otimes$*

$\mathcal{O}_{Y^{red}}$. Then, the map of filtered complexes

$$\phi : (\Omega^*_{X/D}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{red}}, \sigma_{\geq}) \to (A^{\cdot}, F) \tag{2.3.9}$$

is a filtered quasi-isomorphism.

(2)

$$\phi : (\Omega^*_{X/D}(\log Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{Y^{red}}, W) \to (A^{\cdot}, F) \tag{2.3.10}$$

is a filtered quasi-isomorphism.

(3) ϕ is a quasi-isomorphism.

Proof. See [38, §13]. □

It follows from Proposition 2.5 and (2.2.3) that the hypercohomology of the complex $A^{\cdot}$ computes the cohomology of the universal fibre $\tilde{X}^*$. The spectral sequence associated to the filtration L on A^{pq} therefore induces a filtration on $H^r(\tilde{X}^*, \mathbb{C})$ for each $r \in \mathbb{Z}$.

We recall that if $A \subset \mathbb{R}$ is a Noetherian ring such that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field, a A-mixed Hodge structure consists of the following data:

(1) a finitely generated A-module H_A,

(2) a finite increasing filtration W on $H_A \otimes_{\mathbb{Z}} \mathbb{Q}$ and

(3) a finite decreasing filtration F on $H_A \otimes_A \mathbb{C}$ such that for all $p, q \in \mathbb{Z}$ with $p + q = n + 1$,

$$F^p Gr^W_n (H_A \otimes_A \mathbb{C}) \oplus F^q Gr^W_n (H_A \otimes_A \mathbb{C}) = Gr^W_n H_A \otimes_A \mathbb{C} \tag{2.3.11}$$

(using the filtration induced on each $Gr^W_n (H_A \otimes_A \mathbb{C})$ by F).

Theorem 2.6. *Consider the filtration L induced on $H^q(\tilde{X}^*, \mathbb{C})$ by the spectral sequence*

$$ {}_LE_1^{-r,q+r} \Rightarrow Gr^L_{r+q} H^q(\tilde{X}^*, \mathbb{C}) \qquad q \in \mathbb{Z} \tag{2.3.12}$$

Then the pair $(H^q(\tilde{X}^*, \mathbb{Q}), L)$ along with the triple $(H^q(\tilde{X}^*, \mathbb{C}), L, F)$ define a $\mathbb{Q}$-mixed Hodge structure on $H^q(\tilde{X}^*, \mathbb{Q})$ for all $q \in \mathbb{Z}$.

Proof. See [38, 13.27]. □

The terms ${}_LE_1^{-r,q+r}$ of the spectral sequence associated to L can be described more fully. Using the Poincaré residue theorem, we have the decomposition

$$ {}_LE_1^{-r,q+r} = \mathbb{H}^q(Y, Gr^L_r A^{\cdot}) = \bigoplus_{k \geq 0, -r} H^{q-r-2k}(\tilde{Y}^{(r+2k+1)}, \mathbb{C}) \tag{2.3.13}$$

We consider the terms ${}_WE_1^{-r,q+r}$ of the spectral sequence associated to the filtered complex $(\Omega^*_X(\log Y), W, F)$. Then, using the Poincaré residue theorem, we have isomorphisms

$$ {}_WE_1^{-r,q+r} \xrightarrow{\approx} H^q(\tilde{Y}^{(r)}, \Omega_{\tilde{Y}^{(r)}}[-r]) = H^{q-r}(\tilde{Y}^{(r)}, \mathbb{C}) \tag{2.3.14}$$

The differential d_1 on the terms ${}_WE_1^{-r,q+r}$ fits into the following commutative diagram

$$\begin{array}{ccc} {}_WE_1^{-r,q+r} & \xrightarrow{\approx} & H^{q-r}(\tilde{Y}^{(r)}, \mathbb{C}) \\ {\scriptstyle d_1}\big\downarrow & & {\scriptstyle -\gamma^{(r)}}\big\downarrow \\ {}_WE_1^{-r+1,q+r} & \xrightarrow{\approx} & H^{q-r+2}(\tilde{Y}^{(r-1)}, \mathbb{C}) \end{array} \tag{2.3.15}$$

where $\gamma^{(r)} = \sum_{m=1}^r (-1)^{m-1}(\delta_m)_*$ and the δ_m's denote the inclusions of $\tilde{Y}^{(r)}$ into $\tilde{Y}^{(r-1)}$. We set $\rho^{(r)} = \sum_{m=1}^r (-1)^{m-1}(\delta_m)^*$.

We introduce the following complex

$$K_S^{i,j,k} = \begin{cases} H^{i+j-2k+n}(\tilde{Y}^{(2k-i+1)}, \mathbb{C}) & \text{if } k \geq 0, i \\ 0 & \text{otherwise} \end{cases} \tag{2.3.16}$$

Summing over k produces the following complex

$$K_S^{i,j} = \bigoplus_{k \in \mathbb{Z}} K_S^{i,j,k} \tag{2.3.17}$$

We notice that the Poincaré residue map determines isomorphisms

$$Res : {}_L E_1^{-r+1,q+r} \xrightarrow{\approx} K_S^{-r,q-n} \tag{2.3.18}$$

The differentials on the bicomplex K_S^{**} coincide with the two differentials on the spectral sequence ${}_L E_1$

$$d_1' : K_S^{i,j,k} \longrightarrow K_S^{i+1,j+1,k+1} \qquad d_1'' : K_S^{i,j,k} \longrightarrow K_S^{i+1,j+1,k+1} \tag{2.3.19}$$

Moreover the map N on K_S^{***} is induced by

$$N : K_S^{i,j,k} \longrightarrow K_S^{i+2,j,k+1} \tag{2.3.20}$$

Here, we can check that $Res^{-1} d_1' = (\_\_ \wedge \theta)$ and $Res^{-1} d_1'' = -(\gamma)$. Hence, it follows that

$$d_1' = \rho \qquad d_1'' = -\gamma \qquad N = Id \tag{2.3.21}$$

in terms of the cohomologies of the $\tilde{Y}^{(r)}$, i.e. on the other side of the Poincaré residue isomorphism. We will denote the morphisms d_1' and d_1'' simply by d' and d''

respectively. Finally, we recall the following important result:

Theorem 2.7. *If the variety $\tilde{Y}^{(1)}$ is Kahler, then*

(1) $K_S^{i,j}$ carries a $\mathbb{Q}$-Hodge structure of weight $j+i-n$.

(2) The morphism $N : K_S^{i,j} \to K_S^{i+2,j}(-1)$ is a morphism of Hodge structures.

(3) For all $i \geq 0$, N induces an isomorphism

$$N^i : K_S^{-i,j} \longrightarrow K_S^{i,j}(-i) \tag{2.3.22}$$

of Hodge structures.

(4) The primitive part of $K_S^{-i,j}$ is $K_S^{-i,j,0}$, i.e.

$$K_S^{-i,j,0} = Ker(N^{i+1}) \cap K_S^{-i,j} \tag{2.3.23}$$

Proof. See [28, 2.9].

□

2.4 Cohomology of the fibre at infinity: Consani's complex

In Arakelov geometry, the fibre at archimedean infinity is a manifold X of dimension $n \geq 0$, say, over $\mathbb{C}$ or $\mathbb{R}$. Consani [18] has defined a bicomplex $(K^{\cdot\cdot}, d', d'')$ with a monodromy type map N that is analogous to Steenbrink's complex described in the last section.

We denote by $(\Omega_X^{a,b} + \Omega_X^{b,a})_{\mathbb{R}}(p)$ the abelian group of real differentials of type $(a,b)+(b,a)$ on X along with the p-th Hodge-Tate twist, i.e.

$$(\Omega_X^{a,b} + \Omega_X^{b,a})_{\mathbb{R}}(p) = (2\pi\sqrt{-1})^p(\Omega_X^{a,b} + \Omega_X^{b,a})_{\mathbb{R}} \qquad p \in \mathbb{Z} \tag{2.4.1}$$

For all $i, j, k \in \mathbb{Z}$, we define the term

$$K^{i,j,k} = \begin{cases} \bigoplus\limits_{\substack{a \leq b, a+b=j+1 \\ |a-b| \leq 2k-i}} (\Omega_X^{a,b} + \Omega_X^{b,a})_{\mathbb{R}} \left(\frac{n+j-i}{2}\right) & \text{if } k \geq \max\{0, i\} \\ 0 & \text{otherwise} \end{cases} \tag{2.4.2}$$

We denote by ∂ and $\overline{\partial}$ the usual partial differential operators on any $\Omega_X^{a,b}$. Then one introduces the following maps:

$$\begin{aligned} d' &: K^{i,j,k} \longrightarrow K^{i+1,j+1,k+1} & \quad d' &= \partial + \overline{\partial} \\ d'' &: K^{i,j,k} \longrightarrow K^{i+1,j+1,k} & \quad d'' &= i(\partial - \overline{\partial}) \\ N &: K^{i,j,k} \longrightarrow K^{i+2,j,k+1} & \quad N(a) &= (2\pi i)^{-1} a \end{aligned} \tag{2.4.3}$$

We remark that d'' should be considered as composed with a projection onto its range. We will also maintain the notation

$$d = d' + d'' \tag{2.4.4}$$

For $i, j \in \mathbb{Z}$, we write $K^{i,j} = \bigoplus_{k \in \mathbb{Z}} K^{i,j,k}$ and for a fixed index r, we let $K^r = \bigoplus_{i+j=r} K^{i,j}$. Consider the bicomplex $(K^{\cdot\cdot}, d', d'')$ as well as its associated total complex $(K^{\cdot}, d)$.

Note that $[N, d'] = [N, d''] = 0$ where $[a, b] = ab - ba$.

Let $p \in \mathbb{Z}_{\geq 0}$. Recall that the real Deligne-Beilinson cohomology of $X_{\mathbb{C}}$ is the hypercohomology of the complex

$$\mathbb{R}(p)_{\mathcal{D}} : \mathbb{R}(p) \to \Omega_X^0 \to \Omega_X^1 \to \ldots \to \Omega_X^{p-1} \to 0 \tag{2.4.5}$$

i.e. we define $H_{\mathcal{D}}^q(X/\mathbb{C}, \mathbb{R}(p)) = \mathbb{H}^q(X, \mathbb{R}(p)_{\mathcal{D}})$. If X is defined over $\mathbb{R}$, we can still form the complex $\mathbb{R}(p)_{\mathcal{D}}$ above. In this case, the complex $\mathbb{R}(p)_{\mathcal{D}}$ carries a complex conjugation which we refer to as $\bar{F}_{\infty}$. We refer to the invariants under conjugation $\bar{F}_{\infty}$ as the real Deligne-Beilinson cohomology, i.e.

$$H_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) = \mathbb{H}^q(X, \mathbb{R}(p)_{\mathcal{D}})^{\bar{F}_{\infty}=id} \tag{2.4.6}$$

Let (X^i, d_X^i) and (Y^i, d_Y^i) denote two complexes and suppose that $f = (f^i : X^i \to Y^i)$ is a morphism between them. Then the complex $(X^i + Y^{i-1}, (d_X^i + f) + d_Y^{i-1})$ is referred to as the cone of f.

Theorem 2.8. *Let p be a nonnegative integer. Then, the complex $Cone(N : K^{q-2p,q-1} \to K^{q-2p+2,q-1})$ $(q \in \mathbb{Z})$ is quasi-isomorphic to the complex $(C_{\mathcal{D}}^q(p), d_{\mathcal{D}})$, defined as follows*

$$C_{\mathcal{D}}^q(p) := \begin{cases} \left(\bigoplus\limits_{\substack{a+b=q-1 \\ |a-b| \leq 2p-q-1}} \Omega_X^{a,b} \right)_{\mathbb{R}} (p-1) & \text{if } q \leq 2p-1 \\ \\ \left(\bigoplus\limits_{\substack{a+b=q \\ |a-b| \leq q-2p}} \Omega_X^{a,b} \right)_{\mathbb{R}} (p) & \text{if } q \geq 2p \end{cases} \tag{2.4.7}$$

with differentials, which for $a \in C_{\mathcal{D}}^q(p)$ are defined as;

$$d_{\mathcal{D}} = \begin{cases} d''(a) & \text{if } q < 2p-1 \\ 2\pi\sqrt{-1}d'd''(a) & \text{if } q = 2p \\ d'(a) & \text{if } q \geq 2p \end{cases} \tag{2.4.8}$$

The homology of the complex, in each degree q, computes the real Deligne cohomology group of $X_{/\mathbb{C}}$. Taking $\bar{F}_{\infty}$ invariants of the homology gives us $H_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p))$.

Proof. See [18, §4] or [16, §2.19]. □

3 The Steenbrink complex and the Cyclic Homology of the Sheaf of differential operators

Let X be a complex manifold along with a map $f : X \to D$ to the unit disc D. Suppose that the fibres $f^{-1}(t)$ are smooth for all $t \neq 0$, while the fibre $f^{-1}(0)$ is a divisor Y with normal crossings on X. We denote the complement $X - Y$ by X^*. Let $\tilde{D}^*$ be a universal covering for $D^* = D - \{0\}$. Denote the fibre product $X \times_D \tilde{D}^*$ by $\tilde{X}^*$.

As discussed in the previous chapter, Steenbrink (cf. [37], [38]) has shown that the cohomology $H^*(\tilde{X}^*, \mathbb{C})$ can be computed by as the hypercohomology of the bicomplex A^{pq}, $p \geq 0$, defined as

$$A^{pq} = \Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \tag{3.0.9}$$

Furthermore, there exists a monodromy operator N, of bidegree $(1, -1)$ on A^{**}, defined as the projection

$$N : A^{pq} = \Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \longrightarrow \tag{3.0.10}$$

$$\Omega_X^{p+q+1}(\log Y)/W_{p+1}\Omega_X^{p+q+1}(\log Y) = A^{p+1,q-1}$$

In this chapter, we consider the complex of nearby cycles as defined in [22], or more precisely, a resolution ψ^{**} of the nearby cycles complex, which computes the coho-

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
u^2\Omega^0_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^1\Omega^1_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^0\Omega^2_X(\log Y) \arrow[u, "d''"] \\
& u^1\Omega^0_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^0\Omega^1_X(\log Y) \arrow[u, "d''"] \\
& & u^0\Omega^0_X(\log Y) \arrow[u, "d''"]
\end{tikzcd}
\tag{3.0.14}
$$

Figure 1: **The bicomplex** ψ^{**}

mology of $\tilde{X}^*$. The bicomplex ψ^{**} is defined as follows

$$\psi^{-p,q} = \mathbb{C}u^p \otimes \Omega^{q-p}_X(\log Y) \qquad p \geq 0 \tag{3.0.11}$$

along with differentials

$$d'(u^p \otimes \omega) = pu^{p-1} \otimes \theta \wedge \omega \qquad d''(u^p \otimes \omega) = u^p \otimes d(\omega) \tag{3.0.12}$$

where d is the exterior differential and $\theta = f^*(dt/t)$ is the pullback of the logarithmic form dt/t on D as defined in (2.1.6) in Section 2.1. In [28], Guillén and Navarro Aznar define a map $\mu : \psi^{**} \to A^{**}$ of bicomplexes that induces a quasi-isomorphism of complexes. The bicomplex ψ^{**} supports an operator (monodromy) defined as

$$N(u^p \otimes \omega) = pu^{p-1}\omega \tag{3.0.13}$$

The map N does not commute with μ, but $N \circ \mu$ and $\mu \circ N$ are, in fact, homotopy equivalent (see [28, 2.5]).

In this chapter, we show (see Section 3.2, Corollary 3.9) that the (hyper)cohomology of the columns of the nearby cycles complex ψ^{**} is isomorphic to the Hochschild homology $HH_*(\mathscr{D}_{X^*})$ of the sheaf $\mathscr{D}_{X^*}$ of differential operators on X^*. This is defined to be the hypercohomology of the sheafification of the hoschschild complexes associated to the ring of differential operators $\mathscr{D}(U)$ for each open U in X^*. This construction is reviewed in Appendix 1. By a result of Wodzicki (see [46] or Proposition A.3.13), we know that if U is a Stein manifold of dimension n over $\mathbb{C}$, then the Hochschild and cyclic homologies of the ring $\mathscr{D}(U)$ are determined by the de Rham cohomology of U as

$$HH_q(\mathscr{D}(U)) \cong H_{dR}^{2n-q}(U) \tag{3.0.15}$$

$$HC_q(\mathscr{D}(U)) \cong H_{dR}^{2n-q}(U) \oplus H_{dR}^{2n-q+2}(U) \oplus \dots \tag{3.0.16}$$

Also, we have the long exact periodicity sequence of Connes (see [31, 2.2.1]):

$$\dots \longrightarrow HH_q(\mathscr{D}(U)) \xrightarrow{I} HC_q(\mathscr{D}(U)) \xrightarrow{S} HC_{q-2}(\mathscr{D}(U)) \xrightarrow{B} \dots \tag{3.0.17}$$

connecting the Hochschild and cyclic homologies of $\mathscr{D}(U)$. Using the fact that every point on X^* has a fundamental system of neighbourhoods that are Stein, we will establish similar isomorphisms (in Proposition A.3.17), as well as a corresponding long exact sequence periodicity sequence for the sheaf of differential operators $\mathscr{D}_{X^*}$.

Remark 3.1. Since the complexes used in defining the Hochschild and cyclic homology are not bounded below, the notion of *hypercohomology* is not defined in the classical sense. Here we use the notion of hypercohomology obtained by writing an unbounded (below) complex as a limit of its "good" truncations.

Furthermore, we show in Corollary 3.9 that the filtration F'_j, $j \geq 0$ by columns on the total complex $Tot(\psi^{**})$, i.e. $((F'_jTot(\psi^{**}))^* = \bigoplus_{r+s=*,r\geq -j} \psi^{rs}, d' + d'')$ of the bicomplex ψ^{**} coincides with the kernel filtration $F_jTot(\psi^{**}) = Ker(N^j)$, $j \geq 0$ on ψ^{**}. These filtrations are bounded below (i.e. $F_0Tot(\psi^{**}) = 0$, $F_0'Tot(\psi^{**}) = 0$) and exhaustive (i.e. $Tot(\psi^{**}) = \cup_{j\geq 0}F_j\psi^{**}$, $Tot(\psi^{**}) = \cup_{j\geq 0}F_j'\psi^{**}$). Corresponding to the filtration F_j' by columns on $Tot(\psi^{**})$, we have a converging spectral sequence in hypercohomology

$$E_1^{-p,2n+p-q} := \mathbb{H}^{2n-q}(F_p'/F_{p-1}') \Rightarrow \mathbb{H}^{2n-q}(\psi^*) \cong \mathbb{H}^{2n-q}(A^*) \tag{3.0.18}$$

We will show in Section 3.2 that $\mathbb{H}^{2n-q}(F_p'/F_{p-1}') \cong HH_q(\mathscr{D}_{X^*})$. In particular, this implies that the hypercohomology of the graded pieces of the filtration $F_j\psi^* = Ker(N^j)$, $j \geq 0$, compute the Hochschild homology of $\mathscr{D}_{X^*}$.

We construct the cyclic complex for $\mathscr{D}_{X^*}$ and the periodicity long exact sequence associated to it. In Proposition 3.12, we show that the periodicity operator S appearing in the cyclic homology of $\mathscr{D}_{X^*}$ coincides with the operator N acting on the complex ψ^{**}.

In Chapter 4, following Connes and Karoubi [13], we shall define the algebraic, topological and relative K-theory groups of the sheaf $\mathscr{D}_{X^*}$ and show that there exists a long exact sequence of these K-theory groups that maps to the periodicity sequence in the cyclic homology of $\mathscr{D}_{X^*}$.

3.1 The complex of nearby cycles and de Rham cohomology

Let $D = \{z \in \mathbb{C} | |z| < 1\}$ be the unit disc, $D^* = D - \{0\}$ and X be an algebraic variety with a proper morphism $f : X \longrightarrow D$. We suppose that each of the fibres $X_t = f^{-1}(t)$, $t \neq 0$ is nonsingular and of same dimension, while $Y = f^{-1}\{0\}$ is a divisor with normal crossings on X. Let $\tilde{D}^*$ be the universal cover of D^*, the map

$p' : \tilde{D}^* \longrightarrow D$ being given by $p'(u) = e^{2\pi i u}$. Denote by $\tilde{D} = \tilde{D}^* \cup \{0\}$ the space obtained from $\tilde{D}^*$ by adjoining a point 0 such that

(1) $\tilde{D}^*$ is open in $\tilde{D}$.

(2) $p : \tilde{D} \to D$ extends the map $p' : \tilde{D}^* \to D^*$ and $p(0) = 0$.

(3) The system $p^{-1}(U)$, where U runs through all the neighbourhoods of 0 in D, forms a fundamental system of neighbourhoods of 0 in $\tilde{D}$.

Set

$$\tilde{X} = X \times_D \tilde{D} \qquad X^* = X - Y = X \times_D D^* \qquad \tilde{X}^* = X^* \times_D \tilde{D}^* = \tilde{X} - Y \tag{3.1.1}$$

With these definitions, we have the commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\tilde{i}} & \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{X}^* \\ {\scriptstyle id}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X^* \end{array} \tag{3.1.2}$$

cartesian above the diagram

$$\begin{array}{ccccc} \{0\} & \xrightarrow{\tilde{i}} & \tilde{D} & \xleftarrow{\tilde{j}} & \tilde{D}^* \\ {\scriptstyle id}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ \{0\} & \xrightarrow{i} & D & \xleftarrow{j} & D^* \end{array} \tag{3.1.3}$$

Let us denote by $\mathscr{D}$ the category of sheaves of sets on D whose restriction to D^* is locally constant. If Y is any topological space, we can consider the collection of sheaves $Y \times \mathscr{D}$, i.e. the category of sheaves of sets on $Y \times D$ whose restriction to D lies in $\mathscr{D}$. This category of sheaves $Y \times \mathscr{D}$ admits the following alternative descriptions (cf [22, 1.2.4])

(1) $Y \times \mathscr{D}$ is the category of sheaves (of sets) F on $Y \times \mathscr{D}$ such that D^* can be covered

by open sets U such that $F|Y \times U$ is the inverse image of a sheaf on Y.

(2) For $t \in D^*$, $Y \times \mathscr{D}$ is the category of triples (F_0, F_t, α) such that

(a) F_0 is a sheaf on Y.

(b) F_t is a sheaf on Y, provided with an action of $\pi_1(D^*, t)$.

(c) $\alpha : F_0 \to F_t$ is a morphism of sheaves, the image of which is contained in the set of invariants of F_t under the action of $\pi_1(D^*, t)$, i.e. $Im(\alpha) \subset (F_t)^{\pi_1(D^*,t)}$,

(3) $Y \times \mathscr{D}$ is the category of triples (F_0, F_η, α) such that

(a) F_0 is a sheaf on Y.

(b) F_η is a functor from the category of universal coverings of D^* to sheaves on Y.

The category of universal coverings of D^*, consists of a single object, say, for instance, the upper half plane $\mathbb{H}$, while the morphisms are the automorphisms of this space that reduce to identity under the projection to D^*, denoted by $Aut_{D^*}(\mathbb{H})$. As such, an action of $\pi(D^*, t) = Aut_{D^*}(\mathbb{H})$ on objects in a category $\mathscr{C}$ may be encoded as a functor from the category of universal coverings to $\mathscr{C}$.

(c) $\alpha : F_0 \to F_\eta$ is a morphism of functors, where F_0 denotes the constant functor having value F_0.

We also need to consider a category of sheaves $Y \times \mathscr{D}^*$ for which we have three equivalent definitions (cf [22, 1.2.4])

(1)' The topos of sheaves F on $Y \times D^*$ such that D^* can be covered by open sets U such that the restriction $F|Y \times U$ is the inverse image of a sheaf on Y.

(2)' For $t \in D^*$, the topos of sheaves F_t on Y, provided with an action of $\pi_1(D^*, t)$.

(3)' The topos of functors from the category of universal coverings of D^* to the category of sheaves on Y.

Let F be a sheaf on X. In the language of (3) above, we describe a functor

$$\Psi : (\text{Sheaves on } X) \longrightarrow (\text{Sheaves on } Y \times \mathscr{D}) \tag{3.1.4}$$

as

$$\Psi(F) = (i^*F, \tilde{i}^*\tilde{j}_*((jp')^*F), \alpha) \tag{3.1.5}$$

where i^* denotes the inverse image functor (and not the pullback!) and α is the adjunction morphism

$$\alpha : i^*F = \tilde{i}^*p^*F \longrightarrow \tilde{i}^*\tilde{j}_*((jp')^*)F) = \tilde{i}^*\tilde{j}_*\tilde{j}^*p^*F \tag{3.1.6}$$

Similarly, given a sheaf F^* on X^*, in the language of (3)' above, we define a functor

$$\Psi_\eta : (\text{Sheaves on } X^*) \longrightarrow (\text{Sheaves on } Y \times \mathscr{D}^*) \tag{3.1.7}$$

as

$$\Psi_\eta(F^*) = \tilde{i}^*\tilde{j}_*(p'^*F^*) \tag{3.1.8}$$

We are interested in their derived functors

$$R\Psi : D^+(X) \longrightarrow D^+(Y \times \mathscr{D})$$

$$R\Psi_\eta : D^+(X^*) \longrightarrow D^+(Y \times \mathscr{D}^*) \tag{3.1.9}$$

The functor $R\Psi_\eta$ above is referred to as the functor of *nearby cycles.*

We now apply this to the de Rham cohomology. Denote by $\Omega^*_{\tilde{X}^*}$ the de Rham complex of holomorphic differential forms on $\tilde{X}^*$. By the holomorphic Poincaré lemma, $(\Omega^*_{\tilde{X}^*}, d)$ forms a resolution of the constant sheaf $\mathbb{C}$ on $\tilde{X}^*$, which, moreover, is $\tilde{j}_*$-

acyclic. Denote by $D^+(X,\mathbb{C})$ the (bounded below) derived category of sheaves of finite dimensional $\mathbb{C}$-vector spaces on $\tilde{X}$. Then we have an isomorphism of complexes

$$\tilde{j}_*\Omega^*_{\tilde{X}^*} \xrightarrow{\sim} R\tilde{j}_*\mathbb{C} \tag{3.1.10}$$

in $D^+(X,\mathbb{C})$.

Proposition 3.2. *Let V be a complex local system on X^*. The quasi-isomorphism $V \xrightarrow{\sim} \Omega^*_{X^*}(V)$ (which is a consequence of Poincaré lemma) induces a quasi-isomorphism*

$$\tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*}(p'^*V) \xrightarrow{\sim} R\Psi_\eta(V) \tag{3.1.11}$$

Proof. See [22, 4.4] □

We therefore have a quasi-isomorphism

$$\tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*} \xrightarrow{\sim} R\Psi_\eta(\mathbb{C}) \tag{3.1.12}$$

By abuse of notation, we shall still denote by f the restriction $f : X^* \to D^*$. Let F be a sheaf of $f^*\mathcal{O}_{D^*}$-modules on X^* and let $\bar{F}$ denote the inverse image on $\tilde{X}^*$.

We choose a holomorphic local coordinate $z : D \to \mathbb{C}$, with $z(0) = 0$ and on the universal cover $\tilde{D}^*$, we define the function $u = \log(z)$ such that $exp(u) = z$. We can set

$$z^\alpha = exp(\alpha u) = exp(\alpha \log(z))$$

The action of $\pi_1(D^*)$ on the universal covering $\tilde{D}^*$ extends to the space $\tilde{X}^* = X \times_D \tilde{D}^*$ and hence to all locally constant sheaves on $\tilde{X}^*$. Hence, the group $\pi_1(D^*)$ acts on $H^0(\tilde{X}^*, \bar{F})$ and let T denote the positive generator for the group $\pi_1(D^*)$.

Definition 3.3. *An element $x \in H^0(\tilde{X}^*, \bar{F})$ is said to be* finitely determined *if the elements $\{T^i(x)\}_{i\in\mathbb{Z}}$ generate a finite dimensional subspace (over $\mathbb{C}$) of $H^0(\tilde{X}^*, \bar{F})$.*

Moreover, we will say that x is finitely determined unipotent *(resp.* finitely determined quasi-unipotent*) if there exists an integer A (resp. integers A and B) such that $(T-1)^A(x) = 0$ (resp. $(T^B-1)^A(x) = 0$).*

Define $N = \frac{\log T}{2\pi i}$ on the space of finitely determined quasi-unipotent sections $x \in H^0(\tilde{X}^*, \bar{F})$ by the finite sum

$$N(x) = \frac{-1}{2\pi i}\sum_{n>0}\frac{(1-T)^n}{n}x \tag{3.1.13}$$

Now suppose that the sheaf F on X^* is actually the restriction of a sheaf F on X (which we also denote by F). Further, suppose that the sheaf F on X is coherent and its restriction to X^* is locally free. Under these additional hypotheses, we define $\Psi_\eta^m(F)$ (resp. $\Psi_\eta^{mu}(F)$, resp. $\Psi_\eta^{mqu}(F)$) to be the subsheaf of $\tilde{i}^*\tilde{j}_*(\bar{F})$ consisting of sections that are finitely determined (resp. finitely determined unipotent, resp. finitely determined quasi-unipotent).

In particular, consider the case where $F = \Omega^i_{\tilde{X}^*}$, for any given i. Then, we have a natural morphism of complexes

$$\Psi_\eta^{mqu}(\Omega_{X^*}) \hookrightarrow \tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*} \longrightarrow R\Psi_\eta(\mathbb{C}) \tag{3.1.14}$$

induced by the morphism in Proposition 3.2. Then, Deligne proves the following theorem (see [22, Theorem 4.13]):

Proposition 3.4. *The natural morphism*

$$\Psi_\eta^{mqu}(\Omega^*_{X^*}) \longrightarrow R\Psi_\eta(\mathbb{C}) \tag{3.1.15}$$

is a quasi-isomorphism.

Combining this with the isomorphism

$$\tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*} \xrightarrow{\sim} \mathbb{R}\Psi_\eta(\mathbb{C}) \tag{3.1.16}$$

of Proposition 3.2, we obtain quasi-isomorphisms

$$\tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*} \xrightarrow{\sim} \mathbb{R}\Psi_\eta(\mathbb{C}) \xleftarrow{\sim} \Psi^{mqu}_\eta(\Omega^*_{X^*}) \tag{3.1.17}$$

Moreover, the sheaves $\Omega^i_{\tilde{X}^*}$, are $\tilde{j}_*$-acyclic resolution of $\mathbb{C}_{\tilde{X}^*}$ and hence it follows,

$$\mathbb{H}^p(\tilde{X}, \tilde{j}_*\Omega^*_{\tilde{X}^*}) \approx H^p(\tilde{X}^*, \mathbb{C}) \tag{3.1.18}$$

The functor $\tilde{i}^*$ being exact, we have

$$\mathbb{H}^p(Y, \tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*}) \approx H^p(\tilde{X}^*, \mathbb{C}) \tag{3.1.19}$$

and hence all three complexes $\tilde{i}^*\tilde{j}_*\Omega^*_{\tilde{X}^*}$ $\mathbb{R}\Psi_\eta(\mathbb{C})$ and $\Psi^{mqu}_\eta(\Omega^*_{X^*})$ compute the cohomology of $\tilde{X}^*$.

In [38, Section 13], Steenbrink has defined the complex

$$A^{pq} = \mathbb{C}u_{[p+1]} \otimes_{\mathbb{C}} \Omega^{p+q+1}_X(\log Y)/W_p\Omega_X(\log Y)^{p+q+1} \tag{3.1.20}$$

along with the maps

$$\begin{aligned} d' : A^{pq} \longrightarrow A^{p+1,q} \quad & d'(u_{[p+1]} \otimes \omega) = u_{[p+2]} \otimes \theta \wedge \omega \\ d'' : A^{pq} \longrightarrow A^{p,q+1} \quad & d''(u_{[p+1]} \otimes \omega) = u_{[p+1]} \otimes d\omega \end{aligned} \tag{3.1.21}$$

where we denote, following [28],

$$u^{[p]} = u^p/p! \qquad u_{[p]} = (-1)^{p-1}(p-1)!u^{-p}$$

It follows from the theory developed by Deligne [22] and Steenbrink [38], that this complex is, in fact, a resolution of the nearby cycles complex. The monodromy on Steenbrink's complex has the explicit description

$$N : A^{pq} \longrightarrow A^{p+1,q-1} \tag{3.1.22}$$

defined as

$$N(u_{[p+1]} \otimes \omega) = u_{[p+2]} \otimes \omega \tag{3.1.23}$$

Also, $[N, d'] = [N, d''] = 0$. We recall that the maps are as indicated in the following diagram :

$$\begin{array}{ccccc} \tilde{X}^* & \xrightarrow{\tilde{j}} & X & \xleftarrow{i} & Y \\ \big\downarrow & & f\big\downarrow & & \big\downarrow \\ \tilde{D}^* & \xrightarrow{\tilde{p}} & D & \longleftarrow & \{0\} \end{array} \tag{3.1.24}$$

where $\tilde{D}^*$ is a universal cover for the punctured disc D^* with the map $p : \tilde{D}^* \to D^*$ given by $p(u) = e^u$. Further, we have $X^* = f^{-1}(D^*)$. If t is a chosen uniform coordinate for D, we set $\theta = f^*(dt/t)$.

Following [28, 2.2], let us define the following complex of sheaves on X, which is yet another resolution of the nearby cycles complex.

$$\psi^*(\mathbb{C}) = \mathbb{C}[u] \otimes_{\mathbb{C}} \Omega^*_X(\log Y) = \sum_{p \geq 0} \mathbb{C}u^p \otimes_{\mathbb{C}} \Omega^*_X(\log Y) \tag{3.1.25}$$

extending the exterior differential by $d(u\otimes 1) = 1\otimes\theta$. Let $\mathbf{R}\psi^*(\mathbb{C})$ be the the complex of nearby cycles, defined as $\mathbf{R}\psi^*(\mathcal{F}) = i^*\mathbf{R}\tilde{j}_*\tilde{j}^*\mathcal{F}$ for a sheaf $\mathcal{F}$ on X.

Then the double complex $\psi^{**}(\mathbb{C})$ defined by

$$\psi(\mathbb{C})^{-p,q} = \mathbb{C}u^p \otimes_{\mathbb{C}} \Omega_X^{q-p}(\log Y) \qquad p \geq 0 \tag{3.1.26}$$

with the differentials

$$\begin{aligned} d' : \psi(\mathbb{C})^{-p,q} &\longrightarrow \psi(\mathbb{C})^{-p+1,q} \quad d'(u^p \otimes \omega) = (pu^{p-1} \otimes \theta \wedge \omega) \\ d'' : \psi(\mathbb{C})^{-p,q} &\longrightarrow \psi(\mathbb{C})^{-p,q+1} \quad d''(u^p \otimes \omega) = (u^p \otimes d\omega) \end{aligned} \tag{3.1.27}$$

is a resolution of the complex $\psi^*(\mathbb{C})$. On $\psi^*(\mathbb{C})$, the momodromy is expressed by the operator,

$$T : \psi^*(\mathbb{C}) \longrightarrow \psi^*(\mathbb{C}) \tag{3.1.28}$$

given by

$$T(u^p \otimes \omega) = (u + 2\pi i)^p \otimes \omega \tag{3.1.29}$$

If we let $N = \log T/2\pi i$ as before, we note that N has the following description on $\psi^{**}(\mathbb{C})$

$$N(u^{[p]} \otimes \omega) = u^{[p-1]} \otimes \omega \qquad p \geq 0 \tag{3.1.30}$$

By abuse of notation, we shall often refer to ψ^{**} itself as the nearby cycles complex.

Proposition 3.5. *(see [22, Section 4]) There is an isomorphism of hypercohomologies*

$$\mathbb{H}^*(Y, \psi^*(\mathbb{C})) \approx \mathbb{H}^*(Y, \mathbf{R}\psi^*(\mathbb{C})) \approx H^*(\tilde{X}^*, \mathbb{C})$$

Proof. We have mentioned that $\psi^{**}(\mathbb{C})$ is a resolution of the nearby cycles complex

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
u^2\Omega^0_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^1\Omega^1_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^0\Omega^2_X(\log Y) \arrow[u, "d''"] \\
 & u^1\Omega^0_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] \arrow[dr, "N"] & u^0\Omega^1_X(\log Y) \arrow[u, "d''"] \\
 & & u^0\Omega^0_X(\log Y) \arrow[u, "d''"]
\end{tikzcd}
\tag{3.1.31}
$$

Figure 2: **The bicomplex ψ^{**}**

and hence it induces the isomorphism $\mathbb{H}^*(Y, \psi^*(\mathbb{C})) \approx \mathbb{H}^*(Y, \mathbf{R}\psi^*(\mathbb{C}))$. The second part of the statement has already been proved above. $\square$

Proposition 3.6. *(see [28, (2.5)]) The morphism $\mu : \psi^*(\mathbb{C}) \longrightarrow A^*$ defined by*

$$
\mu(u^{[p]} \otimes \omega_p) = \begin{cases} 0 & \textit{if } p \neq 0 \\ (-1)^{|\omega_0|} u_{[1]} \otimes \theta \wedge \omega_0 & \textit{if } p = 0 \end{cases} \tag{3.1.32}
$$

is a quasi-isomorphism of complexes. Further, if $N = \frac{\log T}{2\pi i}$, where T is the monodromy operator (T may be assumed to be unipotent by making an adequate transformation $z \mapsto z^N$), the morphisms $N \circ \mu$ and $\mu \circ N$ are homotopic. The homotopy is induced by the map h of bidegree $(0, -1)$ defined as

$$
h(u^{[p]} \otimes \omega_p) = \begin{cases} 0 & \textit{if } p \neq 0 \\ (-1)^{|\omega_0|} u_{[1]} \otimes \omega_0 & \textit{if } p = 0 \end{cases} \tag{3.1.33}
$$

Proof. It is clear that $[\mu, d''] = [\mu, d'] = 0$ and hence μ induces a morphism of complexes. The proof that μ is actually a quasi-isomorphism may be found in [33]. Further, we note that $h \circ d'' + d'' \circ h = 0$ and $N \circ \mu - \mu \circ N = h \circ d' + d' \circ h$, whence

h defines a homotopy from $N \circ \mu$ to $\mu \circ N$. $\square$

As mentioned above, the maps N and μ commute only up to homotopy. Consider the subcomplex $Ker(N)^*$ of $\psi^*(\mathbb{C})$ and the subcomplex $Im(N)^*$ of A^*. The weight filtration W on $\Omega^*_X(\log Y)$ (see chapter 2) may be extended to $\psi^*(\mathbb{C})$. Following [38]; on the terms A^{pq}, we have two filtrations W and L, given by

$$\begin{aligned} W_r A^{pq} &= \mathbb{C}u_{[p+1]} \otimes_{\mathbb{C}} W_{r+p+1}\Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \\ L_r A^{pq} &= \mathbb{C}u_{[p+1]} \otimes_{\mathbb{C}} W_{r+2p+1}\Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \end{aligned} \tag{3.1.34}$$

It is important to note that the differentials d' and d'' respect the latter filtration L (usually called the monodromy filtration) and that we have

$$Gr_r^L A^* \cong \bigoplus_{k\geq 0,-r} \mathbb{C}u_{[k+1]} \otimes_{\mathbb{C}} Gr^W_{r+2k+1}\Omega_X^{*+1}(\log Y) \tag{3.1.35}$$

Using the Poincaré residue map described in the previous chapter, the right hand side is isomorphic to

$$Gr_r^L A^* \cong \oplus_{k\geq 0,-r}\Omega^{*+1}_{\tilde{Y}^{r+2k+1}}[-r-2k-1] \tag{3.1.36}$$

where the residue map is extended by $Res(u_{[k+1]} \otimes \omega) = Res_{r+2k+1}(\omega)$.

Theorem 3.7. *Denote the monodromy on ψ^{**} by N_ψ and the corresponding operator on A^{**} by N_A. Consider the complexes $(Ker(N_\psi)^*, d'')$ and $(Coker(N_A)^*, d'')$. Then, we have*

$$(Ker(N_\psi)^*, d'')/W_0(Ker(N_\psi)^*, d'') = (Coker(N_A)^*, d'')[-1] \tag{3.1.37}$$

where $W_0(Ker(N_\psi)^*, d'')$ the subcomplex of $(Ker(N_\psi)^*, d'')$ of terms of weight 0.

Proof. The map N_ψ acts by $N_\psi(u^{[p]} \otimes \omega) = u^{[p-1]} \otimes \omega$ and hence it is clear that the kernel of N_ψ consists of the terms $u^{[0]} \otimes \omega$, which form the column ψ^{0q}. Hence $Ker(N_\psi)^i = \mathbb{C}u^0\Omega^i_X(\log Y)$.

Similarly, since N_A acts as $N_A(u_{[p]} \otimes \omega) = u_{[p+1]} \otimes \omega$, the cokernel of N_A consists of the terms $u_{[1]} \otimes \omega$ which form the column $A^{0q} = \mathbb{C}u_{[1]} \otimes \Omega^{q+1}_X(\log Y)/W_0\Omega^{q+1}_X(\log Y)$. Hence, it follows that

$$(Ker(N_\psi)^*, d'')/W_0(Ker(N_\psi)^*, d'') = (\mathbb{C}u^0 \otimes \Omega^q_X(\log Y)/W_0\Omega^q_X(\log Y), d'') \xrightarrow{\sim} \tag{3.1.38}$$

$$(\mathbb{C}u^0 \otimes \Omega^{q+1}_X(\log Y)/W_0\Omega^{q+1}_X(\log Y), d'')[-1] = (A^{0q}, d'')[-1] = (Coker(N_A)^*, d'')[-1] \tag{3.1.39}$$

□

3.2 The complex Ker(N)* and the Hochschild Complex

As recalled in Section 3.1, the bicomplex $\psi^{**}(\mathbb{C})$ is a resolution of the nearby cycles complex. The monodromy N, which is given by $N = \frac{\log T}{2\pi i}$, acts on $\psi^{**}(\mathbb{C})$ as follows; given $u^{[p]} \otimes \omega \in \psi^{-p,q}(\mathbb{C})$, we have

$$N(u^{[p]} \otimes \omega) = (u^{[p-1]} \otimes \omega) \in \psi^{-p+1,q-1}(\mathbb{C}) \qquad p \geq 0$$

Moreover, it is easy to check that $N : \psi^{-p,q}(\mathbb{C}) \to \psi^{-p+1,q-1}(\mathbb{C})$ commutes with the differentials d' and d'' of $\psi^{**}(\mathbb{C})$, i.e.,

$$[N, d'] = [N, d''] = 0$$

Therefore, we can consider the subcomplex of $Tot(\psi^{**}(\mathbb{C}))$ given by $(Ker(N), d' + d'')$, which we shall denote by $Ker(N)^*$. We consider the increasing filtration F_k on $Tot(\psi^{**}(\mathbb{C}))$ by subcomplexes $F_k(Tot(\psi^{**}(\mathbb{C}))) = Ker(N^k)^*$, $k \geq 0$ and the graded pieces $F_{k+1}/F_k = Ker(N^{k+1})^*/Ker(N^k)^*$. Note that the filtration is bounded below, i.e. $F_0(Tot(\psi^{**}(\mathbb{C}))) = 0$ and that it is exhaustive, i.e. $Tot(\psi^{**}(\mathbb{C})) = \cup_{k\geq 0} F_k(Tot(\psi^{**}(\mathbb{C})))$.

Proposition 3.8. *Let $\mathscr{D}_{X^*}$ denote the sheaf of differential operators on X^* as defined in Appendix 1. Then, there are canonical isomorphisms*

$$HH_q(\mathscr{D}_{X^*}) \xrightarrow{\sim} \mathbb{H}^{2n-q}(Ker(N)^*) \tag{3.2.1}$$

where $HH_*(\mathscr{D}_{X^*})$ denotes the Hochschild homology of the sheaf of differential operators on X^* as defined in Appendix 1.

Proof. Since $N(u^{[p]} \otimes \omega) = (u^{[p-1]} \otimes \omega)$, $u^{[p]} \otimes \omega$ lies in $Ker(N)^*$ if and only if $p = 0$. Hence, we have

$$Ker(N)^* = \psi^{0,*}(\mathbb{C}) \tag{3.2.2}$$

From the description of the terms $\psi^{0,*}$, the complex $Ker(N)^*$ can now be written as

$$Ker(N)^i = \mathbb{C}u^0 \otimes \Omega^i_X(\log Y) \tag{3.2.3}$$

Let $j : X^* \to X$ denote the open immersion. From [23, 3.1.8], we know that there is a quasi-isomorphism of filtered complexes

$$(\Omega^*_X(\log Y), W) \leftarrow (\Omega^*_X(\log Y), \tau) \hookrightarrow (j_*\Omega^*_{X^*}, \tau) \tag{3.2.4}$$

where W is the weight filtration on the de Rham complex with logarithmic poles and τ is the canonical filtration (see [23, 3.1.7] for details). Both W and τ are finite filtrations (on each term of the complex) and hence we have a quasi-isomorphism of complexes and an isomorphism on hypercohomologies

$$\mathbb{H}^*(Ker(N)^*) \xrightarrow{\sim} \mathbb{H}^*(\Omega^*_X(\log Y)) \xrightarrow{\sim} \mathbb{H}^*(X, j_*\Omega^*_{X^*}) \tag{3.2.5}$$

Every point in X^* has a fundamental system of neighbourhoods that are Stein. Hence, for any analytic coherent sheaf $\mathcal{F}$ on X^*, we have $R^ij_*\mathcal{F} = 0$ for $i > 0$. Hence the de Rham complex on X^* is a resolution of the constant sheaf $\mathbb{C}$ on X^* by a complex of sheaves that are j_* acyclic. This implies that (see [23, 3.1.7.1])

$$H^*(X^*, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^*(X^*, \Omega^*_{X^*}) \xleftarrow{\sim} \mathbb{H}^*(X, j_*\Omega^*_{X^*}) \tag{3.2.6}$$

From Theorem A.3.17, we obtain

$$HH_q(\mathscr{D}_{X^*}) \xrightarrow{\sim} \mathbb{H}^{2n-q}(X^*, \Omega_{X^*}) \tag{3.2.7}$$

Hence it follows that

$$HH_q(\mathscr{D}_{X^*}) \xrightarrow{\sim} \mathbb{H}^{2n-q}(Ker(N)^*) \tag{3.2.8}$$

□

Corollary 3.9. *The filtration by columns on $Tot(\psi^{**})$ coincides with the filtration by $F_j(Tot(\psi^{**}(\mathbb{C}))) = Ker(N^j)^*$, $j \geq 0$ and moreover, the Hochschild homology of $\mathscr{D}_{X^*}$ can be computed as (for $k \geq 0$)*

$$HH_q(\mathscr{D}_{X^*}) \cong \mathbb{H}^{2n-q}(Ker(N^{k+1})^*/Ker(N^k)^*) \cong \mathbb{H}^{2n-q}(gr^k_I Tot(\psi^{**})) \tag{3.2.9}$$

where $gr_I^k\psi^{**}(\mathbb{C})$ denote the graded pieces corresponding to the filtration of $\psi^{**}(\mathbb{C})$ by columns.

Proof. For any $k \geq 0$, we have that $F_k(Tot(\psi^{**}(\mathbb{C}))) = Ker(N^k)^*$ comprises the terms $\psi^{p,q}$ where $p \geq -k+1$. Consequently, we have

$$Ker(N^{k+1})^*/Ker(N^k)^* = \psi^{-k,*} \tag{3.2.10}$$

We see that $Ker(N^k)$ consists of all terms of $Tot(\psi^{**})$ that come from columns 0, 1,...,k. Therefore F_k coincides with the filtration by columns on $Tot(\psi^{**})$ and, using Proposition 3.8, we have

$$HH_q(\mathscr{D}_{X^*}) \cong \mathbb{H}^{2n-q}(Ker(N^{k+1})^*/Ker(N^k)^*) \cong \mathbb{H}^{2n-q}(gr_I^k\psi^{**}) \tag{3.2.11}$$

□

3.3 The Connes periodicity operator and the monodromy operator

In the previous section, we have seen that the graded pieces of the filtration on $Tot(\psi^{**})$ by $F_jTot(\psi^{**}(\mathbb{C})) = Ker(N^j)$, $j \geq 0$ compute the Hochschild homology of $\mathscr{D}_{X^*}$. Furthermore, we have shown in Corollary 3.9 that the filtration by $Ker(N^j)$, $j \geq 0$ coincides with the filtration by columns of ψ^{**}. This yields an increasing filtration on the associated total complex ψ^*, which, moreover, bounded below and exhaustive. Consequently, we have a spectral sequence in hypercohomology

$$HH_q(\mathscr{D}_{X^*}) \overset{\simeq}{=} E_1^{-p,2n+p-q} \Rightarrow \mathbb{H}^{2n-q}(\psi^*) \tag{3.3.1}$$

We will now define a triple complex $\mathcal{BC}^{***}$ by "vertically stacking" copies of the bi-complexes $\psi^{**}(\mathbb{C})$ in a certain way. Then, a "column" of the triple complex is a bicomplex $\mathcal{BC}^{p**}$ obtained by fixing the first index. The "column" $\mathcal{BC}^{p**}$ therefore consists of a column from each copy of ψ^{**} that has been stacked to form the triple complex $\mathcal{BC}^{p**}$.

In Theorem 3.10, we shall show that each of the "columns" $\mathcal{BC}^{p**}$ is quasi-isomorphic to the complex computing cyclic homology of $\mathscr{D}_{X^*}$. This is a counterpart to Corollary 3.9, where we showed that the columns of $\psi^{**}(\mathbb{C})$ are quasi-isomorphic to Hochschild complexes for $\mathscr{D}_{X^*}$.

The $\psi^{**}(\mathbb{C})$ complexes stacked vertically to form the complex $\mathcal{BC}^{***}$ induce a monodromy operator N on $\mathcal{BC}^{***}$ and hence on each column $\mathcal{BC}^{p**}$. We have just mentioned that the column $\mathcal{BC}^{p**}$ is quasi-isomorphic to the cyclic homology complex of $\mathscr{D}_{X^*}$. Therefore the hypercohomology of each column $\mathcal{BC}^{p**}$ carries a periodicity operator S. We will also show in Theorem 3.10 that, in this formulation, the monodromy operator N coincides with the Connes periodicity operator S in each column $\mathcal{BC}^{p**}$.

Finally, in Theorem 3.11, we show that the filtration by columns $\mathcal{BC}^{p**}$ on the total complex $Tot(\mathcal{BC}^{***})$ yields a spectral sequence converging to the hypercohomology of $\mathcal{BC}^{***}$ whose E_1-terms are cyclic homologies of $\mathscr{D}_{X^*}$. This is a counterpart to (3.3.1) above.

Consider the following triple complex of sheaves;

$$\mathcal{BC}^{p,q,r} = \psi^{p+r,q-2r} = \mathbb{C}u^{-p-r} \otimes \Omega_X^{q+p-r}(\log Y) \qquad p,q,r \in \mathbb{Z}, p+r \leq 0 \tag{3.3.2}$$

along with maps

$$\begin{aligned} d' : \mathcal{BC}^{p,q,r} \to \mathcal{BC}^{p+1,q,r} \qquad & d'(u^p \otimes \omega) = pu^{p-1} \otimes \theta \wedge \omega \\ d'' : \mathcal{BC}^{p,q,r} \to \mathcal{BC}^{p,q+1,r} \qquad & d''(u^p \otimes \omega) = u^p \otimes d\omega \end{aligned} \tag{3.3.3}$$

(Recall here the definition of θ, given immediately after Proposition 2.3)

where the morphisms are understood as composed with the projection onto their image. The third differential of the triple complex, i.e. the map from $\mathcal{BC}^{p,q,r} \to \mathcal{BC}^{p,q,r+1}$ is taken to be 0. This is supposed to reflect the fact that the cyclic homology of $\mathscr{D}_{X^*}$ splits as a direct sum of Hochschild homologies of $\mathscr{D}_{X^*}$. In general one would expect only a filtration on the cyclic homology, determined by the Hochschild to Cyclic spectral sequence. The monodromy on the complex ψ^{**} extends to a monodromy operator on $\mathcal{BC}^{***}$ defined as

$$N : \mathcal{BC}^{p,q,r} \longrightarrow \mathcal{BC}^{p+1,q-1,r} \qquad N(u^{[t]} \otimes \omega) = u^{[t-1]} \otimes \omega \tag{3.3.4}$$

For any fixed $p \in \mathbb{Z}$, the bicomplex $\mathcal{BC}^{p,*,*}$ may be described explicitly as in the diagram (3.3.5).

While dealing with Hochschild homology of $\mathscr{D}_{X^*}$ in Section 3.2, we filtered the complex $\psi^{p,q}(\mathbb{C})$ by its columns fixing the value of p. We will now deal with one "column" of the triple complex $\mathcal{BC}^{p,q,r}$ at a time, again by considering the terms in $\mathcal{BC}^{p,q,r}$ with a fixed value of p.

Theorem 3.10. *(a) For each fixed $p \in \mathbb{Z}$, the bicomplex $\mathcal{BC}^{p,*,*}$ computes the cyclic homology of $\mathscr{D}_{X^*}$, i.e., there is a quasi-isomorphism of complexes:*

$$Tot(CC_*(\mathscr{D}_{X^*})) \longrightarrow Tot(\mathcal{BC}^{p**})[2n].$$

$$
\begin{array}{lccccccc}
r=-p: & & & & & \overbrace{\mathbb{C}u^0\Omega^0_X(\log Y)}^{\mathcal{BC}^{p,-2p,-p}} & \xrightarrow{d''} & \dots \\
& & & & & \uparrow{\scriptstyle 0} & & \\
r=-p-1: & & & \overbrace{\mathbb{C}u^1\Omega^0_X(\log Y)}^{\mathcal{BC}^{p,-2p-1,-p-1}} & \xrightarrow{d''} & \mathbb{C}u^1\Omega^1_X(\log Y) & \xrightarrow{d''} & \dots \\
& & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} & & \\
r=-p-2: & \overbrace{\mathbb{C}u^2\Omega^0_X(\log Y)}^{\mathcal{BC}^{p,-2p-2,-p-2}} & \xrightarrow{d''} & \mathbb{C}u^2\Omega^1_X(\log Y) & \xrightarrow{d''} & \mathbb{C}u^2\Omega^2_X(\log Y) & \xrightarrow{d''} & \dots
\end{array}
\tag{3.3.5}
$$

Figure 3: **The bicomplex $\mathcal{BC}^{p,*,*}$**

and hence a canonical isomorphism:

$$HC_k(\mathscr{D}_{X^*}) \cong \mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p,*,*})) \qquad k \in \mathbb{N}, \quad n = dim(X^*) \tag{3.3.6}$$

where the total complex $Tot(\mathcal{BC}^{p,*,*})$ is described by

$$Tot(\mathcal{BC}^{p,*,*})^l = \bigoplus_{q+r=-p+l} \mathcal{BC}^{p,q,r} = \bigoplus_{q+r=-p+l} \mathbb{C}u^{-p-r}\Omega_X^{q+p-r}(\log Y) \tag{3.3.7}$$

(b) Using the isomorphism in (3.3.6)*, the monodromy N on the complex $\mathcal{BC}^{p,*,*}$ coincides with the periodicity operator S on the cyclic homology of $\mathscr{D}_{X^*}$, i.e.*

$$N : \mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p,*,*})) \cong HC_k(\mathscr{D}_{X^*}) \xrightarrow{S} \tag{3.3.8}$$

$$HC_{k-2}(\mathscr{D}_{X^*}) \cong \mathbb{H}^{2n-k+2}(Tot(\mathcal{BC}^{p,*,*}))$$

Proof. (a) Notice that each of the rows with d'' differentials in the diagram for $\mathcal{BC}^{p**}$ is simply the logarithmic de Rham complex $(\Omega^*_X(\log Y), d'')$ (upto a shift). We know from before (see for instance, the proof of Proposition 3.8), that this logarithmic de

Rham complex computes the de Rham cohomology of X^*. Hence, if we take the total complex $Tot(\mathcal{BC}^{p,*,*})$, it follows that

$$\mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p,*,*})) \cong H_{DR}^{2n-k}(X^*) \oplus H_{DR}^{2n-k+2}(X^*) \oplus \dots$$

But, from Theorem A.3.17 in Appendix 1, we know that

$$HC_k(\mathscr{D}_{X^*}) \cong H_{DR}^{2n-k}(X^*) \oplus H_{DR}^{2n-k+2}(X^*) \oplus \dots \tag{3.3.9}$$

The isomorphism $H^{2n-k}(Tot(\mathcal{BC}^{p,*,*})) \cong HC_k(\mathscr{D}_{X^*})$ is canonical because, on the one hand, the rows of $\mathcal{BC}^{p**}$ with d''-differentials are logarithmic de Rham complexes and hence are canonically isomorphic to de Rham complex on X^*. Hence, we have a quasi-isomorphism of complexes

$$Tot(\mathcal{BC}^{p**}) = \bigoplus_{k=0}^{\infty} \Omega_X^*(\log Y)[2k] \xrightarrow{q.i.} \bigoplus_{k=0}^{\infty} \Omega_{X^*}^*[2k] \tag{3.3.10}$$

It now follows from the proof of Theorem A.3.17 that the cyclic complex $Tot(CC(\mathscr{D}_{X^*}))$ is quasi-isomorphic to a shift of the right hand side of (3.3.10), i.e.

$$Tot(CC(\mathscr{D}_{X^*})) \xrightarrow{q.i.} \left(\bigoplus_{k=0}^{\infty} \Omega_{X^*}^*[2k]\right)[2n] \xrightarrow{q.i.} Tot(\mathcal{BC}^{p**})[2n] \qquad (n = dim(X^*)) \tag{3.3.11}$$

(b) From the definition of the operator N, it follows that $N(\mathcal{BC}^{p,*,*}) \cong \mathcal{BC}^{p,*,*}[0,1,1]$. Hence,

$$\mathbb{H}^{2n-k}(N(\mathcal{BC}^{p,*,*})) = \mathbb{H}^{2n-k+2}(Tot(\mathcal{BC}^{p,*,*})) \tag{3.3.12}$$

Therefore, the operator N coincides with the projection

$$H^{2n-k}_{DR}(X^*) \oplus H^{2n-k+2}_{DR} \oplus H^{2n-k+4}_{DR}(X^*) \cdots \to H^{2n-k+2}_{DR} \oplus H^{2n-k+4}_{DR}(X^*)$$

The operator S can be identified with the same projection, as we show in Corollary A.3.18 in Appendix 1. $\square$

Theorem 3.11. *Let $\mathcal{BC}^*$ denote the total complex associated to the triple complex $\mathcal{BC}^{***}$. Let F'_k, $k \in \mathbb{Z}$ denote the filtration induced on the total complex $\mathcal{BC}^*$ by "columns" $\mathcal{BC}^{p**}$, i.e.*

$$F'_k\mathcal{BC}^* = \left(\bigoplus_{q+r=-p+*, p\geq -k} \mathcal{BC}^{p,q,r}, d' + d'' + d''' \right) \tag{3.3.13}$$

There is a converging spectral sequence in hypercohomology

$$E_1^{-p,2n+p-q} := \mathbb{H}^{2n-q}(F'_p/F'_{p-1}) \Rightarrow \mathbb{H}^{2n-q}(\mathcal{BC}^*) \tag{3.3.14}$$

The E_1-terms are, moreover, canonically isomorphic to cyclic homologies of $\mathscr{D}_{X^*}$, i.e

$$HC_q(\mathscr{D}_{X^*}) \cong E_1^{-p,2n+p-q} \tag{3.3.15}$$

Proof. For each fixed $p \in \mathbb{Z}$, the bicomplexes $\mathcal{BC}^{p**}$ form the graded pieces of a "filtration by columns" on the triple complex $\mathcal{BC}^{***}$. Hence, the complexes $Tot(\mathcal{BC}^{p**})$ form the graded pieces of a filtration on the total complex $\mathcal{BC}^*$, i.e.

$$F'_p/F'_{p-1} = Tot(\mathcal{BC}^{p**}) \tag{3.3.16}$$

Further, this filtration is bounded below and exhaustive. From Theorem 3.10 above, we know that

$$HC_k(\mathscr{D}_{X^*}) \cong \mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p,*,*})) \qquad k \in \mathbb{N} \tag{3.3.17}$$

From the spectral sequence associated to the filtration by $Tot(\mathcal{BC}^{p**})$, $p \in \mathbb{Z}$, we get

$$HC_q(\mathscr{D}_{X^*}) = E_1^{-p,2n+p-q} \Rightarrow \mathbb{H}^{2n-q}(\mathcal{BC}^*) \tag{3.3.18}$$

□

Finally, we have

Proposition 3.12. *We have the following commutative diagram of long exact sequences in which the vertical maps are isomorphisms.*

$$\begin{tikzcd}
... \arrow[r] & \mathbb{H}^{2n-k}(Ker(N)^*) \arrow[r, "I"] \arrow[d, "\cong"'] & \mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p**})) \arrow[r, "N"] \arrow[d, "\cong"'] & \mathbb{H}^{2n-k+2}(Tot(\mathcal{BC}^{p**})) \arrow[r] \arrow[d, "\cong"'] & ... \\
... \arrow[r] & HH_k(\mathscr{D}_{X^*}) \arrow[r, "I"] & HC_k(\mathscr{D}_{X^*}) \arrow[r, "S"] & HC_{k-2}(\mathscr{D}_{X^*}) \arrow[r] & ...
\end{tikzcd}$$

Proof. The isomorphism $\mathbb{H}^{2n-k}(Ker(N)^*) \cong HH_k(\mathscr{D}_{X^*})$ was proved in Proposition 3.8 in Section 3.2 while the isomorphisms $\mathbb{H}^{2n-k}(Tot(\mathcal{BC}^{p**})) \cong HC_k(\mathscr{D}_{X^*})$ and $\mathbb{H}^{2n-k+2}(Tot(\mathcal{BC}^{p**})) \cong HC_{k-2}(\mathscr{D}_{X^*})$ follow from Theorem 3.10(a) above. The morphisms S and N have been identified in Theorem 3.10(b). The lower sequence in the diagram above has been shown to be long exact in Corollary A.3.18 in Appendix 1. This proves the result.

□

Appendix 1: *Cyclic Homology of the Ring of Differential Operators*

A manifold U is said to be *Stein* if the following conditions hold:

(1) U is *holomorphically convex*, i.e., for every compact subset K of U, the holomorphic convex hull $\overline{K}$ of K:

$$\overline{K} = \{z \in U \mid |f(z)| \leq \sup_{\mathrm{K}} |f| \ \forall\ f \in \mathcal{O}(U)\}$$

is a compact subset of U. Here $\mathcal{O}(U)$ denotes the ring of holomorphic functions on U.

(2) U is *holomorphically separated*, i.e. given two points $x \neq y$ in U, there exists a holomorphic function $f \in \mathcal{O}(U)$ such that

$$f(x) \neq f(y)$$

(3) For every point $x \in U$, there exist holomorphic functions f_1,...,f_n in $\mathcal{O}(U)$ forming a local coordinate system at x.

In fact, the key property of Stein manifolds is that if $\mathcal{F}$ is an analytic quasi-coherent sheaf on a Stein manifold U, then $H^i(U, \mathcal{F}) = 0$ for all $i > 0$. In the GAGA set of analogies of manifolds with algebraic varieties, Stein manifolds correspond to affine varieties.

Let us denote by $\mathscr{D}(U)$ the ring of holomorphic differential operators on a Stein manifold U. If $(z_1, z_2, ..., z_n)$ is a system of holomorphic local coordinates for U ($dim_{\mathbb{C}} U = n$), then the ring $\mathscr{D}(U)$ consists of all finite sums of terms of the form

$$f_I \frac{\partial}{\partial z_{i_1}} \frac{\partial}{\partial z_{i_2}} \cdots \frac{\partial}{\partial z_{i_k}}$$

where $I = \{i_1 \leq i_2 \leq ... \leq i_k\} \subseteq \{1, 2, ..., n\}$ and $f_I \in \mathcal{O}(U)$. Henceforth, we shall implicitly assume that the spaces $H^*_{dR}(U)$, $HH_*(\mathscr{D}(U))$ and $HC_*(\mathscr{D}(U))$ are finite dimensional. The Hochschild and cyclic homologies of $\mathscr{D}(U)$ (considered as a $\mathbb{C}$-algebra), as determined in [46] (as also for the ring of differential operators on an affine variety), are described as follows:

Proposition A. 3.13. *(see [46, Theorem 3]) If U is a Stein manifold and $dim(U) = n$, then*

$$\begin{aligned} &(1) \quad HC_q(\mathscr{D}(U)) \simeq H_{dR}^{2n-q}(U) \oplus H_{dR}^{2n-q+2}(U) \oplus H_{dR}^{2n-q+4}(U) \oplus ... \\ &(2) \qquad\qquad\qquad HH_q(\mathscr{D}(U)) \simeq H_{dR}^{2n-q}(U) \end{aligned} \tag{3.3.19}$$

In particular, the analogous result holds for the ring of algebraic differential operators on an affine variety.

Proof. The ring $\mathscr{D}(U)$ has a natural filtration by order of the differential operator. The filtration gives us a spectral sequence E^r_{pq} associated to the mixed complex $Tot(\mathcal{B}_{**}(\mathscr{D}(U)))$ that converges to $HC_{p+q}(\mathscr{D}(U))$. This spectral sequence E^r_{pq} is a priori located in the region $p \geq 0$ and $p+q \geq 0$. Wodzicki[46] shows that the $E^r_{pq} = 0$ $(r \geq 1)$ if either $p \geq 1$ and $q \geq n$ or if $p \geq 1$ and $p+q \geq 2n$. For $p = 0$, we get

$$E^1_{0q} \simeq H^q_{dR}(U) \qquad (q \geq n)$$

Since the columns of the mixed complex $\mathcal{B}_{**}(\mathscr{D}(U))$ compute the Hochschild homology of $\mathscr{D}(U)$, we can use the spectral sequence associated to the filtration of $\mathcal{B}_{**}(\mathscr{D}(U))$ by columns to reduce the proof of statement (1) to the proof of (2) above. The terms E^1_{0q} show that

$$dimHC_q(\mathscr{D}(U)) \leq \sum_{i=0}^{[q/2]} HH_{q-2i}(\mathscr{D}(U))$$

for $q \geq 2n-1$ whence it follows from [46, Lemma 6] that the long exact periodicity sequence splits into short exact sequences

$$0 \longrightarrow HH_q(\mathscr{D}(U)) \xrightarrow{I} HC_q(\mathscr{D}(U)) \xrightarrow{S} HC_{q-2}(\mathscr{D}(U)) \longrightarrow 0$$

Finally, if ${}'E^r_{pq}$ denotes the spectral sequence associated to the filtration on the Hochschild complex of $\mathscr{D}(U)$, ${}'E^r_{pq}$ degenerates at the E^2-term, i.e. ${}'E^2_{pq} = 0$ unless $q = n$ and the differential ${}'d^1_{pq}$ corresponds to the de Rham differential. Then,

$$HH_k(\mathscr{D}(U)) \simeq {}'E^2_{k-n,n} = H^{2n-k}_{dR}(U)$$

□

In Proposition A.3.13 above, the ring $\mathscr{D}(U)$ is considered as a topological algebra. Therefore, in the complexes defining the Hochschild and cyclic homologies of $\mathscr{D}(U)$, it should be understood that the tensor product $\otimes_{\mathbb{C}}$ has been replaced by the projective topological tensor product $\hat{\otimes}_{\pi,\mathbb{C}}$.

The isomorphism in Proposition A.3.13 above is canonical and functorial with respect to embeddings of codimension zero, i.e. with respect to embeddings of the form $U \hookrightarrow V$ with $dim(U) = dim(V)$ (since the expressions $H^{2n-q}_{dR}(U)$ involve the dimension n of the manifold U, the isomorphism $HH_q(\mathscr{D}(U)) \approx H^{2n-q}_{dR}(U)$ cannot be functorial with respect to an embedding $U \hookrightarrow V$, unless U and V have the same dimension).

In the notation of Section 3.1, we have a proper morphism $f : X \to D$, X being an algebraic variety and D being the open unit disc. If $Y = f^{-1}(\{0\})$ is the (closed) fibre over over $\{0\}$, we have set $X^* = X - Y$. Since X^* is open in X, any $x \in X^*$ has a fundamental system of neighbourhoods that are Stein. We can therefore form a basis of X^*, say $\{U_i\}_{i \in I}$, such that each U_i is Stein. Since all open sets U_i and their finite intersections have the same topological dimension and are also Stein open sets, the functoriality of the isomorphisms $HH_q(\mathscr{D}(U), \mathscr{D}(U)) \simeq H^{2n-q}_{dR}(U)$ can be employed to obtain an isomorphism in hypercohomology.

The following lemma will be useful both in this as well as in later chapters. If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves on a manifold W, we know that, in general, isomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ on the open sets $U \subseteq W$ do not imply the existence of an isomorphism (or even a morphism) of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$. However, if the isomorphisms φ_U are functorial, in a way to be made precise in the following lemma, this is true.

Lemma A. 3.14. *(a) Let $\mathcal{F}$ and $\mathcal{G}$ be two separated presheaves (of sets or abelian groups) on a topological space W, such that W has a basis consisting of open sets $\{U_\alpha\}_{\alpha \in A}$ and there exist isomorphisms (of sets, or abelian groups resp.)*

$$\varphi_\alpha : \mathcal{F}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha)$$

which are (contravariantly) functorial with respect to inclusions of the open sets U_α, $\alpha \in A$. Then, there exists an isomorphism $\varphi : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ of their sheafifications.

(b) Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves (of sets or abelian groups) on a topological space W, such that W has a basis consisting of open sets $\{U_\alpha\}_{\alpha \in A}$ and there exist isomorphisms (of sets, or abelian groups resp.)

$$\varphi_\alpha : \mathcal{F}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha)$$

which are (contravariantly) functorial with respect to inclusions of the open sets U_α, $\alpha \in A$. Then, there exist morphisms

$$\varphi^s_\alpha : \mathcal{F}^s(U_\alpha) \longrightarrow \mathcal{G}^s(U_\alpha)$$

which are contravariantly functorial with respect to inclusions of open sets U_α, $\alpha \in A$, where $\mathcal{F}^s$ and $\mathcal{G}^s$ refer to the separated presheaves associated to $\mathcal{F}$ and $\mathcal{G}$ respectively.

Proof. (a) Choose an open set $U \subseteq W$ such that $\tilde{\mathcal{F}}(U) \neq \phi$ and consider $s \in \tilde{\mathcal{F}}(U)$. There exists a collection of open sets $\{U_i\}_{i \in I}$ covering U such that $I \subseteq A$, i.e. the U_i are elements of the basis and $s \in \tilde{\mathcal{F}}(U)$ corresponds to sections $s_i \in \mathcal{F}(U_i)$ of $s \in \mathcal{F}(U)$ and their images $t_i = \varphi_i(s_i) \in \mathcal{G}(U_i)$.

For any two i, $j \in I$, consider a collection of basis open sets $\{V_{ij\beta}\}_{\beta \in B_{ij}}$ covering $U_i \cap U_j$. Denote by s_{ij} (resp. $s_{ij\beta}$) the restriction of s_i to $U_i \cap U_j$ (resp. to each $V_{ij\beta}$). Denote by t_{ij} and t'_{ij} (resp. $t_{ij\beta}$ and $t'_{ij\beta}$) the restriction of t_i and t_j to $U_i \cap U_j$ (resp. $V_{ij\beta}$) respectively. The morphisms φ_α are functorial with respect to inclusions of the sets of the basis. From the inclusion $V_{ij\beta} \hookrightarrow U_i$, we get $\varphi_{ij\beta}(s_{ij\beta}) = t_{ij\beta}$ and from the inclusion $V_{ij\beta} \hookrightarrow U_j$, we get $\varphi_{ij\beta}(s_{ij\beta}) = t'_{ij\beta}$.

Let $i_{G,W'} : \mathcal{G}(W') \to \tilde{\mathcal{G}}(W')$ be the morphism of presheaves due to sheafification of $\mathcal{G}$ on an open set $W' \subseteq W$. Since $t_{ij\beta} = t'_{ij\beta}$, it follows that,

$$i_{G,U_{ij}}(t_{ij})|V_{ij\beta} = i_{G,V_{ij\beta}}(t_{ij\beta}) = i_{G,V_{ij\beta}}(t'_{ij\beta}) = i_{G,U_{ij}}(t'_{ij})|V_{ij\beta}$$

and hence it follows that

$$i_{G,U_i}(t_i)|U_{ij} = i_{G,U_{ij}}(t_{ij}) = i_{G,U_{ij}}(t'_{ij}) = i_{G,U_j}(t_j)|U_{ij}$$

From this it follows that the collection $i_{G,U_i}(t_i) \in \tilde{\mathcal{G}}(U_i)$ gives a well defined element $t \in \tilde{\mathcal{G}}(U)$. We define $\varphi_U(s) = t \in \tilde{\mathcal{G}}(U)$. It is clear that φ is an isomorphism.

Finally, suppose that $\tilde{\mathcal{F}}(U) = \phi$. If $\tilde{\mathcal{G}}(U) = \phi$ as well, there is nothing to prove. However, if $\mathcal{G}(U) \neq \phi$, we can find sections $t_\alpha \in \mathcal{G}(U_\alpha)$, U_α being open sets of the basis; such that, for any $\alpha, \beta \in A$ with $U_\alpha, U_\beta \subseteq U$, t_α and t_β have the same restriction to a collection of basis open sets covering the intersection $U_\alpha \cap U_\beta$. Then, using the isomorphisms $\varphi_\alpha^{-1} : \mathcal{G}(U_\alpha) \longrightarrow \mathcal{F}(U_\alpha)$ and their functoriality with respect to inclusions of basis open sets, we can obtain sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that, for any $\alpha, \beta \in A$ with $U_\alpha, U_\beta \subseteq U$, s_α and s_β have the same restriction to a collection of basis open sets covering the intersection $U_\alpha \cap U_\beta$. The sections s_α can now be glued together to give a section $s \in \tilde{\mathcal{F}}(U)$ and hence $\tilde{\mathcal{F}}(U) \neq \phi$.

Of course, if $\mathcal{F}$ and $\mathcal{G}$ are presheaves of abelian groups (as they will be in all our applications), the identity 0 is always a section over any open set and hence $\tilde{\mathcal{F}}(U) \neq \phi$ for all U.

(b) Take any open set U_β with $\beta \in A$. Suppose that s and t are two sections of $\mathcal{F}(U_\beta)$ that are identified in $\mathcal{F}^s(U_\beta)$. Then, since $\{U_\alpha\}_{\alpha \in A}$ is a basis for W, there exists a cover $\{U_i\}_{i \in I \subseteq A}$ of U_β such that $s|U_i = t|U_i$ for each $i \in I$. Consider the sections $\varphi_\beta(s)$, $\varphi_\beta(t) \in G(U_\beta)$. We have commutative diagrams

$$\begin{array}{ccc} \mathcal{F}(U_\beta) & \xrightarrow{\varphi_\beta} & \mathcal{G}(U_\beta) \\ {\scriptstyle res}\downarrow & & {\scriptstyle res}\downarrow \\ \mathcal{F}(U_i) & \xrightarrow{\varphi_i} & \mathcal{G}(U_i) \end{array} \tag{3.3.20}$$

for each $i \in I$, whence it follows that $\varphi_\beta(s)|U_i = \varphi_\beta(t)|U_i$ for each $i \in I$. Hence, $\varphi_\beta(s)$ and $\varphi_\beta(t)$ coincide in $\mathcal{G}^s(U_\beta)$. Hence, the morphisms $\varphi_\alpha : \mathcal{F}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha)$ for all $\alpha \in A$ descend to morphisms $\varphi_\alpha^s : \mathcal{F}^s(U_\alpha) \longrightarrow \mathcal{G}^s(U_\alpha)$. That the φ_α^s are contravariantly functorial with respect to inclusions of open sets in $\{U_\alpha\}_{\alpha \in A}$ is clear. □

Corollary A. 3.15. *Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves (of sets or abelian groups) on a topological space W, such that W has a basis consisting of open sets $\{U_\alpha\}_{\alpha \in A}$ and* *there exist isomorphisms (of sets, or abelian groups resp.)*

$$\varphi_\alpha : \mathcal{F}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha)$$

which are (contravariantly) functorial with respect to inclusions of the open sets U_α*,* $\alpha \in A$*. Then, there exists an isomorphism* $\varphi : \tilde{\mathcal{F}} \to \tilde{\mathcal{G}}$ *of their sheafifications.*

Proof. This is clear from Lemma A.3.14 above. □

Denote by $\mathscr{D}_{X^*}$ the sheaf of differential operators on X^*, i.e. $\mathscr{D}_{X^*}$ is the sheaf associated to the presheaf $U \mapsto \mathscr{D}(U)$ for every open set $U \subseteq X^*$.

Definition A. 3.16. *Let* $C_*^h(\mathscr{D}_{X^*})$ *and* $\mathcal{B}_{**}(\mathscr{D}_{X^*})$ *denote the sheafification of the complexes defining the Hochschild and cyclic homologies of* $\mathscr{D}(U)$ *(for each open* $U \subseteq X^*$*) resp. The Hochschild and cyclic homology of* $\mathscr{D}_{X^*}$ *are defined to be the respective hypercohomlogies, i.e*

$$HH_*(\mathscr{D}_{X^*}) = \mathbb{H}^{-*}(C_*^h(\mathscr{D}_{X^*}))\ HC_*(\mathscr{D}_{X^*}) = \mathbb{H}^{-*}(Tot(\mathcal{B}_{**}(\mathscr{D}_{X^*})))$$

Once again, we recall here that the complexes $C_*^h(\mathscr{D}_{X^*})$ and $Tot(\mathcal{B}_{**}(\mathscr{D}_{X^*}))$ are unbounded below and therefore the hypercohomology groups are defined as in Weibel [42].

Theorem A. 3.17. *Let* X^* *be a locally Stein complex manifold and let* $\mathscr{D}_{X^*}$ *be the sheaf of differential operators on* X^**. Then, there are isomorphisms*

(1) $HC_q(\mathscr{D}_{X^*}) \simeq H_{dR}^{2n-q}(X^*) \oplus H_{dR}^{2n-q+2}(X^*) \oplus H_{dR}^{2n-q+4}(X^*) \oplus ...$

(2) $HH_q(\mathscr{D}_{X^*}) \simeq H_{dR}^{2n-q}(X^*)$

Proof. Let $\mathcal{HH}_*(\mathscr{D}_{X^*})$ and $\mathcal{HC}_*(\mathscr{D}_{X^*})$ denote the respective total homology sheaves associated to the complexes $C^h_*(\mathscr{D}_{X^*})$ and $\mathcal{B}_{**}(\mathscr{D}_{X^*})$ as defined in Definition A.3.16. For each open set $V \subseteq X^*$, the de Rham complex is the hypercohomology of the complex of sheaves with de Rham differential:

$$0 \longrightarrow \Omega^0_V \longrightarrow \Omega^1_V \longrightarrow ...$$

where Ω^i_V denotes as usual the sheaf of holomorphic differential forms of degree i on V. When V is Stein, the sheaves Ω^i_V are coherent and the higher cohomologies vanish and hence the hypercohomology can be calculated by taking the homology of the complex of global sections. Let H^* denote the cohomology presheaf of the complex of sheaves

$$0 \to \Omega^0_{X^*} \to \Omega^1_{X^*} \to \dots$$

and let $\mathcal{H}^*$ denote the cohomology sheaf associated to H^*.

Consider the basis $\{U_i\}_{i \in I}$ of X^* such that each U_i is Stein. Then, using the functoriality of the isomorphisms in Proposition A.3.13 (for embeddings of codimension 0), we deduce the existence of isomorphisms f_i

$$f_i : HH_q(\mathscr{D}(U_i)) \xrightarrow{\sim} H^{2n-q}(U_i)$$

Since the intersections of Stein manifolds are Stein, the morphisms f_i also satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Since the Stein open sets form a basis, by applying Lemma A.3.14 we obtain an isomorphism of sheaves

$$\mathcal{HH}_q(\mathscr{D}_{X^*}) \xrightarrow{\sim} \mathcal{H}^{2n-q}$$

We conclude, using the hypercohomology spectral sequence for the Hochschild homology, that

$$E_2^{pq} = H^p(X^*, \mathcal{HH}_q) \Rightarrow HH_{q-p}(\mathscr{D}_{X^*})$$

For the de Rham complex, we use the hypercohomology spectral sequence to conclude

$$H^p(X^*, \mathcal{HH}_q) \xrightarrow{\sim} H^p(X^*, \mathcal{H}^{2n-q}) = E_2^{p,2n-q} \Rightarrow H_{dR}^{p+2n-q}(X^*)$$

By applying the Poincaré lemma, we deduce that the sheaf $\mathcal{H}^{2n-q}$ is non zero only when $q = 2n$ and hence $\mathcal{HH}_q$ is non zero only when $q = 2n$ as well. Hence both spectral sequences are degenerate and we obtain isomorphisms

$$H^p(X^*, \mathcal{HH}_{2n}) \xrightarrow{\sim} HH_{2n-p}(\mathscr{D}_{X^*}) \qquad H^p(X^*, \mathcal{H}^0) \xrightarrow{\sim} H_{dR}^p(X^*)$$

and hence we have

$$HH_{2n-p}(\mathscr{D}_{X^*}) \xrightarrow{\sim} H_{dR}^p(X^*)$$

This proves (2).

To prove (1), we proceed as follows: We recall the bicomplex $CC(\mathscr{D}(U))$ that defines cyclic homology from Definition 1.1 for each open set U in X^*. The odd numbered columns of $CC(\mathscr{D}(U))$, i.e. those with "b'-differentials" are chain homotopic to 0. The even numbered columns of $CC(\mathscr{D}(U))$, i.e., those with "b-differentials" compute the Hochschild homology of $\mathscr{D}(U)$.

From [6] and [7] we know that the Hochschild complex of the sheaf of differential operators $\mathscr{D}_{X^*}$ is quasi-isomorphic to $\mathbb{C}[2n]$ (the constant sheaf $\mathbb{C}$ shifted $2n$-places, where $n = dim(U)$). Since the periodicity operator is defined by dropping the first two columns of $CC(\mathscr{D}(U))$, i.e. one b-column (computing Hochschild homology) and one b'-column (which is chain homotopic to 0), it follows that (the total complex of) the sheafification of $CC(\mathscr{D}(U))$ is quasi-isomorphic to

$$\mathbb{C}[2n] \oplus \mathbb{C}[2n+2] \oplus ...$$

and that the periodicity operator acts by dropping the first summand $\mathbb{C}[2n]$. Hence, we have the isomorphism of hypercohomologies

$$HC_q(\mathscr{D}_{X^*}) \simeq H_{dR}^{2n-q}(X^*) \oplus H_{dR}^{2n-q+2}(X^*) \oplus H_{dR}^{2n-q+4}(X^*) \oplus ...$$

□

Corollary A. 3.18. *There exists a long exact sequence*

$$\cdots \to HH_q(\mathscr{D}_{X^*}) \to HC_q(\mathscr{D}_{X^*}) \xrightarrow{S} HC_{q-2}(\mathscr{D}_{X^*}) \to \ldots$$

The periodicity operator

$$S : HC_q(\mathscr{D}_{X^*}) \longrightarrow HC_{q-2}(\mathscr{D}_{X^*})$$

acts by dropping the summand $HH_q(\mathscr{D}_{X^*}) \cong H_{DR}^{2n-q}(X)$.

Proof. We have the short exact sequence of complexes of sheaves

$$0 \to \mathcal{C}_*^h(\mathscr{D}_{X^*}) \to Tot(\mathcal{B}_{**}(\mathscr{D}_{X^*})) \to Tot(\mathcal{B}_{**}(\mathscr{D}_{X^*}))[2] \to 0$$

The long exact periodicity sequence now follows from the fact that hypercohomology is a hyperderived functor. The last statement follows from the proof of Proposition A.3.17 in which we showed that the sheafification of the double complex $CC(\mathscr{D}(U))$

is quasi-isomorphic to $\mathbb{C}[2n] \oplus \mathbb{C}[2n+2] \oplus ...$ and that the periodicity operator acts by dropping the first summand $\mathbb{C}[2n]$. $\square$

4 Consani's complex and the Cyclic Cohomology of the ring of differential operators

The formalism of nearby cycles associated to an algebraic degeneration over a disc can be extended to the case of the reduced and irreducible fibre at infinity of an arithmetic variety. In 2.3, we described a complex $K^{i,j,k}$ that is used to make explicit the E_1-terms of the spectral sequence associated to the L-filtration on Steenbrink's complex that converges to the cohomology of the universal fibre $\tilde{X}^*$. We also described an endomorphism $N : K^{i,j,k} \to K^{i+2,j,k+1}$. Thereafter, in Chapter 3, we considered a resolution $\psi^{**}(\mathbb{C})$ of the nearby cycles complex (which is quasi-isomorphic to Steenbrink's complex); we identified the homology of the graded pieces of filtration induced on the former by the kernel of N^j, $j \geq 0$ with the Hochschild homology of the sheaf of differential operators on X^* and finally showed that, the endomorphism N induced on the cohomology can be identified with the periodicity operator S in cyclic homology.

Consani, in her PhD. thesis (see [18],[19]), introduced a complex $K^{i,j,k}$, again with monodromy N that is the "archimedean" analogue of the Steenbrink complex at archimedean infinity. In this new setup, the complex (or real) points of an arithmetic variety over a number field play the role of the nearby fibre over infinity. The complex points of the nearby fibre at infinity form a smooth complex manifold $X(\mathbb{C})$ of dimension, say n over $\mathbb{C}$. The manifold $X(\mathbb{C})$ may also be extended from $\mathbb{R}$, i.e. there exists a real manifold of dimension n $X(\mathbb{R})$ such that $X(\mathbb{C}) = X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. The images of N^j, $j \geq 0$ are direct summands of $K^* = \oplus_{i+j=*} K^{i,j}$, with $K^{i,j} = \oplus_k K^{i,j,k}$.

In this chapter, we show that the homology of the graded pieces of the filtration defined by them correspond naturally to Hochschild cohomology of the ring of differential operators $\mathscr{D}(X)$ on X. Thereafter, we construct a complex that computes the cyclic cohomology of $\mathscr{D}(X)$. This complex is, in fact, a deck of Hochschild complexes. Thereafter, we show that the endomorphism N on the hypercohomology corresponds to the periodicity operator S in cyclic cohomology.

We understand this "interchange" of roles between the framework of cyclic homology and that of cyclic cohomology between the two chapters as a natural consequence of the fact that that the $K^{i,j,k}$'s appearing in Consani's complex are an analogue of the graded pieces of the L-filtration on Steenbrink's complex A^*, which appears as the target of the quasi-isomorphism $\mu : \psi^*(\mathbb{C}) \longrightarrow A^*$ in Proposition 3.6. Then, the interchange between the kernel and the cokernel of N appearing on $\psi^*(\mathbb{C})$ and N appearing on A^* which was already made explicit in Corollary 3.7, will be understood as an interchange between cyclic homology and cyclic cohomology. Therefore, in section 4 of this chapter, we will construct a complex φ^{**} which is the analogue of the nearby cycles complex ψ^{**} at archimedian infinity. Correspondingly, we will assemble the graded pieces $K^{i,j,k}$ to form a complex B^{**} which replaces the Steenbrink complex A^{**} of Proposition 3.6 and also a morphism of complexes $\mu : \varphi^* \to B^*$. Both The complexes φ and B^{**} are equipped with an endomorphism N, and, as in Proposition 3.6, $\mu \circ N$ is chain homotopic to the operator $N \circ \mu$.

4.1 Consani's complex

Let X be a smooth compact manifold of dimension m over $\mathbb{C}$ or $\mathbb{R}$. If X is defined over $\mathbb{R}$, then we let $X(\mathbb{C})$ denote the complex manifold $X(\mathbb{C}) = X \otimes_{\mathbb{R}} \mathbb{C}$. Then, in either case, $dim_{\mathbb{C}}(X(\mathbb{C})) = m$ and, by abuse of notation, we will continue to refer to

$X(\mathbb{C})$ simply as X, whenever X is defined over $\mathbb{R}$. We let $\Omega^M_{\mathbb{R}}$ denote the module of real differentials on X, i.e. the direct sum $\bigoplus_{a+b=M}(\Omega^{a,b}+\Omega^{b,a})_{\mathbb{R}}$. We denote the direct sums $(\Omega^{a,b}_X+\Omega^{b,a}_X)_{\mathbb{R}}$ by $\Omega^{a,b}_{X,\mathbb{R}}$. We have the following setup (see [18, Section 4]): For all $i,j,k \in \mathbb{Z}$, we define

$$K^{i,j,k} = \begin{cases} \bigoplus\limits_{\substack{a \leq b \\ a+b=j+m, |a-b| \leq 2k-i}} \Omega^{a,b}_{X,\mathbb{R}}\left(\frac{m+j-i}{2}\right) & k \geq max\{0,i\} \\ 0 & \text{otherwise} \end{cases} \tag{4.1.1}$$

where the term $\Omega^{a,b}_{X,\mathbb{R}}(r)$ refers to real differentials of type (a,b) multiplied by a factor of $(2\pi i)^r$. We refer to the number r in $\Omega^{a,b}_{X,\mathbb{R}}(r)$ as the (Tate)twist of the real differential forms.

We will now define the maps d', d'' and N as follows. Denote by ∂ and $\overline{\partial}$ the usual partial differential operators on $\Omega^{a,b}_{X,\mathbb{R}}$. We define

$$\begin{aligned} d' &: K^{i,j,k} \longrightarrow K^{i+1,j+1,k+1} & d' &= \partial + \overline{\partial} \\ d'' &: K^{i,j,k} \longrightarrow K^{i+1,j+1,k} & d'' &= i(\partial - \overline{\partial}) \\ N &: K^{i,j,k} \longrightarrow K^{i+2,j,k+1} & N(a) &= (2\pi i)^{-1} a \end{aligned} \tag{4.1.2}$$

We remark that d'' is be considered as composed with a projection onto its range. We will also maintain

$$d = d' + d'' \tag{4.1.3}$$

For $i,j \in \mathbb{Z}$, write $K^{i,j} = \bigoplus_{k\in\mathbb{Z}} K^{i,j,k}$. We will consider the bicomplex $(K^{\cdot\cdot}, d', d'')$ and the associated total complex $K^* = \bigoplus_{i+j=*} K^{i,j}$. It is easy to check that $[d',N] = [d'',N] = 0$. Therefore, we can form bicomplexes $Ker(N)^{**}$, $Coker(N)^{**}$ and $Cone(N)^{**}$. Following the definition given in Section 2.4, the complex $Cone(N) = K^* \oplus K^*[-1]$ endowed with the differential $D(a,b) = (da, Na - db)$. Following [18], we introduce (for $q \geq 0$):

$$gr_{2p}^W H^q(\tilde{X}^*) = \frac{Ker(d : K^{q-2p,q-n} \to K^{q-2p+1,q-m+1})}{Im(d : K^{q-2p-1,q-m-1} \to K^{q-2p,q-m})}$$

$$gr_{2p}^W H^q(Y) = \frac{Ker(d : Ker(N)^{q-2p,q-m} \to Ker(N)^{q-2p+1,q-m+1})}{Im(d : Ker(N)^{q-2p-1,q-m-1} \to Ker(N)^{q-2p,q-m})}$$

$$gr_{2p}^W H_Y^q(X) = \frac{Ker(d : Coker(N)^{q-2p,q-m} \to Coker(N)^{q-2p+1,q-m+1})}{Im(d : Coker(N)^{q-2p-1,q-m-1} \to Coker(N)^{q-2p,q-m})}$$

$$gr_{2p}^W H^q(X^*) = \frac{Ker(d : Cone(N)^{q-2p,q-m} \to Cone(N)^{q-2p+1,q-m+1})}{Im(d : Cone(N)^{q-2p-1,q-m-1} \to Cone(N)^{q-2p,q-m})} \tag{4.1.4}$$

Following, [16, Section 2.3], we define $H^q(\tilde{X}^*) = \mathbb{H}^q(K^\cdot)$, $H^q(Y) = \mathbb{H}^q(Ker(N)^\cdot)$, $H_Y^q(X) = \mathbb{H}^q(Coker(N)^\cdot)$ and $H^q(X^*) = \mathbb{H}^q(Cone(N)^\cdot)$. Clearly, we have the decompositions

$$\begin{array}{ll} H^q(\tilde{X}^*) = \bigoplus_{p\in\mathbb{Z}} gr_{2p}^W H^q(\tilde{X}^*) & H^q(Y) = \bigoplus_{p\in\mathbb{Z}} gr_{2p}^W H^q(Y) \\ H_Y^q(X) = \bigoplus_{p\in\mathbb{Z}} gr_{2p}^W H_Y^q(X) & H^q(X^*) = \bigoplus_{p\in\mathbb{Z}} gr_{2p}^W H^q(X^*) \end{array} \tag{4.1.5}$$

We also recall the following exact sequences (see [16, Corollary 2.13]):

$$\begin{array}{l} q \geq 2p: \\ \qquad 0 \longrightarrow gr_{2p}^W H^q(X^*) \longrightarrow gr_{2p}^W H^q(\tilde{X}^*) \xrightarrow{N} gr_{2(p-1)}^W H^q(\tilde{X}^*) \longrightarrow 0 \\ q \leq 2(p-1): \\ \qquad 0 \longrightarrow gr_{2p}^W H^q(\tilde{X}^*) \xrightarrow{N} gr_{2(p-1)}^W H^q(\tilde{X}^*) \longrightarrow gr_{2p}^W H^q(X^*) \longrightarrow 0 \end{array} \tag{4.1.6}$$

Following Remark 2.14 of [16], we note that the short exact sequence for $q \geq 2p$, is actually split exact.

4.2 The de Rham complex and the Ring of differential operators

Given X as above, we can consider X as a smooth compact $\mathbb{C}$-manifold X^{an} of dimension n, which we shall continue to denote by X. Let $\mathscr{D}(X)$ denote the ring of C^∞ differential operators on X. We have the following result of Wodzicki [46].

Proposition 4.1. *If X is a compact complex manifold of dimension n, the cyclic homology of the ring $\mathscr{D}(X)$ of C^∞-differential operators on X decomposes canonically as:*

$$HC_q(\mathscr{D}(X)) \simeq H_{dR}^{2n-q}(X) \bigoplus H_{dR}^{2n-q+2}(X) \bigoplus H_{dR}^{2n-q+4}(X) \bigoplus ... \tag{4.2.1}$$

The Hochschild homology of $\mathscr{D}(X)$ is given by

$$HH_q(\mathscr{D}(X), \mathscr{D}(X)) \simeq H_{dR}^{2n-q}(X) \tag{4.2.2}$$

Moreover, the isomorphisms are functorial with respect to embeddings of codimension zero.

Now, suppose that $(C^\cdot, d_1)$ and $(D^\cdot, d_2)$ are two complexes of $\mathbb{C}$-modules and let $f : C^\cdot \to D^\cdot$ be a morphism of complexes (of degree 0). We assume that each of the modules $C^\cdot$ and $D^\cdot$ is provided with a natural conjugation (denoted $x \mapsto x^c$ for x belonging to any C^n or D^m) compatible with complex conjugation, i.e. $(\alpha x)^c = \bar{\alpha} x^c$ for any $\alpha \in \mathbb{C}$ and x belonging to any C^m or D^n and let $C^\cdot_{\mathbb{R}}$ and $D^\cdot_{\mathbb{R}}$ denote the $\mathbb{R}$-submodule of $C^\cdot$ and $D^\cdot$ resp. of elements fixed under this conjugation.

Assume that the differentials d_i, $i = 1, 2$ commute with the conjugation and so do the maps f_i, $i \in \mathbb{Z}$ of the morphism $f : C^{\cdot} \to D^{\cdot}$. This means we have the equalities:

$$d_i(x^c) = d_i(x)^c \qquad f_i(x^c) = f_i(x)^c \tag{4.2.3}$$

Then, we have two new complexes $(C^{\cdot}_{\mathbb{R}}, d_1)$ and $(D^{\cdot}_{\mathbb{R}}, d_2)$ and an induced morphism connecting them, which we also denote by $f : (C^{\cdot}_{\mathbb{R}}, d_1) \to (D^{\cdot}_{\mathbb{R}}, d_2)$. Finally, we set $H^*(C^{\cdot}_{\mathbb{R}}) = H^*(C^{\cdot}_{\mathbb{R}}, d_1)$ and similarly for $(D^{\cdot}_{\mathbb{R}}, d_2)$.

Lemma 4.2. *(1) If $f_* : H^*(C^{\cdot}) \to H^*(D^{\cdot})$ is a monomorphism, then so is $f_* : H^*(C^{\cdot}_{\mathbb{R}}) \to H^*(D^{\cdot}_{\mathbb{R}})$.*

(2) If each $f_* : H^*(C^{\cdot}) \to H^*(D^{\cdot})$ is an epimorphism, then so is $f_* : H^*(C^{\cdot}_{\mathbb{R}}) \to H^*(D^{\cdot}_{\mathbb{R}})$.

Proof. (1) Let $z_1 \in C^n_{\mathbb{R}}$ be a cycle such that $f_n(z_1) = 0$ in $H^*(D^{\cdot}_{\mathbb{R}})$. As such, there exists $y_2 \in D^{n-1}_{\mathbb{R}}$ such that $d_2(y_2) = f_n(z_1)$. Since $f_n : H^n(C^{\cdot}) \to H^n(D^{\cdot})$ is a monomorphism, there exists $y_1 \in C^{n-1}$ such that $d_1(y_1) = z_1$. Then, $d_1((y_1 + y_1^c)/2) = z_1$ and hence z_1 is a boundary in $C^n_{\mathbb{R}}$.

(2) Choose a cycle $z_2 \in D^n_{\mathbb{R}}$. Since $f_n : H^n(C^{\cdot}) \to H^n(D^{\cdot})$ is an epimorphism, there exists a cycle $z_1 \in C^n$ and a boundary $b_2 \in D^n$ such that $f_n(z_1) + b_2 = z_2$. Hence, $f_n((z_1 + z_1^c)/2) + (b_2 + b_2^c)/2 = z_2$ and the class of $(z_1 + z_1^c)/2$ in $H^n(C^{\cdot}_{\mathbb{R}})$ maps to the class of z_2 in $H^n(D^{\cdot}_{\mathbb{R}})$.

□

Lemma 4.3. *Let $(E^{\cdot}, d_1)$ be an exact sequence of $\mathbb{C}$-modules where each module E^m, $m \in \mathbb{Z}$ is equipped with a conjugation $x \mapsto x^c$ compatible with the complex conjugation and with the differential d_1, i.e. $d_1(x^c) = (d_1(x))^c$. Then, the sequence $(E^{\cdot}_{\mathbb{R}}, d_1)$, where d_1 denotes the induced differential, is also exact.*

Proof. This fact may be checked directly as in the proof of the lemma above. Alternatively, the follows from the fact that the sequence $(E^{\cdot}, d_1)$ is exact if and only if it is quasi-isomorphic to the zero complex. Then, applying Lemma 4.2, it follows that the complex $(E^{\cdot}_{\mathbb{R}}, d_1)$ is also quasi-isomorphic to zero.

□

Lemma 4.4. *Let $(C^{\cdot}, d_1)$ be a complex of $\mathbb{C}$-modules where each module C^m, $m \in \mathbb{Z}$ is equipped with a conjugation $x \mapsto x^c$ compatible with the complex conjugation and with the differential d_1, (i.e. $d_1(x^c) = (d_1(x))^c$). Then, the homology groups $H^*(C^{\cdot})$ carry a conjugation compatible with complex conjugation. Let $H^*(C^{\cdot})_{\mathbb{R}}$ denote the submodule of $H^*(C^{\cdot})$ invariant under this conjugation. Then:*

$$H^*(C^{\cdot}_{\mathbb{R}}) \approx H^*(C^{\cdot})_{\mathbb{R}}. \tag{4.2.4}$$

Proof. Let $d_1^m : C^m \to C^{m+1}$, $(m \in \mathbb{Z})$ be the differential. If $z \in C^m$ is such that $d_1^m(z) = 0$, we define the conjugate of its class in homology by

$$(z + Im(d_1^{m-1}))^c = z^c + Im(d_1^{m-1})$$

This class is well defined since d_1^{m-1} is compatible with the conjugation $z \mapsto z^c$. Let $w \in C^m_{\mathbb{R}}$ (hence $w = w^c$) be such that $d_1^m(w) = 0$. Then define the homomorphism:

$$\phi_m : H^m(C^{\cdot}_{\mathbb{R}}) \longrightarrow H^m(C^{\cdot})_{\mathbb{R}} \qquad \phi_m(w + Im(d_1^{m-1})) = w + Im(d_1^{m-1}) \in H^m(C^{\cdot})_{\mathbb{R}} \tag{4.2.5}$$

Suppose that $\phi_m(w + Im(d_1^{m-1})) = 0$. Then $w = d_1^{m-1}(x)$ for some $x \in C^{m-1}$. But then $w = w^c = d_1^{m-1}(x^c)$ and hence $w = d_1^{m-1}((x + x^c)/2)$. However, since

$(x+x^c)/2 \in C_{\mathbb{R}}^{m-1}$ we obtain $w + Im(d_1^{m-1}) = 0$ in $H^m(C_{\mathbb{R}}^{\cdot})$. This proves that ϕ_m is one one.

We now choose $z + Im(d_1^{m-1}) \in H^m(C^{\cdot})_{\mathbb{R}}$ such that $(z + Im(d_1^{m-1}))^c = z + Im(d_1^{m-1})$. Then, there exists $x \in C^{m-1}$ such that $z - z^c = d_1^{m-1}(x)$. Consider $w = z - d_1^{m-1}(x/2) = z - (z-z^c)/2 = (z+z^c)/2$. Hence $w^c = w$ and it is clear that $\phi_m(w + Im(d_1^{m-1})) = z + Im(d_1^{m-1})$. This proves that ϕ_m is also onto and hence an isomorphism.

□

The above formalism applies to the complexes defining the cyclic homology $HC_*(\mathscr{D}(X))$ of the ring $\mathscr{D}(X)$ of C^∞-differential operators on X and the de Rham cohomology of X. We shall denote the homologies of the subcomplexes fixed under conjugation by $HC_*(\mathscr{D}(X))_{\mathbb{R}}$ and $H^*_{dR}(X)_{\mathbb{R}}$ respectively. We shall refer to $HC_*(\mathscr{D}(X))_{\mathbb{R}}$ as the *real cyclic homology* of $\mathscr{D}(X)$, while we shall refer to $H^*_{dR}(X)_{\mathbb{R}}$ as the *real de Rham cohomology*. By defining a similar conjugation on the complex defining the Hochschild homology of $\mathscr{D}(X)$, we can define the *real Hochschild homology* of $\mathscr{D}(X)$, which we will denote by $HH_*(\mathscr{D}(X))_{\mathbb{R}}$.

Note moreover, that, if M is a $\mathbb{C}$-module equipped with a complex conjugation $m \mapsto m^c$ for all $m \in M$, compatible with complex conjugation, there is a conjugation $f \mapsto f^c$ for $f \in Hom(M,\mathbb{C})$ which is

$$f^c(m) = \overline{f(m^c)} \qquad m \in M, f \in Hom(M,\mathbb{C}) \tag{4.2.6}$$

It is easy to check that

$$f^c(\alpha m) = \overline{f((\alpha m)^c)} = \overline{f(\bar{\alpha} m^c)} = \alpha \overline{f(m^c)} = \alpha f^c(m) \qquad \alpha \in \mathbb{C} \tag{4.2.7}$$

In particular, we will apply this to the modules $Hom(HC_q(\mathscr{D}(X), \mathbb{C})$.

Proposition 4.5. *If X has dimension n, we have canonical isomorphisms of $\mathbb{C}$-vector spaces:*

$$HC_q(\mathscr{D}(X))_{\mathbb{R}} \simeq H_{dR}^{2n-q}(X)_{\mathbb{R}} \bigoplus H_{dR}^{2n-q+2}(X)_{\mathbb{R}} \bigoplus H_{dR}^{2n-q+4}(X)_{\mathbb{R}} \bigoplus ... \tag{4.2.8}$$

$$HH_q(\mathscr{D}(X), \mathscr{D}(X))_{\mathbb{R}} \simeq H_{dR}^{2n-q}(X)_{\mathbb{R}} \tag{4.2.9}$$

Proof. From Proposition 4.1, we have canonical isomorphisms:

$$HH_q(\mathscr{D}(X), \mathscr{D}(X)) \simeq H_{dR}^{2n-q}(X) \tag{4.2.10}$$

The groups $HH_q(\mathscr{D}(X))$ are obtained from the Hochschild complex of the ring $\mathscr{D}(X)$ and this complex carries a natural conjugation. The complex that computes de Rham cohomology also carries a natural conjugation and the map between the Hochschild complex and the de rham complex is compatible with this conjugation. Hence from the lemmae above, and the canonical and functorial isomorphisms $HH_q(\mathscr{D}(X)) \approx H_{dR}^{2n-q}(X)$, we have an isomorphism of their invariants $HH_q(\mathscr{D}(X), \mathscr{D}(X))_{\mathbb{R}} \simeq H_{dR}^{2n-q}(X)_{\mathbb{R}}$. This proves the second isomorphism in the proposition.

Again, for cyclic homology, from Proposition 4.1, we have the canonical isomorphism

$$HC_q(\mathscr{D}(X)) \simeq H_{dR}^{2n-q}(X) \bigoplus H_{dR}^{2n-q+2}(X) \bigoplus H_{dR}^{2n-q+4}(X) \bigoplus ... \tag{4.2.11}$$

The bicomplex $CC(\mathscr{D}(X))$ that is used to compute cyclic homology also carries a natural conjugation, which is compatible with the conjugation on the de Rham complex.

Hence, we have the isomorphism

$$HC_q(\mathscr{D}(X))_{\mathbb{R}} \simeq H^{2n-q}_{dR}(X)_{\mathbb{R}} \bigoplus H^{2n-q+2}_{dR}(X)_{\mathbb{R}} \bigoplus H^{2n-q+4}_{dR}(X)_{\mathbb{R}} \bigoplus ... \tag{4.2.12}$$

$\square$

We now consider the cohomological viewpoint. Given an algebra A over a commutative ring k, recall from Section 1.1 that there exists a Kronecker product pairing between Hochschild homology and cohomology

$$< .,. >: HH^q(A) \times HH_q(A) \longrightarrow A^* \otimes_{A^e} A \longrightarrow k \tag{4.2.13}$$

where A^e is the enveloping algebra $A \otimes_k A^{op}$ of A and $A^* = Hom(A, k)$. This product can be extended to the cyclic theory and it induces a pairing (see [31, § 2.4.8])

$$< .,. >_{HC}: HC^q(A) \times HC_q(A) \longrightarrow k \tag{4.2.14}$$

This means that one obtains a natural map

$$HC^q(A) \to Hom(HC_q(A), k) \tag{4.2.15}$$

On the other hand, the Poincare duality isomorphism for the compact manifold X gives us:

$$Hom(H^{n-q}_{dR}(X), \mathbb{C}) \xrightarrow{\simeq} H^q_{dR}(X) \qquad \forall\ q \geq 0 \tag{4.2.16}$$

Proposition 4.6. *(a) There is a commutative diagram of long exact sequences*

$$
\begin{array}{ccccc}
\to HC^{q+n-2}(\mathscr{D}(X)) & \xrightarrow{S} & HC^{q+n}(\mathscr{D}(X)) & \xrightarrow{I} & HH^{q+n}(\mathscr{D}(X)) \to \\
\downarrow & & \downarrow & & \downarrow \\
\to Hom(HC_{q+n-2}(\mathscr{D}(X)),\mathbb{C}) & \xrightarrow{S} & Hom(HC_{q+n}(\mathscr{D}(X)),\mathbb{C}) & \longrightarrow & Hom(HH_{q+n}(\mathscr{D}(X)),\mathbb{C}) \to \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\to H^{q-2}_{dR}(X) \oplus H^{q-4}_{dR}(X) \oplus ... & \xrightarrow{S} & H^{q}_{dR}(X) \oplus H^{q-2}_{dR}(X) \oplus ... & \longrightarrow & H^{q}_{dR}(X) \to
\end{array}
\tag{4.2.17}
$$

(b) By conjugating and taking invariants, the vertical maps in the diagram above give us morphisms:

$$
\begin{aligned}
&HC^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \longrightarrow H^{q}_{dR}(X)_{\mathbb{R}} \oplus H^{q-2}_{dR}(X)_{\mathbb{R}} \oplus \dots \\
&HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \longrightarrow H^{q}_{dR}(X)_{\mathbb{R}}
\end{aligned}
\tag{4.2.18}
$$

Further, we have a long exact periodicity sequence

$$
\dots \to HC^{q+n-2}(\mathscr{D}(X))_{\mathbb{R}} \xrightarrow{S} HC^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \xrightarrow{I} HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \to \dots
\tag{4.2.19}
$$

Proof. Since the pairing of HC^q and HC_q extends the pairing on Hochschild homology, the natural morphisms induced by the (not necessarily perfect) pairings: $HC^{q+n}(\mathscr{D}(X)) \to Hom(HC_{q+n}(\mathscr{D}(X)),\mathbb{C})$ and $HH^{q+n}(\mathscr{D}(X)) \to Hom(HH_{q+n}(\mathscr{D}(X)),\mathbb{C})$ commute with the maps in the periodicity sequence. The isomorphisms

$$
Hom(HH_{q+n}(\mathscr{D}(X),\mathbb{C}) \xrightarrow{\simeq} Hom(H^{n-q}_{dR}(X),\mathbb{C}) \xrightarrow{\simeq} H^{q}_{dR}(X)
\tag{4.2.20}
$$

follow from Poincare duality. This proves (a). For (b), we consider a sample of a vertical column in the diagram above:

$$HH^q(\mathscr{D}(X)) \longrightarrow Hom(HH_q(\mathscr{D}(X)), \mathbb{C}) \longrightarrow H^q_{dR}(X)$$

All three homology modules $HH^{q+n}(\mathscr{D}(X))$, $Hom(HH_{q+n}(\mathscr{D}(X)), \mathbb{C})$ and $H^q_{dR}(X)$ carry a natural conjugation we consider the sequence of induced maps between the invariants:

$$HH^{q+n}(\mathscr{D}(X))^{c=id} \longrightarrow Hom(HH_{q+n}(\mathscr{D}(X)), \mathbb{C})^{c=id} \longrightarrow H^q_{dR}(X)^{c=id}$$

By applying Lemma 4.4, it follows that $HH^q(\mathscr{D}(X))^{c=id} \cong HH^q(\mathscr{D}(X))_{\mathbb{R}}$ and $H^q_{dR}(X)^{c=id} \cong H^q_{dR}(X)_{\mathbb{R}}$. Therefore, we have morphisms

$$HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \longrightarrow H^q_{dR}(X)_{\mathbb{R}} \tag{4.2.21}$$

and so also for the other vertical columns. Finally, the fact that the complex

$$\cdots \to HC^{q+n-2}(\mathscr{D}(X))_{\mathbb{R}} \xrightarrow{\ S\ } HC^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \xrightarrow{\ I\ } HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \to \dots \tag{4.2.22}$$

is exact follows from Lemma 4.3. This proves (b). □

Remark: Note that if A is a $\mathbb{C}$-algebra, the pairing

$$HC^q(A) \times HC_q(A) \longrightarrow \mathbb{C} \tag{4.2.23}$$

is perfect only in a few special cases. For instance, if A is also a finite dimensional vector space over $\mathbb{C}$, the pairing is perfect.

Lemma 4.7. *For any given $t \in \mathbb{Z}$ and $k \geq 0$ and $j \in \mathbb{Z}$, the projection $p^t_{j,k} : \Omega^{j+n/2}_{\mathbb{R}} \to K^{j-t',j,k}$ induces a map of complexes (where $t' = 2t - n/2$),*

$$p^t_{*,k} : (\Omega^{*+n/2}_{\mathbb{R}}, -d^c) \to (K^{*-t',*,k}, d'') \tag{4.2.24}$$

Proof. We note that if $j - t' \geq k$, then the target of $p^t_{j+1,k}$ is always zero and there is nothing to prove. Therefore, we may suppose that $j - t' < k$. In order to check that the projections $p^t_{j,k}$ induce a map of complexes, it suffices to show that, for $x \neq 0$, $x \in \Omega^{j+n/2}_{\mathbb{R}}$, if $p^t_{j,k}(x) = 0$, then $p^t_{j+1,k}(d^c(x)) = 0$. For sake of convenience, we can assume that x lies in some $(\Omega^{a,b} + \Omega^{b,a})_{\mathbb{R}}$ where $a + b = j + n/2$. Then, if $p^t_{j,k}(x) = 0$, we must have $|a - b| > 2k - i$, where $i = j - t' = j - 2t + n/2$. On the other hand, if $p^t_{j+1,k}(d^c(x)) \neq 0$, it must follow that $|a - b| - 1 \leq 2k - i - 1$, which is a contradiction. $\square$

For each fixed value of the Tate twist $t \in \mathbb{Z}$, we define the following bicomplex (K^{**}_t, d', d'') as

$$K^{p,q}_t = \mathbb{R}u^{-q-1} \otimes K^{p+q,p+q+t',q} \qquad p \leq 0,\ q \geq 0 \tag{4.2.25}$$

where $t' = 2t - n/2$. We introduce the differentials:

$$\begin{aligned} d' : K^{p,q}_t \to K^{p,q+1}_t \qquad & d'(u_{[q+1]} \otimes (2\pi i)^t \omega) = u_{[q+2]} \otimes (2\pi i)^t (\partial + \bar{\partial})\omega \\ d'' : K^{p,q}_t \to K^{p+1,q}_t \qquad & d''(u_{[q+1]} \otimes (2\pi i)^t \omega) = u_{[q+1]} \otimes (2\pi i)^t i(\partial - \bar{\partial})\omega \end{aligned} \tag{4.2.26}$$

The monodromy N acts on (K^{**}_t, d', d'') as follows:

$$N : K^{pq}_t \longrightarrow K^{p+1,q+1}_{t-1} \qquad N(u_{[q+1]} \otimes (2\pi i)^t \omega) = u_{[q+2]} \otimes (2\pi i)^{t-1} \omega \tag{4.2.27}$$

$$
\begin{array}{cccccc|c|c}
 & & & & & & \vdots & \vdots \\
 & \uparrow & & \uparrow\, {\scriptstyle d'} & & \uparrow & & \\
\to & u^{-3}K^{0,t',2} & \xrightarrow{d''} & u^{-3}K^{1,t'+1,2} & \xrightarrow{d''} & u^{-3}K^{2,t'+2,2} & & = Im(N^2) \\
\hline
 & \uparrow & & \uparrow\, {\scriptstyle d'} & & \uparrow & = Im(N) & \\
\to & u^{-2}K^{-1,t'-1,1} & \xrightarrow{d''} & u^{-2}K^{0,t',1} & \xrightarrow{d''} & u^{-2}K^{1,t'+1,1} & & \\
\hline
 & \uparrow & & \uparrow\, {\scriptstyle d'} & & \uparrow & & \\
\to & u^{-1}K^{-2,t'-2,0} & \xrightarrow{d''} & u^{-1}K^{-1,t'-1,0} & \xrightarrow{d''} & u^{-1}K^{0,t',0} & &
\end{array}
\tag{4.2.29}
$$

Figure 4: **The bicomplex** K_t^{**}

In fact, we shall also consider powers of N; for any $l \geq 0$, we have maps

$$
\begin{aligned}
N^{l+1} : K^{**}_{t+l+1} &\to K_t^{*+l+1,*+l+1} \\
N^{l} : K^{**}_{t+l} &\to K_t^{*+l,*+l}
\end{aligned}
\tag{4.2.28}
$$

By convention, it is understood that N^0 refers to the identity map.

For each fixed $l \geq 0$, we consider the complexes $Im(N^{l+1})^*$ and $Im(N^l)^*$ both of which are subcomplexes of K_t^{**} as well as the corresponding graded piece, i.e. the quotient $(Im(N^l)/Im(N^{l+1}))^*$.

Proposition 4.8. *For each fixed $l \in \mathbb{Z}_{\geq 0}$ and for each fixed value $t \in \mathbb{Z}$ of the Tate twist, there are natural maps connecting the real Hochschild cohomology of $\mathscr{D}(X)$ to the homology of the quotient complexes $(Im(N^l)/Im(N^{l+1}))^*$,*

$$
HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \longrightarrow H^{q-2t}((Im(N^l)/Im(N^{l+1}))^*) \tag{4.2.30}
$$

Proof. We let $t' = 2t - n/2$. Then, by definition, we have the map N:

$$
N : K^{pq}_{t+1} = \mathbb{R}u^{-q-1} \otimes K^{p+q,p+q+t'+2,q} \to \mathbb{R}u^{-q-2} \otimes K^{p+q+2,p+q+t'+2,q+1} = K_t^{p+1,q+1} \tag{4.2.31}
$$

$$N(u_{[q+1]} \otimes (2\pi i)^t \omega) = u_{[q+2]} \otimes (2\pi i)^{t-1} \omega \tag{4.2.32}$$

If $K^{p+q+2,p+q+t'+2,q+1} \neq 0$, then $p+q+2 \leq q+1$ and hence $p+q < q$. Thus, $K^{p+q,p+q+(t'+2),q} \neq 0$ is a preimage for $K^{p+q+2,p+q+t'+2,q+1}$, unless $q+1=0$. It follows that the cokernel of N consists solely of the terms $K_t^{p,0} = \mathbb{R}u^{-1} \otimes K^{p,p+t',0}$ whereas the image of N consists of the terms K_t^{pq} with $q > 0$.

From these considerations, we deduce that the graded pieces $Im(N^l)/Im(N^{l+1})$, $(l \geq 0)$ have the following description

$$(Im(N^l)/Im(N^{l+1}))^* = K_t^{*,l} \tag{4.2.33}$$

By applying Lemma 4.7, we obtain morphisms

$$p^t_{*-n/2,l} : (\Omega^*_{\mathbb{R}}, -d^c) \to (K^{*-2t,*-n/2,l}, d'') = (K_t^{*-l-2t,l}, d'') \tag{4.2.34}$$

By composing with the maps of Proposition 4.6, we get a morphism of homologies

$$\begin{aligned} HH^{q+n}(\mathscr{D}(X))_{\mathbb{R}} \longrightarrow H^q_{dR}(X)_{\mathbb{R}} \xrightarrow{p^t_{q-n/2,l}} \\ H^{q-2t}((Im(N^l)/Im(N^{l+1}))^*) \end{aligned} \tag{4.2.35}$$

□

4.3 The Connes periodicity operator and the monodromy operator

We recall the definition of the bicomplexes $(K_t^{\cdot\cdot}, d', d'')$, $t \in \mathbb{Z}$, which were introduced in Section 4.2 in (4.2.25) as a modified version of Consani's complex $(K^{**}, d' = \partial + \bar{\partial}, d'' = i(\partial - \bar{\partial}))$. For any $t \in \mathbb{Z}$ and $n = dim(X)$, we set

$$K_t^{p,q} = \mathbb{R}u^{-q-1} \otimes K^{p+q,p+q+t',q} \qquad p \leq 0,\, q \geq 0 \tag{4.3.1}$$

where $t' = 2t - n$. The differentials on K_t^{**} are described by the maps:

$$\begin{aligned} d' : K_t^{p,q} &\to K_t^{p,q+1} & d'(u_{[q+1]} \otimes \omega) &= u_{[q+2]} \otimes (2\pi i)^t (\partial + \bar{\partial})\omega \\ d'' : K_t^{p,q} &\to K_t^{p+1,q} & d''(u_{[q+1]} \otimes \omega) &= u_{[q+1]} \otimes (2\pi i)^t i(\partial - \bar{\partial})\omega \end{aligned} \tag{4.3.2}$$

The operator N on K_t^{pq} acts as follows:

$$N : K_t^{p,q} \to K_{t-1}^{p+1,q+1} \qquad N(u_{[q+1]} \otimes (2\pi i)^t \omega) = u_{[q+2]} \otimes (2\pi i)^{t-1} \omega \tag{4.3.3}$$

In Proposition 4.8, we have studied the relation between the homology of the graded pieces $Im(N^l)/Im(N^{l+1}))$ and the Hochschild cohomology of the ring $\mathscr{D}(X)$. This leads to the following two conclusions:

(1) The filtration on the complex K_t^{**} by rows coincides with the filtration by the images $Im(N^l)$, $l \geq 0$.

(2) The spectral sequence associated to either of these filtrations converges to the total homology of K_t^{**}. The E_1-terms of this spectral sequence are homologies of the complexes $Im(N^l)/Im(N^{l+1})$. It follows from Proposition 4.8 that there exist natural morphisms from the Hochschild cohomology of $\mathscr{D}(X)$ to the homology of the graded

pieces.

We will now define a complex which carries a filtration with the property that the cyclic homology of $\mathscr{D}(X)$ maps to the homology of its associated graded pieces. We introduce the following complex ; for $p, q, r \in \mathbb{Z}$ let $\mathcal{K}^{***}$ be

$$\mathcal{K}^{p,q,r} = \mathbb{R}u_{[q+r+1]} \otimes K^{p+q+r,p+q-n-r,q+r} \qquad p \leq 0,\ q+r \geq 0 \tag{4.3.4}$$

with differentials (here the twist $t = -r$)

$$\begin{aligned} d' : \mathcal{K}^{p,q,r} \to \mathcal{K}^{p,q+1,r} \qquad & d'(u_{[q+r+1]}) \otimes (2\pi i)^t\omega) = u_{[q+r+2]} \otimes (2\pi i)^t(\partial + \bar{\partial})\omega \\ d'' : \mathcal{K}^{p,q,r} \to \mathcal{K}^{p+1,q,r} \qquad & d''(u_{[q+r+1]} \otimes (2\pi i)^t\omega) = u_{[q+r+1]} \otimes (2\pi i)^t i(\partial - \bar{\partial})\omega \end{aligned} \tag{4.3.5}$$

The third differential $\mathcal{K}^{p,q,r} \to \mathcal{K}^{p,q,r+1}$ is taken to be zero. The operator N is described on this complex as

$$N : \mathcal{K}^{p,q,r} \to \mathcal{K}^{p+1,q,r+1} \qquad N(u_{[q+r+1]} \otimes (2\pi i)^t\omega) = u_{[q+r+2]} \otimes (2\pi i)^{t-1}\omega \tag{4.3.6}$$

Proposition 4.9. *For each fixed $q \in \mathbb{Z}$, there are natural maps from the cyclic cohomology of the ring $\mathscr{D}(X)$ to the homology of the bicomplex $\mathcal{K}^{*q*}$*

$$HC^{j+n}(\mathscr{D}(X)) \to H^j(\mathcal{K}^{*q*}) \qquad j \geq 0 \tag{4.3.7}$$

which are induced by the projections of Lemma 4.7.

Proof. For a fixed $q \in \mathbb{Z}$, we know from Lemma 4.7, that the complex $\mathcal{K}^{*q*}$ is a quotient of the following complex (where $d^c = i(\bar{\partial} - \partial)$)

$$
\begin{tikzcd}
 & & & \vdots & \vdots & \\
 & & u^{-3}\Omega^0_{\mathbb{R}} \arrow[r, "-d^c"] & u^{-3}\Omega^1_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u] & u^{-3}\Omega^2_{\mathbb{R}} \arrow[r] \arrow[u] & \ldots \\
 & u^{-2}\Omega^0_{\mathbb{R}} \arrow[r, "-d^c"] & u^{-2}\Omega^1_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u, "0"] & u^{-2}\Omega^2_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u, "0"] & u^{-2}\Omega^3_{\mathbb{R}} \arrow[r] \arrow[u, "0"] & \ldots \\
u^{-1}\Omega^0_{\mathbb{R}} \arrow[r, "-d^c"] & u^{-1}\Omega^1_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u, "0"] & u^{-1}\Omega^2_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u, "0"] & u^{-1}\Omega^3_{\mathbb{R}} \arrow[r, "-d^c"] \arrow[u, "0"] & u^{-1}\Omega^4_{\mathbb{R}} \arrow[r] \arrow[u, "0"] & \ldots
\end{tikzcd} \tag{4.3.8}
$$

The bicomplex $\Omega^{**}_{\mathbb{R}}[u^{-1}]$

where the projections p^{-1}_{**} of Lemma 4.7 map the lowest row of $\Omega^{**}_{\mathbb{R}}[u^{-1}]$ to the terms $\mathcal{K}^{*,q,-q}$ (note that we impose $r + q \geq 0$). For sake of convenience, we shall denote this bicomplex by $\Omega^{**}_{\mathbb{R}}[u^{-1}]$. The j-th total homology of the bicomplex $\Omega^{**}_{\mathbb{R}}[u^{-1}]$ is therefore given by the direct sum

$$H^j(\Omega^{**}_{\mathbb{R}}[u^{-1}]) = H^j_{dR}(X)_{\mathbb{R}} \oplus H^{j-2}_{dR}(X)_{\mathbb{R}} \oplus \ldots \tag{4.3.9}$$

Hence we have the following composition of maps: we have maps

$$HC^{j+n}(\mathscr{D}(X)) \longrightarrow H^j_{dR}(X)_{\mathbb{R}} \oplus H^{j-2}_{dR}(X)_{\mathbb{R}} \oplus \ldots \to H^j(\mathcal{K}^{*q*}) \tag{4.3.10}$$

The first map in (4.3.10) comes from Proposition 4.6 while the second follows from the fact that $\mathcal{K}^{*q*}$ is a quotient of $\Omega^{**}_{\mathbb{R}}[u^{-1}]$.

□

In what follows, we shall continue to use $\Omega^{**}_{\mathbb{R}}[u^{-1}]$ to denote the bicomplex of (4.3.8).

Proposition 4.10. *For any $q \in \mathbb{Z}$, the periodicity operator S appearing on the cyclic cohomology of $\mathscr{D}(X)$ (see Proposition 4.5) can be identified with the monodromy operator N appearing on the complex $\mathcal{K}^{*q*}$, in other words, for any $j \in \mathbb{Z}$ and $n = dim(X)$, we have a commutative diagram*

$$\begin{array}{ccccc} HC^{j+n-2}(\mathscr{D}(X)) & \xrightarrow{S} & HC^{j+n}(\mathscr{D}(X)) & \longrightarrow & HH^{j+n}(\mathscr{D}(X)) \\ \downarrow & & \downarrow & & \downarrow \\ H^{j-2}(\mathcal{K}^{*q*}) & \xrightarrow{N} & H^{j}(\mathcal{K}^{*q*}) & \longrightarrow & H^{j}(Coker(N)^{*}) \end{array} \tag{4.3.11}$$

Proof. Consider the bicomplex $\Omega_{\mathbb{R}}^{**}[u^{-1}]$ of (4.3.8). For sake of convenience, we introduce a map $\tilde{N}$ on $\Omega_{\mathbb{R}}^{**}[u^{-1}]$ that acts as

$$\tilde{N}(u_{[q+1]} \otimes \omega) = u_{[q+2]} \otimes \omega \tag{4.3.12}$$

Notice that $\tilde{N}$ lies above the monodromy operator N on $\mathcal{K}^{*q*}$, since the latter is a quotient of $\Omega_{\mathbb{R}}^{**}[u^{-1}]$. Furthermore, we note that $\tilde{N}$ is injective on $\Omega_{\mathbb{R}}^{**}[u^{-1}]$.

We therefore have a short exact sequence of bicomplexes

$$0 \longrightarrow \Omega_{\mathbb{R}}^{**}[u^{-1}][-1,-1] \xrightarrow{\tilde{N}} \Omega_{\mathbb{R}}^{**}[u^{-1}] \longrightarrow Coker(\widetilde{N})^{*} \longrightarrow 0 \tag{4.3.13}$$

which gives rise to a long exact sequence of associated homologies

$$\begin{array}{l} \dots \longrightarrow H^{l-2}(\Omega_{\mathbb{R}}^{**}[u^{-1}]) \xrightarrow{\tilde{N}} H^{l}(\Omega_{\mathbb{R}}^{**}[u^{-1}]) \longrightarrow \\ \qquad\qquad\qquad\qquad\qquad H^{j}(Coker(N)^{*}) \longrightarrow \dots \end{array} \tag{4.3.14}$$

The isomorphisms in Proposition 4.5 are canonical and hence the periodicity operator on $HC^{j+n}(\mathscr{D}(X))$ acts by dropping the top summand as follows:

$$\begin{array}{ccc} HC^{j+n}(\mathscr{D}(X)) & \longrightarrow & H^j_{dR}(X)\oplus H^{j-2}_{dR}(X)\oplus H^{j-4}_{dR}(X)\oplus ... \\ {\scriptstyle S}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ HC^{j+n-2}(\mathscr{D}(X)) & \longrightarrow & H^{j-2}_{dR}(X)\oplus H^{j-4}_{dR}(X)\oplus ... \end{array} \tag{4.3.15}$$

where the horizontal arrows are the morphisms of Proposition 4.7. Combining (4.3.15) with the fact that $\mathcal{K}^{*q*}$ is a quotient of $\Omega^{**}_{\mathbb{R}}[u^{-1}]$ and considering the maps $HC^{j+n}(\mathscr{D}(X)) \to H^j(\mathcal{K}^{*q*})$ described in Proposition 4.9, we have the commutative diagram

$$\begin{array}{ccccc} HC^{j+n-2}(\mathscr{D}(X)) & \xrightarrow{S} & HC^{j+n}(\mathscr{D}(X)) & \longrightarrow & HH^{j+n}(\mathscr{D}(X)) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^{j-2}(\mathcal{K}^{*q*}) & \xrightarrow{N} & H^j(\mathcal{K}^{*q*}) & \longrightarrow & H^j(Coker(N)^*) \end{array} \tag{4.3.16}$$

□

4.4 An analogue of the complex of nearby cycles at archimedean infinity

In this section, our objective is to define a complex φ^{**} endowed with an operator N that is in direct analogy to the complex of nearby cycles ψ^{**} of Section 3.1. Let X be a compact manifold defined over $\mathbb{C}$ or $\mathbb{R}$. Again, if X has total dimension n, $dim_{\mathbb{C}}(X(\mathbb{C})) = n/2$.

The idea is as follows: in the notation of Section 2.3, we recall from (2.3.13) and (2.3.16) the terms $K_S^{i,j,k}$ defined by Steenbrink; For any $i, j, k \in \mathbb{Z}$;

$$
\begin{aligned}
{}_L E_1^{-r,q+r} &= \mathbb{H}^q(Y, Gr_r^L A^{\cdot}) = \bigoplus_{k\geq 0,-r} H^{q-r-2k}(\tilde{Y}^{(r+2k+1)}, \mathbb{C}) \\
K_S^{i,j,k} &= \begin{cases} H^{i+j-2k+n/2}(\tilde{Y}^{(2k-i+1)}, \mathbb{C}) & \text{if } k \geq 0, i \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{4.4.1}
$$

We have also seen that the $K_S^{i,j,k}$ denote the homologies of the graded pieces of the L-filtration on Steenbrink's complex A^{**}. Proposition 3.6 states that there is a quasi-isomorphism $\mu : \psi^* \longrightarrow A^*$.

This suggests that the terms $K^{i,j,k}$ appearing in Consani's complex (4.1.2) may be assembled, in place of the terms $K_S^{i,j,k}$, to define a complex B^{**} that plays the role, at archimedean infinity, of Steenbrink complex A^{**}

$$
A^{pq} = \mathbb{C}u_{[p+1]} \otimes \Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y) \qquad p \geq 0 \tag{4.4.2}
$$

Similarly, we will assemble the terms $K^{i,j,k}$ to form a complex φ^{**} that plays the role, in this context, of the nearby cycles complex.

Both complexes φ^{**} and B^{**} are equipped with an operator N and we will show that there exists a morphism $\mu : \varphi^{**} \to B^{**}$ of bicomplexes which is compatible with N upto homotopy, in other words, $\mu \circ N - N \circ \mu$ is homotopic to 0.

From (2.3.13) and (2.3.16), we know that the $K_S^{i,j,k}$ are direct summands of the E_1-term of the spectral sequence associated to the Picard Lefschetz filtration (also called the L-filtration in the previous chapter) on A^{**}. This spectral sequence converges to the hypercohomology of A^{**}. Therefore, by considering appropriate direct sums of the terms $K^{i,j,k}$, we shall construct a bicomplex B^{**} that plays the role of A^{**} in the archimedean setup.

For easy comparison, we present the complexes ψ^{**} and A^{**} side by side in the following diagram. Notice that, along each row with d'-differentials, the terms on the ψ^{**} side differ from the terms on the A^{**} side only in that the differentials of weight "at most p" have been removed by considering the quotient $A^{pq} = \mathbb{C}u_{[p+1]} \otimes \Omega_X^{p+q+1}(\log Y)/W_p\Omega_X^{p+q+1}(\log Y)$. Hence, the associated graded object $Gr_*^W\psi^{**}$ of A^{**} is a direct summand of the associated graded object $Gr_*^W\psi^{**}$ of ψ^{**}.

ψ^{**}	μ	A^{**}
$\begin{tikzcd} & {} & {} \\ {} \arrow[r, "d'"] & u^1\Omega^1_X(\log Y) \arrow[u] \arrow[r, "d'"] & u^0\Omega^2_X(\log Y) \arrow[u] \\ & u^1\Omega^0_X(\log Y) \arrow[u, "d''"] \arrow[r, "d'"] & u^0\Omega^1_X(\log Y) \arrow[u, "d''"] \\ & & u^0\Omega^0_X(\log Y) \arrow[u, "d''"] \end{tikzcd}$	$\begin{tikzcd} {} \arrow[r, "d'"] & {} \\ {} \arrow[r, "-d'"] & {} \\ {} \arrow[r, "d'"] & {} \end{tikzcd}$	$\begin{tikzcd} {} & {} \\ u^{-1}\Omega^3_X(\log Y)/W_0 \arrow[u] \arrow[r, "d'"] & u^{-2}\Omega^4_X(\log Y)/W_1 \arrow[u] \\ u^{-1}\Omega^2_X(\log Y)/W_0 \arrow[u, "d''"] \arrow[r, "d'"] & u^{-2}\Omega^3_X(\log Y)/W_1 \arrow[u, "d''"] \\ u^{-1}\Omega^1_X(\log Y)/W_0 \arrow[u, "d''"] \arrow[r, "d'"] & u^{-2}\Omega^2_X(\log Y)/W_1 \arrow[u, "d''"] \end{tikzcd}$

Table 4.1

The connecting map μ between the nearby cycles complex ψ^{**} and Steenbrink's complex A^{**} is simply the composite of the differential d' with alternating signs with the projection over the terms of weight > 0. If we apply the weight filtration to both of these complexes and consider the associated graded objects, then, using the isomorphism from Poincaré lemma ($(Gr_k^W\Omega_X^*(\log Y) \xrightarrow{\sim} \Omega^*_{\tilde{Y}^k}[-k])$ as in the notation of Chapter 3) we obtain

ψ^{**}	μ	A^{**}
$\begin{array}{ccccc} & & \uparrow & & \uparrow \\ \xrightarrow{d'} & & \Omega^1_{\tilde{Y}^0} \oplus \Omega^0_{\tilde{Y}^1} & \xrightarrow{d'} & \Omega^2_{\tilde{Y}^0} \oplus \Omega^1_{\tilde{Y}^1} \oplus \Omega^0_{\tilde{Y}^2} \\ & & d''\uparrow & & d''\uparrow \\ & & \Omega^0_{\tilde{Y}^0} & \xrightarrow{d'} & \Omega^1_{\tilde{Y}^0} \oplus \Omega^0_{\tilde{Y}^1} \\ & & & & d''\uparrow \\ & & & & \Omega^0_{\tilde{Y}^0} \end{array}$	$\begin{array}{c} \xrightarrow{d'} \\ \\ \xrightarrow{-d'} \\ \\ \xrightarrow{d'} \end{array}$	$\begin{array}{ccc} \uparrow & & \uparrow \\ \Omega^2_{\tilde{Y}^1} \oplus \Omega^1_{\tilde{Y}^2} \oplus \Omega^0_{\tilde{Y}^3} & \xrightarrow{d'} & \Omega^2_{\tilde{Y}^2} \oplus \Omega^1_{\tilde{Y}^3} \oplus \Omega^0_{\tilde{Y}^4} \\ d''\uparrow & & d''\uparrow \\ \Omega^1_{\tilde{Y}^1} \oplus \Omega^0_{\tilde{Y}^2} & \xrightarrow{d'} & \Omega^1_{\tilde{Y}^2} \oplus \Omega^0_{\tilde{Y}^3} \\ d''\uparrow & & d''\uparrow \\ \Omega^0_{\tilde{Y}^1} & \xrightarrow{d'} & \Omega^0_{\tilde{Y}^2} \end{array}$

Table 4.2

Table 4.2 shows clearly how the graded object associated to A^{**} is constructed from graded object associated to ψ^{**} by the summands of top degree. We now incorporate the same idea to define an object that plays the role of ψ^{**} at archimedian infinity.

The definition of the K^{***} is as in (4.1.2). Fix any initial $i, j, k \in \mathbb{Z}$ (the natural choice would be $i = 0$, $j = -n/2$ and $k = 0$, where n is the total dimension of X). We define, then, the bicomplex

$$\varphi(i,j,k)^{-p,q} = \begin{cases} \mathbb{R}u^{[p]} \otimes \bigoplus_{l=0}^{q-p} K^{i+q-3p-2l, j+q-p, k-p} & \text{(if } q \geq p \geq 0) \\ = \mathbb{R}u^{[p]} \otimes \bigoplus_{l=0}^{q-p} \bigoplus_{\substack{|a-b| \leq 2k-i+p-q+1, \\ a+b=j+q-p+n/2, a \leq b}} (\Omega^{a,b} + \Omega^{b,a})_{\mathbb{R}} \left(\frac{n/2+j-i}{2} + p + l \right) & \\ 0 & \text{(otherwise)} \end{cases} \tag{4.4.3}$$

For sake of convenience, we will denote the bicomplex $\varphi(i,j,k)^{**}$ by φ^{**}. Then $\varphi^{-p,q}$ consists of sums of terms of the form $u^{[p]} \otimes (2\pi i)^t \omega$, where $(2\pi i)^t \omega \in K^{i+q-3p-2l, j+q-p, k-p}$, $0 \leq l \leq q-p$ (from which, by definition of $K^{i+q-3p-2l, j+q-p, k-p}$, it follows that

$t = \left(\frac{n/2+j-i}{2} + p + l\right)$). On the bicomplex φ^{**}, we introduce the two differentials:

$$\begin{aligned} d' : \varphi^{-p,q} \to \varphi^{-p+1,q} \qquad & d'(u^{[p]} \otimes (2\pi i)^t \omega) = u^{[p-1]} \otimes (2\pi i)^t (\partial + \bar{\partial})\omega \\ d'' : \varphi^{-p,q} \to \varphi^{-p,q+1} \qquad & d''(u^{[p]} \otimes (2\pi i)^t \omega) = u^{[p]} \otimes (2\pi i)^t i(\partial - \bar{\partial})\omega \end{aligned} \tag{4.4.4}$$

and the operator

$$N : \varphi^{-p,q} \to \varphi^{-p+1,q-1} \qquad N(u^{[p]} \otimes (2\pi i)^t \omega) = u^{[p-1]} \otimes (2\pi i)^{t-1} \omega \tag{4.4.5}$$

For the same fixed values i, j, k, we define the "Steenbrink Complex" B^{**} as

$$B(i,j,k)^{pq} = \begin{cases} \mathbb{R}u^{-p-1} \otimes \bigoplus_{l=0}^{q} K^{i+p+q+1-2l, j+p+q+1, k+p+1} & \text{if } p \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.4.6}$$

For sake of convenience, we will denote $B(i,j,k)^{**}$ simply by B^{**}. On the bicomplex B^{**} we introduce the differentials

$$\begin{aligned} d' : B^{pq} \to B^{p+1,q} \qquad & d'(u_{[p+1]} \otimes (2\pi i)^t \omega) = u_{[p+2]} \otimes (2\pi i)^t (\partial + \bar{\partial})\omega \\ d'' : B^{pq} \to B^{p,q+1} \qquad & d''(u_{[p+1]} \otimes (2\pi i)^t \omega) = u_{[p+1]} \otimes (2\pi i)^t i(\partial - \bar{\partial})\omega \end{aligned} \tag{4.4.7}$$

and the operator

$$N : B^{pq} \to B^{p+1,q-1} \qquad N(u_{[p+1]} \otimes (2\pi i)^t \omega) = u_{[p+2]} \otimes (2\pi i)^{t-1} \omega \tag{4.4.8}$$

To illustrate the analogy, compare the following tables with Table 4.2: (Compare Table 4.3(1) to the left side of Table 4.2 and Table 4.3(2) to the right side of Table 4.2)

$\varphi(i,j,k)^{**}$
$$\begin{tikzcd} \underset{\substack{\| \\ \varphi^{-2,2}}}{K^{i-4,j,k-2}} \arrow[r, "d'"] & K^{i-1,j+1,k-1} \oplus K^{i-3,j+1,k-1} \arrow[r, "d'"] & K^{i+2,j+2,k} \oplus K^{i,j+2,k} \oplus K^{i-2,j+2,k} \\ & \underset{\substack{\| \\ \varphi^{-1,1}}}{K^{i-2,j,k-1}} \arrow[u, "d''"] \arrow[r, "d'"] & K^{i+1,j+1,k} \oplus K^{i-1,j+1,k} \arrow[u, "d''"] \\ & & \underset{\substack{\| \\ \varphi^{0,0}}}{K^{i,j,k}} \arrow[u, "d''"] \end{tikzcd}$$
Table 4.3(1)

$B(i,j,k)^{**}$
$$\begin{tikzcd} K^{i+3,j+3,k+1} \oplus K^{i+1,j+3,k+1} \oplus K^{i-1,j+3,k+1} \arrow[r, "d'"] & K^{i+4,j+4,k+2} \oplus K^{i+2,j+4,k+2} \oplus K^{i,j+4,k+2} \\ K^{i+2,j+2,k+1} \oplus K^{i,j+2,k+1} \arrow[u, "d''"] \arrow[r, "d'"] & K^{i+3,j+3,k+2} \oplus K^{i+1,j+3,k+2} \arrow[u, "d''"] \\ \underset{\substack{\| \\ B^{0,0}}}{K^{i+1,j+1,k+1}} \arrow[u, "d''"] \arrow[r, "d'"] & \underset{\substack{\| \\ B^{1,0}}}{K^{i+2,j+2,k+2}} \arrow[u, "d''"] \end{tikzcd}$$
Table 4.3(2)

Proposition 4.11. *Consider the morphism* $\mu : \varphi^* \to B^*$ *defined as*

$$\mu(u^{[p]} \otimes (2\pi i)^t \omega) = \begin{cases} (-1)^{|\omega|} u_{[1]} \otimes (2\pi i)^t (\partial + \bar{\partial})\omega & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases} \tag{4.4.9}$$

Then, the morphisms $\mu \circ N$ *and* $N \circ \mu$ *are homotopic, by means of the homotopy* $h : \varphi^{**} \to B^{**}$ *of bidegree* $(0, -1)$ *defined by*

$$h(u^{[p]} \otimes (2\pi i)^t \omega) = \begin{cases} 0 & \text{if } p \neq 0 \\ (-1)^{|\omega|} u_{[1]} \otimes (2\pi i)^{t-1} \omega & \text{if } p = 0 \end{cases} \tag{4.4.10}$$

Proof. The following diagram describes the setup.

$$\begin{tikzcd}
 & & \varphi^{0,0} \arrow[r, "\mu"] \arrow[d, "d''"'] & B^{0,0} \arrow[r, "d'"] \arrow[d, "d''"'] & B^{1,0} \arrow[r] \arrow[d, "d''"'] & \dots \\
 & \varphi^{-1,1} \arrow[r, "d'"] \arrow[d, "d''"'] & \varphi^{0,1} \arrow[r, "\mu"] \arrow[d, "d''"'] & B^{0,1} \arrow[r, "d'"] \arrow[d, "d''"'] & B^{1,1} \arrow[r] \arrow[d, "d''"'] & \dots \\
\varphi^{-2,2} \arrow[r, "d'"] \arrow[d] & \varphi^{-1,2} \arrow[r, "d'"] \arrow[d] & \varphi^{0,2} \arrow[r, "\mu"] \arrow[d] & B^{0,2} \arrow[r, "d'"] \arrow[d] & B^{1,2} \arrow[r] \arrow[d] & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{4.4.11}$$

The maps d' and d'' describe the differentials of the bicomplexes φ^{**} and B^{**}, the map μ induces a morphism of bicomplexes. This means that, while the (d', d'') squares anti-commute, the (μ, d'') squares commute. For any $(u^{[0]} \otimes (2\pi i)^t \omega)$ in $\varphi^{0,q}$, we have

$$\begin{aligned} d''h(u^{[0]} \otimes (2\pi i)^t \omega) &= d''((-1)^{|\omega|} u_{[1]} \otimes (2\pi i)^{t-1} \omega) = (-1)^{|\omega|} u_{[1]} \otimes (2\pi i)^{t-1} d''(\omega) \\ hd''(u^{[0]} \otimes (2\pi i)^t \omega) &= h(u^{[0]} \otimes (2\pi i)^t d''(\omega)) = (-1)^{|d''(\omega)|} u_{[1]} \otimes (2\pi i)^{t-1} d''(\omega) \end{aligned} \tag{4.4.12}$$

It follows that $d'' \circ h + h \circ d'' = 0$.

Now, note that given any $\varphi^{-p,q}$, the morphism μ is zero (by definition) unless $p = 0$. Hence $N \circ \mu - \mu \circ N$ is non-trivial only on terms $\varphi^{-1,*}$ and $\varphi^{0,*}$. Given $u^{[1]} \otimes (2\pi i)^t \omega_1 \in \varphi^{-1,q}$ and $u^{[0]} \otimes (2\pi i)^{t'} \omega_0 \in \varphi^{0,q'-1}$, we get

$$\begin{aligned}
&(N \circ \mu - \mu \circ N)(u^{[0]} \otimes (2\pi i)^{t'} \omega_0 + u^{[1]} \otimes (2\pi i)^t \omega_1) \\
&= (-1)^{|\omega_0|} u_{[2]} \otimes (2\pi i)^{t'-1} (\partial + \bar{\partial}) \omega_0 - (-1)^{|\omega_1|} u_{[1]} \otimes (2\pi i)^{t-1} (\partial + \bar{\partial}) \omega_1 \\
&\qquad = (d' \circ h + h \circ d')(u^{[0]} \otimes (2\pi i)^{t'} \omega_0 + u^{[1]} \otimes (2\pi i)^t \omega_1)
\end{aligned} \tag{4.4.13}$$

Therefore the morphisms $N \circ \mu$ and $\mu \circ N$ are homotopic.

$\square$

5 The Connes-Karoubi exact sequence of K-theories

In Chapter 3, we have shown that the periodicity operator S appearing in the cyclic homology of the sheaf $\mathscr{D}_{X^*}$ of differential operators can be identified with the monodromy operator N as in theory of the nearby cycles complex. In Chapter 4, we have shown that the periodicity operator S appearing in the cyclic cohomology of the ring of differential operators $\mathscr{D}(X)$ on a compact complex manifold X corresponds to the operator N on Consani's complex. Thereafter, we developed this idea to define an analogue of the nearby cycles complex at Archimedean infinity, endowed with an operator N.

One also knows that the periodicity long exact sequence in cyclic homology (see (1.1.19)) for a locally convex algebra A can be lifted to a long exact sequence of K-theory groups of A (see Connes & Karoubi [13])

$$\begin{tikzcd}
K_{i+1}(A) \arrow[r] \arrow[d, "D_{i+1}"] & K^{top}_{i+1}(A) \arrow[r] \arrow[d, "ch_{i+1}"] & K^{rel}_{i}(A) \arrow[r] \arrow[d, "ch^{rel}_{i}"] & K_i(A) \arrow[r] \arrow[d, "D_i"] & K^{top}_{i}(A) \arrow[d, "ch_i"] \\
H_{i+1}(A,A) \arrow[r] & HC_{i+1}(A) \arrow[r] & HC_{i-1}(A) \arrow[r] & H_i(A,A) \arrow[r] & HC_i(A)
\end{tikzcd}$$

Here $K^{alg}_n(A)$ (resp. $K^{top}_n(A)$, $K^{rel}_n(A)$) denotes the algebraic (resp. topological, relative) K-theory groups of A. The map D_i is the Dennis trace while $ch^{top}_{i+1} : K^{top}_{i+1}(A) \to HC_{i+1}(A)$ and $ch^{rel}_i : K^{rel}_i(A) \to HC_{i-1}(A)$ are the corresponding Chern maps. For the construction and properties of the Chern maps, we refer to [13]. For the construction of the Dennis trace map, see [31, §8.4.3].

It is well known that, if X is a compact manifold, the ring $\mathscr{D}(X)$ of differential operators on X is a locally convex algebra. By taking hypercohomologies, when X is a locally Stein space, we have considered the cyclic homology of the sheaf of differential operators $\mathscr{D}_X$ in Chapter 3. In this chapter we will define K theory groups (algebraic, topological and relative) for the sheaf of differential operators $\mathscr{D}_X$ and prove that they fit into a long exact sequence similar to the one described above. Then, we will define the corresponding Chern maps $ch_{i+1}^{top} : K_{i+1}^{top}(\mathscr{D}_X) \to HC_{i+1}(\mathscr{D}_X)$, $ch_i^{rel} : K_i^{rel}(\mathscr{D}_X) \to HC_{i-1}(\mathscr{D}_X)$ and also a Dennis trace map $D_i : K_i^{alg}(\mathscr{D}_X) \to HH_i(\mathscr{D}_X)$ connecting the K-theory long exact sequence to the periodicity sequence. In doing so, we will use the definition of generalized sheaf cohomology for simplicial sheaves due to Brown and Gersten [5].

5.1 Preliminaries from Algebraic Topology

From Chapter 1, we recall that a simplicial set is a contravariant functor X from the simplicial category Δ to the category of sets, i.e.

$$X : \Delta^{op} \longrightarrow (\text{Sets}) \qquad [n] \mapsto X([n]) \tag{5.1.1}$$

Recall that the category Δ consists of objects $[n] = \{0, 1, 2, ..., n\}$, $n \geq 1$, a morphism $[m] \to [n]$ being an order increasing map $\{0, 1, 2, ..., m\} \to \{0, 1, 2, ..., n\}$.The sets $X([n])$ are denoted by X_n. Hence, given a morphism $\phi : [m] \to [n]$ in Δ, we have a map

$$\phi^* : X_n \longrightarrow X_m \tag{5.1.2}$$

The standard n-simplex, denoted $\mathbf{\Delta}^n$, is the representable functor

$$\mathbf{\Delta}^n := Hom_{\Delta}(\_\_, [n]) : \Delta \longrightarrow (\text{Sets}) \tag{5.1.3}$$

and hence a simplicial set whose k-simplices are the elements of the set $Hom_{\Delta}([k],[n])$ for any $k \in \mathbb{N}$. Hence, given $\phi : [m] \longrightarrow [n]$ in Δ, there is a morphism of functors

$$\phi_* : \mathbf{\Delta}^m \longrightarrow \mathbf{\Delta}^n \qquad Hom_{\Delta}([k],[m]) \longrightarrow Hom_{\Delta}([k],[n]) \qquad \text{for each } k \in \mathbb{N} \tag{5.1.4}$$

The geometric realization of $\mathbf{\Delta}^n$ is the geometric n-simplex

$$\Delta^n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0\} \tag{5.1.5}$$

The *geometric realization* of the simplicial set X, denoted $|X|$, is defined as the quotient of the disjoint union

$$\coprod_{n=1}^{\infty} X_n \times \Delta^n \tag{5.1.6}$$

by the relations

$$(x, \phi_*(y)) \approx (\phi^*(x), y) \text{ for any } x \in X_n,\ y \in \Delta^m \text{ and any } \phi : [m] \to [n] \text{ in category } \Delta \tag{5.1.7}$$

For a given simplicial set X, we denote by $\pi_i(X)$ the i-th homotopy group of its geometric realization $|X|$.

Given a category $\mathcal{C}$, we can define a simplicial set $N\mathcal{C}$, whose n-th simplex is the set of all diagrams in $\mathcal{C}$ of the form

$$X_0 \longrightarrow X_1 \longrightarrow .... \longrightarrow X_n \tag{5.1.8}$$

The i-th face (resp. i-th degeneracy) of this simplex is defined by deleting the object X_i (resp. replacing the object X_i by the identity $1 : X_i \to X_i$). We refer to $N\mathcal{C}$ as the *nerve of the category* $\mathcal{C}$. If we treat a group G as a category consisting of a single object $*$, the morphisms of which are elements of G (with composition defined by multiplication), the nerve of this category is denoted by BG. The geometric realization of BG is also called the *classifying space* of G.

Recall that a group N is said to be *perfect* if it equals its own commutator subgroup, i.e. $[N, N] = N$. Suppose that a connected topological space X is such that its fundamental group $\pi_1(X)$ is perfect. Then Quillen's +-construction attaches to the space X, a collection of 2-cells and 3-cells to form a new space X^+, such that there exists an inclusion $i : X \to X^+$ so that $\pi_1(X^+) = 1$ and

$$i_* : H_*(X, \mathbb{Z}) \longrightarrow H_*(X^+, \mathbb{Z}) \tag{5.1.9}$$

is an isomorphism. Furthermore, the space X^+ has the following universal property: if $f : X \to Y$ is a morphism of topological spaces such that $f_*(\pi_1(X)) = 1$, then the morphism f factors uniquely through X^+.

Now, suppose that X is a connected CW-complex and let N be a perfect normal subgroup of $\pi_1(X)$. Then, we can consider a covering space $\tilde{X}_N$ of X with $\pi_1(\tilde{X}_N) = N$ and apply the +-construction above to the space $\tilde{X}_N$ to define $\tilde{X}_N^+$. Finally, we let X^+ denote the pushout

$$\begin{array}{ccc} \widetilde{X}_N & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \widetilde{X}_N^+ & \longrightarrow & X^+ \end{array} \tag{5.1.10}$$

Then, it follows from van Kampen's theorem, that $\pi_1(X^+) = \pi_1(X)/N$. The space X^+ also has the universal property that a morphism $f : X \to Y$ of topological spaces such that $f_*(N) = 1$ factors through X^+.

It is important to note that the +-construction is not canonical and therefore may not always be functorial for all choices of X^+.

5.2 Definitions and Exact Sequences

If A is an algebra over $\mathbb{C}$, we define the group $GL(A)$ to be the direct limit of the general linear groups

$$GL(A) = \varinjlim GL_N(A)$$

The algebraic K theory of A (as defined by Quillen in [34]) is given by

$$K_n^{alg}(A) = \pi_n(BGL(A)^+) \qquad n \geq 0 \tag{5.2.1}$$

Here $BGL(A)$ denotes the classifying space of $GL(A)$ considered as a discrete group. Then $\pi_1(BGL(A)) = GL(A)$. We recall here that the normal subgroup $E(A)$ of $GL(A)$, generated by elementary matrices, is perfect and that

$$[GL(A), GL(A)] = [E(A), E(A)] = E(A) \tag{5.2.2}$$

Then $BGL(A)^+$ is the + construction of Quillen with respect to $E(A)$, which is a perfect normal subgroup of $GL(A) = \pi_1(BGL(A))$. For our purposes, it is important to ensure that the construction $BGL(A)^+$ is functorial. We will therefore assume that $BGL(A)^+$ is defined as the pushout

$$\begin{array}{ccc} BE(\mathbb{Z}) & \longrightarrow & BE(\mathbb{Z})^+ \\ \downarrow & & \downarrow \\ BGL(A) & \longrightarrow & BGL(A)^+ \end{array} \tag{5.2.3}$$

This setup uses the canonical morphism $\mathbb{Z} \to A$ for all (unital) rings. It also follows directly from this construction that the +-construction commutes with the direct limit.

Furthermore, suppose that A has a topological structure, for instance, A is a Banach algebra (or more generally, a locally convex algebra). Then, one can associate to A a simplicial ring A_* such that $A_n = C^\infty(\Delta^n)\hat{\otimes}A$. Here $C^\infty(\Delta^n)$ refers to the set of C^∞-functions on geometric n-simplex Δ^n and $\hat{\otimes}$ is the completed tensor product (see Section 1.2 for details). This leads to a simplicial group $GL(A_*)$ and an inclusion $GL(A) \cong GL(A_0) \hookrightarrow GL(A_*)$. The continuous inclusion of groups of CW type gives a sequence of homotopy fibrations (see [30, §6.12])

$$GL(A) \to GL(A_*) \to GL(A_*)/GL(A) \xrightarrow{\theta} BGL(A) \to BGL(A_*) \tag{5.2.4}$$

Here θ is defined on the simplices σ of dimension n by the formula

$$\theta(\sigma) = (\alpha_1, ..., \alpha_n) \text{ where } \alpha_i = \tilde{\sigma}(i-1)\tilde{\sigma}(i)^{-1}, 0 \leq i \leq n \tag{5.2.5}$$

where $\tilde{\sigma}$ is the class of an element $\sigma \in GL(A_n)/GL(A)$. By applying the + construction of Quillen to the last three spaces above, we have a homotopy fibration

$$(GL(A_*)/GL(A))^+ \to BGL(A)^+ \to BGL(A_*)^+ = BGL(A_*) \tag{5.2.6}$$

The last equality in the diagram above follows from the fact that $\pi_n(BGL(A_*)) = \pi_{n-1}(GL(A_*))$. By definition,

$$K_n^{top}(A) = \pi_{n-1}(GL(A_*)) = \pi_n(BGL(A_*)) \tag{5.2.7}$$

We define the relative K-theory $K_n^{rel}(A)$ to be

$$K_n^{rel}(A) := \pi_n((GL(A_*)/GL(A))^+) \tag{5.2.8}$$

We have the following long exact sequence of K theories/homotopy groups corresponding to the fibration (5.2.6).

$$\begin{array}{ccccccc} \longrightarrow & \pi_n((GL(A_*)/GL(A))^+) & \longrightarrow & \pi_n(BGL(A)^+) & \longrightarrow & \pi_n(BGL(A_*)^+) & \longrightarrow \\ & || & & || & & || & \\ & K_n^{rel}(A) & & K_n^{alg}(A) & & K_n^{top}(A) & \end{array} \tag{5.2.9}$$

In [13, §3.7] it is shown that one can define Chern characters $ch_{i+1}^{top} : K_{i+1}^{top}(A) \to HC_{i+1}(A)$ and $ch_i^{rel} : K_i^{rel}(A) \to HC_{i-1}(A)$ such that, along with the Dennis trace $D_i : K_n(A) \to H_i(A, A)$, the long exact sequence of homotopy groups lifts the periodicity long exact sequence for the algebra A

$$\begin{array}{ccccccccc} K_{i+1}(A) & \longrightarrow & K_{i+1}^{top}(A) & \longrightarrow & K_i^{rel}(A) & \longrightarrow & K_i(A) & \longrightarrow & K_i^{top}(A) \\ \downarrow{\scriptstyle D_{i+1}} & & \downarrow{\scriptstyle ch_{i+1}} & & \downarrow{\scriptstyle ch_i^{rel}} & & \downarrow{\scriptstyle D_i} & & \downarrow{\scriptstyle ch_i} \\ H_{i+1}(A, A) & \longrightarrow & HC_{i+1}(A) & \longrightarrow & HC_{i-1}(A) & \longrightarrow & H_i(A, A) & \longrightarrow & HC_i(A) \end{array} \tag{5.2.10}$$

Our objective is to carry out the same construction in sheaf hypercohomology, where we work with the sheaf of differential operators $\mathscr{D}_X$ on X and finally, for the structure sheaf of an algebraizable Noetherian formal scheme. In the next section, we review the definition of generalized sheaf cohomology due to Brown and Gersten (see [5]). We use this theory to define the K-theories of the sheaf of differential operators and construct the analogous long exact sequence involving algebraic, topological and relative K-theory. Finally, we define Chern maps from the terms in this long exact sequence to the periodicity sequence for the cyclic homology of $\mathscr{D}_X$.

5.3 Simplicial Sheaves and Generalized Sheaf Cohomology

Let X be a topological space. A simplicial sheaf on X is a sheaf of simplicial sets on X; in other words, it is a simplicial object in the category of sheaves on X. If K is a simplicial sheaf, we denote by $\Gamma(U, K)$ the simplicial set of sections of K over an open set U in X.

A map of $f : F \to F'$ of simplicial sheaves on X induces a morphism on stalks, which are simplicial sets. Hence for each $x \in X$, we have a morphism

$$f_x : F_x \longrightarrow F'_{f(x)}$$

of simplicial sets. The map f is said to be a *weak equivalence* if the morphism on stalks induces a weak equivalence of simplicial sets, in other words, for each $x \in X$, the maps

$$\pi_0(f_x) : \pi_0(F_x) \longrightarrow \pi_0(F'_{f(x)}) \qquad \pi_i(f_x) : \pi_i(F_x) \longrightarrow \pi_i(F'_{f(x)}) \text{ for } i > 0$$

are isomorphisms of groups for $i > 0$ and isomorphisms of sets for $i = 0$. Here the homotopy groups of a simplicial set refer to the homotopy groups of its geometric realization.

A map $p : E_1 \to E_2$ of simplicial sets is said to be a *Kan fibration* if it satisfies the following property: Given a collection of $n+1$ n-simplices $x_0, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}$ of E_1 which satisfy the compatibility condition $d_i x_j = d_{j-1} x_i$, $i < j$, $i \neq k$, $j \neq k$ (the d_i being the face maps of the respective simplicial sets) and for every $n+1$ simplex y

of E_2 such that $d_i y = p(x_i)$, there exists an $n+1$-simplex x of E_1 such that $d_i x = x_i$ for each i and $p(x) = y$. Kan fibrations admit the following alternative description : Given a commutative square

$$\begin{tikzcd} \mathbf{\Delta}^{n,k} \arrow[r] \arrow[d, "i"] & E_1 \arrow[d, "p"] \\ \mathbf{\Delta}^n \arrow[r, "g"] & E_2 \end{tikzcd} \tag{5.3.1}$$

in which $\mathbf{\Delta}^n$ is the standard n-simplex and $\mathbf{\Delta}^{n,k}$ is the union of all faces but the k-th face of $\mathbf{\Delta}^n$, there exists a lift $\tilde{g} : \mathbf{\Delta}^n \to E_1$ that makes the diagram commutative. Suppose that $p : E_1 \to E_2$ is a Kan fibration and that E_2 has only one 0-simplex $*$. Then $F = p^{-1}(*)$ is the fibre of the map p and we have a long exact sequence

$$\to \pi_q(F) \to \pi_q(E_1) \to \pi_q(E_2) \to \pi_{q-1}(F) \to ... \tag{5.3.2}$$

A map $p : E \to B$ of simplicial sheavs is said to be a *global fibration* if for open sets $U \subset V$ the map

$$\Gamma(V, E) \xrightarrow{(\Gamma(V,p),res)} \Gamma(V, B) \times_{\Gamma(U,B)} \Gamma(U, E) \tag{5.3.3}$$

of simplicial sets is a Kan fibration. If we take $U = \phi$, it means that $\Gamma(V, E) \to \Gamma(V, B)$ is a Kan fibration for any open set V. A monomorphism of simplicial sheaves is said to be a *cofibration*. Together with the notions of weak equiavalence, cofibration and global fibration defined above, the category $S(X)$ of simplicial sheaves on X is a model category in the sense of Quillen [35].

The simplicial sheaf $*$ defined by $\Gamma(U, *) = *$ is a final object in the above category. Accordingly, an object K of $S(X)$ is said to be *flasque* if the unique morphism $K \to *$ is a global fibration. By the properties of a model category, given any object L of

$S(X)$, the map $L \to *$ factors as

$$L \xrightarrow{i} R(L) \xrightarrow{p} * \tag{5.3.4}$$

where i is a trivial cofibration (i.e. a cofibration as well as a weak equivalence) and p is a global fibration. The sheaf $R(L)$ along with the morphism $i : L \to R(L)$ is said to be a *flasque resolution* of L.

Remark 5.1. *The association $L \mapsto R(L)$ defines a functor on $S(X)$, known as the fibrant replacement functor. This follows from the fact that the factorization of the map $L \to *$ into the composition $L \to R(L) \to *$ is functorial. We also note that the existence of functorial factorizations is not part of the original definition of a model category by Quillen, but is now commonly assumed.*

If K is a simplicial sheaf on X and $i : K \to R(K)$ is a flasque resolution of K, we define $R\Gamma(X, K)$ to be $\Gamma(X, R(K))$. The generalized sheaf cohomology of K, as defined by Brown and Gersten (see [5, Section 2]) is

$$H^q(X, K) = \pi_{-q}(R\Gamma(X, K)) = \pi_{-q}(\Gamma(X, R(K))) \tag{5.3.5}$$

If $Y \subset X$ is a closed subset, or, more generally, if Φ is a family of supports, we can similarly define $R\Gamma_Y(X, K)$ and $R\Gamma_\Phi(X, K)$. The generalized sheaf cohomology with supports is now defined as

$$H^q_\Phi(X, K) = \pi_{-q}(R\Gamma_\Phi(X, K)) \tag{5.3.6}$$

Further, given a simplicial sheaf K, we define the q-th homotopy sheaf $\pi_q K$ to be the sheaf associated to the presheaf $U \mapsto \pi_q(\Gamma(U, K))$. We conclude this section with the Brown spectral sequence.

Proposition 5.2. *(Brown Spectral Sequence) If X is a scheme and K is a simplicial sheaf on X, we have a spectral sequence*

$$E_1^{p,q} = H^q(X, \pi_{-p}(K)) \Rightarrow H^{p+q}(X, K) \tag{5.3.7}$$

More generally, if Φ is a family of supports on X, we have a spectral sequence

$$E_1^{p,q} = H^q_\Phi(X, \pi_{-p}(K)) \Rightarrow H_\Phi(X, K) \tag{5.3.8}$$

Proof. See [4] and [5]. □

5.4 The exact sequence of K-theories for the sheaf $\mathscr{D}_X$

Let X be a complex manifold of dimension n. For any open subset U of X, let $\mathscr{D}(U)$ denote the ring of holomorphic differential operators on U. If $(z_1, ..., z_n)$ is a system of holomorphic local coordinates at a point p in U, any element of $\mathscr{D}(U)$ can be expressed locally as a finite sum

$$\sum_{I=\{i_1<i_2<...<i_k\}\subseteq\{1,2,...,n\}} f_I \frac{\partial}{\partial z_{i_1}} \frac{\partial}{\partial z_{i_2}} \cdots \frac{\partial}{\partial z_{i_k}} \tag{5.4.1}$$

where each f_I is a holomorphic function on U. Furthermore, if U is Stein(or, more generally, holomorphically convex), it is well known (see [24], for instance) that the ring of holomorphic functions on U is a Fréchet algebra. Therefore, if U is Stein, the ring of differential operators $\mathscr{D}(U)$ on U is a locally convex algebra and we can define on it the projective topological tensor product $\otimes_\pi$.

Hence, applying the results (5.2.9) and (5.2.10) to the locally convex algebra $\mathscr{D}(U)$; there are well defined maps from the algebraic, topological and relative K-theories of $\mathscr{D}(U)$ into its Hochschild and cyclic homologies that are compatible with the periodicity sequence.

In this section, we shall first obtain an analogous Connes-Karoubi long exact sequence for the sheaf of differential operators $\mathscr{D}_X$ on the manifold X. The proof of the existence and properties of this sequence will involve the notion of generalized sheaf cohomology described in Section 5.3. Finally, using the fact that the manifold X has a basis consisting of Stein open sets, we will obtain Chern maps and a Dennis trace map that are compatible with the periodicity sequence in cyclic homology as in (5.2.10).

Definition 5.3. *Choose an open set U contained in X and consider the ring $\mathscr{D}(U)$. Then $\mathscr{D}(U)$ is a topological ring and we can consider the simplicial ring $\mathscr{D}(U)_*$ defined as:*

$$\mathscr{D}(U)_n = C^\infty(\Delta^n)\hat{\otimes}\mathscr{D}(U) \tag{5.4.2}$$

Δ^n being the standard n-simplex. Denote by $\mathbf{BGL}$ *(resp.* $\mathbf{BGL}^{top}$*) the sheaf associated to the presheaf that takes the open set U to $BGL(\mathscr{D}(U))$ (resp. $BGL(\mathscr{D}(U)_*)$).*

The inclusion $GL(\mathscr{D}(U)) = GL(\mathscr{D}(U)_0) \hookrightarrow GL(\mathscr{D}(U)_*)$ induces a (Kan) fibration $BGL(\mathscr{D}(U)) \to BGL(\mathscr{D}(U)_*)$ of simplicial sets. We will show that these morphisms induce a *local fibration* of simplicial presheaves, i.e. morphisms of presheaves that are fibrations on each stalk.

Proposition 5.4. *There is a local fibration of presheaves*

$$(U \mapsto BGL^+(\mathscr{D}(U))) \longrightarrow (U \mapsto BGL^{top}(\mathscr{D}(U))) \tag{5.4.3}$$

Here $(U \mapsto BGL(\mathscr{D}(U))^+)$ refers to the presheaf that associates the simplicial set $BGL(\mathscr{D}(U))^+$ to every open set U in X and $(U \mapsto BGL^{top}(\mathscr{D}(U)))$ refers to the presheaf that associates the simplicial set $BGL(\mathscr{D}(U)_*)$ to every open set U of X.

Proof. For any open set U, we know that

$$BGL(\mathscr{D}(U)) \longrightarrow BGL(\mathscr{D}(U)_*) \tag{5.4.4}$$

is a fibration. For any point $p \in X$, we consider the rings $\mathscr{D}_p = \varinjlim_{U_p} \mathscr{D}(U_p)$ and $\mathscr{D}_{p*} = \varinjlim_{U_p} C^\infty(\Delta^*)\hat{\otimes}\mathscr{D}(U_p)$, where U_p varies over all open sets of X containing p. Then, we have an injection

$$GL(\mathscr{D}_p) \hookrightarrow GL(\mathscr{D}_{p*}) \tag{5.4.5}$$

As in (5.2.6), this gives a sequence of fibrations

$$GL(\mathscr{D}_p) \to GL(\mathscr{D}_{p*}) \to GL(\mathscr{D}_{p*})/GL(\mathscr{D}_p) \longrightarrow BGL(\mathscr{D}_p) \to BGL(\mathscr{D}_{p*}) \tag{5.4.6}$$

Since $\pi_1(BGL(\mathscr{D}_{p*})) = \pi_0(GL(\mathscr{D}_{p*})) = 0$, it follows (see [1]) that applying the Quillen plus construction preserves the fibration, i.e. we have a fibration

$$(GL(\mathscr{D}_{p*})/GL(\mathscr{D}_p))^+ \longrightarrow BGL(\mathscr{D}_p)^+ \longrightarrow BGL(\mathscr{D}_{p*}) \tag{5.4.7}$$

□

Definition 5.5. *Let $\mathbf{Z}$ denote the constant sheaf on X given by $\mathbb{Z}$. Denote by $\mathscr{D}_X$ the sheafification of the presheaf that associates any open set U in X to the ring of* *differential operators $\mathscr{D}(U)$. We define the algebraic K-theory of $\mathscr{D}_X$ to be*

$$K_n^{alg}(\mathscr{D}_X) = H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^+) \tag{5.4.8}$$

Consider the sheaf $[\mathbf{BGL}^+, \mathbf{BGL}^{top}]$ associated to the presheaf defined by the homotopy fibres $U \mapsto [\mathbb{Z}\times BGL(\mathscr{D}(U))^+, \mathbb{Z}\times BGL(\mathscr{D}(U)_*)]$ which is identical to the sheafification of the presheaf that associates the simplicial set $[BGL(\mathscr{D}(U)^+), BGL(\mathscr{D}(U)_*)] = (GL(\mathscr{D}(U))_*/GL(\mathscr{D}(U)))^+$ to any open set U in X. We shall also denote this sheaf by $\mathbf{GL}^{rel+}$.

Let the sheaf associated to the presheaf $U \mapsto (GL(\mathscr{D}(U))_*/GL(\mathscr{D}(U)))$ be denoted $\mathbf{GL}^{rel}$. We define the topological and relative K theories of $\mathscr{D}_X$ to be

$$K_n^{top}(\mathscr{D}_X) = H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^{top}) \qquad K_n^{rel}(\mathscr{D}_X) = H^{-n}(X, \mathbf{GL}^{rel+}) \tag{5.4.9}$$

Proposition 5.6. *There is a long exact sequence of K-theory groups*

$$\dots \longrightarrow K_n^{rel}(\mathscr{D}_X) \to K_n^{alg}(\mathscr{D}_X) \to K_n^{top}(\mathscr{D}_X) \to K_{n-1}^{rel}(\mathscr{D}_X) \longrightarrow \dots \tag{5.4.10}$$

Proof. From Proposition 5.4, we have a local fibration of presheaves

$$(U \mapsto BGL^+(\mathscr{D}(U))) \longrightarrow (U \mapsto BGL^{top}(\mathscr{D}(U)))$$

Since local fibrations are defined stalkwise, sheafification preserves local fibrations and hence we have a local fibration of the associated sheaves

$$\mathbf{GL}^{rel+} \longrightarrow \mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top} \tag{5.4.11}$$

Therefore, we have a local fibration of sheaves

$$\mathbf{GL}^{rel+} \longrightarrow \mathbf{Z} \times \mathbf{BGL}^{+} \longrightarrow \mathbf{Z} \times \mathbf{BGL}^{top} \tag{5.4.12}$$

Using [4, Theorem 7], the local fibration (5.4.12) gives rise to a long exact sequence

$$\cdots \to H^m(X, \mathbf{GL}^{rel}) \to H^m(X, \mathbf{Z} \times \mathbf{BGL}^{+}) \to \tag{5.4.13}$$

$$H^m(X, \mathbf{Z} \times \mathbf{BGL}^{top}) \to H^{m+1}(X, \mathbf{GL}^{rel+}) \to \dots$$

By Definition 5.5, $K_m^{alg}(\mathscr{D}_X) = H^{-m}(X, \mathbf{Z} \times \mathbf{BGL}^{+})$, $K_m^{top}(\mathscr{D}_X) = H^{-m}(X, \mathbf{Z} \times \mathbf{BGL}^{top})$ and $K_m^{rel}(\mathscr{D}_X) = H^{-m}(X, \mathbf{GL}^{rel+})$ and hence the required long exact sequence (5.4.10) is a restatement of (5.4.13). $\square$

5.5 Chern characters and Dennis Trace

As mentioned in Section 5.2, for a locally convex Algebra A, we have Chern maps $ch_{i+1}^{top} : K_{i+1}^{top}(A) \to HC_{i+1}(A)$ and $ch_i^{rel} : K_i^{rel}(A) \to HC_{i-1}(A)$ and the Dennis trace $D_i : K_i^{alg}(A) \to HH_i(A)$ which fit into a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc}
K_{i+1}^{alg}(A) & \longrightarrow & K_{i+1}^{top}(A) & \longrightarrow & K_i^{rel}(A) & \longrightarrow & K_i^{alg}(A) & \longrightarrow & K_i^{top}(A) \\
\downarrow{\scriptstyle D_{i+1}} & & \downarrow{\scriptstyle ch_{i+1}} & & \downarrow{\scriptstyle ch_i^{rel}} & & \downarrow{\scriptstyle D_i} & & \downarrow{\scriptstyle ch_i} \\
H_{i+1}(A, A) & \longrightarrow & HC_{i+1}(A) & \longrightarrow & HC_{i-1}(A) & \longrightarrow & H_i(A, A) & \longrightarrow & HC_i(A)
\end{array}$$

We shall now prove the same result for the K-theory groups (algebraic, topological and relative) of the sheaf $\mathscr{D}_X$ of differential operators on X described in Section 5.4.

Definition 5.7. *Consider the sheafification* $(\widetilde{C^h_*(\mathscr{D}_X)}, b)$ *of the presheaf of complexes that associates to an open set* U *in* X *the Hochschild complex* $(C^h_*(\mathscr{D}(U)), b)$ *of the ring* $\mathscr{D}(U)$*. Define the Hochschild homology* $HH_q(\mathscr{D}_X)$ *of the sheaf* $\mathscr{D}_X$ *to be the* $(-q)$*-th hypercohomology of the complex*

$$HH_q(\mathscr{D}_X) = \mathbb{H}^{-q}((\widetilde{C^h_*(\mathscr{D}_X)}, b)) \tag{5.5.1}$$

Similarly, we consider the sheafification $(\widetilde{BC_*(\mathscr{D}_X)}, b)$ *of the presheaf of complexes that associates to an open set* U *in* X *the "mixed complex"* $(BC_*(\mathscr{D}(U)), b)$ *of the ring* $\mathscr{D}(U)$*, whose homology is the cyclic homology of the ring* $\mathscr{D}(U)$*. Define the cyclic homology* $HC_q(\mathscr{D}_X)$ *of the sheaf* $\mathscr{D}_X$ *to be the* $(-q)$*-th hypercohomology of the complex*

$$HC_q(\mathscr{D}_X) = \mathbb{H}^{-q}((\widetilde{BC_*(\mathscr{D}_X)}, b)) \tag{5.5.2}$$

From the Definition 5.7 and general properties of hypercohomology, it follows (see [42, (0.4)]) that there is a periodicity long exact sequence analogous to (1.1.19) involving $HH_q(\mathscr{D}_X)$ and $HC_q(\mathscr{D}_X)$.

$$\ldots \longrightarrow HH_q(\mathscr{D}_X) \xrightarrow{I} HC_q(\mathscr{D}_X) \xrightarrow{S} HC_{q-2}(\mathscr{D}_X) \xrightarrow{B} \ldots \tag{5.5.3}$$

Once again, we note that the definition of hypercohomology used in Definition 5.7 is the one used in Chapter 3.

We shall now construct the Dennis trace map $D_i : K_i^{alg}(\mathscr{D}_X) \to HH_i(\mathscr{D}_X)$ as well as the Chern maps $ch_{i+1}^{top} : K_{i+1}^{top}(\mathscr{D}_X) \to HC_{i+1}(\mathscr{D}_X)$ and $ch_i^{rel} : K_i^{rel}(\mathscr{D}_X) \to$

$HC_{i-1}(\mathscr{D}_X)$. We start with the Dennis trace. By Definition 5.5,

$$K_i^{alg}(\mathscr{D}_X) = H^{-i}(X, \mathbf{Z} \times \mathbf{BGL}^+) \tag{5.5.4}$$

Let us denote by $R\mathbf{BGL}^+$ the flasque resolution of $\mathbf{BGL}^+$. Then, by definition, the morphism $R\mathbf{BGL}^+ \to *$ is a global fibration and we have a weak equivalence $i_R : \mathbf{BGL}^+ \longrightarrow R\mathbf{BGL}^+$.

From Definition 5.5 and (5.3.5), we have

$$K_i^{alg}(\mathscr{D}_X) = H^{-i}(X, \mathbf{Z} \times R\mathbf{BGL}^+) = \pi_i(\Gamma(X, \mathbf{Z} \times R\mathbf{BGL}^+)) \tag{5.5.5}$$

On the other hand, for each i, consider the three presheaves $\mathscr{F}_{i0}^{alg}$, $\mathscr{F}_{i1}^{alg}$ and $\mathscr{F}_{i2}^{alg}$ on X defined as

$$\begin{gathered} \mathscr{F}_{i0}^{alg}(U) = \pi_i(\mathbb{Z} \times BGL^+(\mathscr{D}(U))) \qquad \mathscr{F}_{i1}^{alg}(U) = \pi_i(\Gamma(U, \mathbf{Z} \times \mathbf{BGL}^+)) \\ \mathscr{F}_{i2}^{alg}(U) = \pi_i(\Gamma(U, \mathbf{Z} \times R\mathbf{BGL}^+) \end{gathered} \tag{5.5.6}$$

for any open set U in X. There are obvious morphisms of presheaves

$$\mathscr{F}_{i0}^{alg} \xrightarrow{j_0} \mathscr{F}_{i1}^{alg} \xrightarrow{j_1} \mathscr{F}_{i2}^{alg} \tag{5.5.7}$$

The presheaf defined by $(U \mapsto BGL(\mathbb{Z} \times \mathscr{D}(U))^+)$ for each open set U in X and its sheafification $\mathbf{Z} \times \mathbf{BGL}^+$ have isomorphic stalks and hence j_0 induces an isomorphism of stalks of $\mathscr{F}_{i0}^{alg}$ and $\mathscr{F}_{i1}^{alg}$. Hence, the corresponding sheafifications $\widetilde{\mathscr{F}_{i0}^{alg}}$ and $\widetilde{\mathscr{F}_{i1}^{alg}}$ of $\mathscr{F}_{i0}^{alg}$ and $\mathscr{F}_{i1}^{alg}$ respectively, i.e. we have an isomorphism of sheaves

$$\tilde{j}_0 : \widetilde{\mathscr{F}_{i0}^{alg}} \xrightarrow{\simeq} \widetilde{\mathscr{F}_{i1}^{alg}} \tag{5.5.8}$$

The morphism j_1 is induced by the inclusion $i_R : \mathbf{Z} \times \mathbf{BGL}^+ \to \mathbf{Z} \times R\mathbf{BGL}^+$ which is a weak equivalence of simplicial sheaves and hence a weak equivalence of their stalks. Hence the stalks of $\mathscr{F}_{i1}^{alg}$ and $\mathscr{F}_{i2}^{alg}$ are isomorphic and we have an isomorphism of sheaves

$$\tilde{j_1} : \widetilde{\mathscr{F}_{i1}^{alg}} \xrightarrow{\simeq} \widetilde{\mathscr{F}_{i2}^{alg}} \tag{5.5.9}$$

Let us denote the common sheafification of $\mathscr{F}_{i0}^{alg}$, $\mathscr{F}_{i1}^{alg}$ and $\mathscr{F}_{i2}^{alg}$ by $\mathscr{F}_{i}^{alg}$. On each Stein open set U, we have, by definition, $K_i^{alg}(\mathscr{D}(U)) = \pi_i(BGL^+(\mathscr{D}(U))$ and therefore, a Dennis trace map

$$D_i : K_i^{alg}(\mathscr{D}(U)) = \pi_i(BGL^+(\mathscr{D}(U)) \to HH_i(\mathscr{D}(U)) \tag{5.5.10}$$

Since X has a basis consisting of Stein open sets, if $\mathcal{HH}_i(\mathscr{D}_X)$ denotes the sheaf associated to the presheaf $U \mapsto HH_i(\mathscr{D}(U))$, the map D_i, at each Stein open set, gives us a morphism of sheaves (as in the result of Lemma A.3.14), also denoted by

$$D_i : \mathscr{F}_i^{alg} = \widetilde{\mathscr{F}_{i0}^{alg}} \longrightarrow \mathcal{HH}_i(\mathscr{D}_X)$$

(at the presheaf level, on U open in X, we compose the map $\pi_i(\mathbb{Z} \times BGL^+(\mathscr{D}(U))) \to \pi_i(\{0\} \times BGL^+(\mathscr{D}(U)))$ with the Dennis trace).

The morphism $\mathscr{F}_i^{alg} \to \mathcal{HH}_i(\mathscr{D}_X)$ of sheaves induces a morphism of global sections $\Gamma(X, \mathscr{F}_i^{alg}) \to \Gamma(X, \mathcal{HH}_i(\mathscr{D}_X))$ and, by abuse of notation, we still denote this induced morphism

$$\Gamma(X, \mathscr{F}_i^{alg}) \to \Gamma(X, \mathcal{HH}_i(\mathscr{D}_X)) \tag{5.5.11}$$

by D_i. Since $\mathscr{F}_i^{alg}$ is also the sheafification of $\mathscr{F}_{i2}^{alg}$, we have a morphism

$$K_i^{alg}(\mathscr{D}_X) = \pi_i(\Gamma(X, \mathbf{Z} \times R\mathbf{BGL}^+)) = \Gamma(X, \mathscr{F}_{i2}^{alg}) \to \Gamma(X, \mathscr{F}_i^{alg}) \to \Gamma(X, \mathcal{HH}_i(\mathscr{D}_X)) \tag{5.5.12}$$

Finally, the Hochschild homology $HH_i(\mathscr{D}_X)$ being defined as the hypercohomology of the complex $(C_h^*(\mathscr{D}_X), b)$ (notation as in Definition 5.7), we have a morphism from $\Gamma(X, \mathcal{HH}_i(\mathscr{D}_X))$ to $HH_i(\mathscr{D}_X)$. We compose this sequence of maps to obtain the Dennis trace

$$D_i : K_i^{alg}(\mathscr{D}_X) \longrightarrow HH_i(\mathscr{D}_X) \tag{5.5.13}$$

(still denoted by D_i). Using an identical process, for each i, we can construct the presheaves $\mathscr{F}_{i0}^{top}$, $\mathscr{F}_{i1}^{top}$ and $\mathscr{F}_{i2}^{top}$ (resp. $\mathscr{F}_{i0}^{rel}$, $\mathscr{F}_{i1}^{rel}$ and $\mathscr{F}_{i2}^{rel}$) and their common sheafification $\mathscr{F}_i^{top}$ (resp. $\mathscr{F}_i^{rel}$). Once again, using the fact that X has a basis consisting of Stein open sets, we can define the Chern maps

$$\begin{aligned} ch_{i+1}^{top} &: K_{i+1}^{top}(\mathscr{D}_X) \to HC_{i+1}(\mathscr{D}_X) \\ ch_i^{rel} &: K_i^{rel}(\mathscr{D}_X) \to HC_{i-1}(\mathscr{D}_X) \end{aligned} \tag{5.5.14}$$

Proposition 5.8. *There is a commutative diagram of long exact sequences*

$$\begin{array}{ccccc} ... \to K_{i+1}^{rel}(\mathscr{D}_X) & \longrightarrow & K_{i+1}^{alg}(\mathscr{D}_X) & \longrightarrow & K_{i+1}^{top}(\mathscr{D}_X) \to ... \\ \downarrow{\scriptstyle ch_i^{rel}} & & \downarrow{\scriptstyle D_i} & & \downarrow{\scriptstyle ch_i^{top}} \\ ... \to HC_i(\mathscr{D}_X) & \longrightarrow & HH_{i+1}(\mathscr{D}_X) & \longrightarrow & HC_{i+1}(\mathscr{D}_X) \to ... \end{array} \tag{5.5.15}$$

Proof. The periodicity long exact sequence for $\mathscr{D}_X$ follows from the definition of cyclic homology for the sheaf $\mathscr{D}_X$ as mentioned in (5.5.3). We already know that the maps ch_i^{top}, ch_i^{rel} and D_i are compatible with the maps in the long exact sequence on each open set U in X. Since the Chern maps and the Dennis trace in the above setup have been obtained by patching over all such open sets, the diagram is commutative. □

5.6 The long exact sequence for formal schemes

The constructions described in the previous sections may also be carried out for the structure sheaves of formal schemes. Furthermore, in the setting of formal schemes, (where we use Zariski topologies instead of complex topology), we show that the sheaf **BGL** is flasque, i.e. is its own flasque resolution.

Definition 5.9. *(see [29, II.9]) A Noetherian formal scheme is a locally ringed space* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ *which can be covered by a finite open cover* $\{\mathfrak{U}_i\}$ *such that for each* i*, the pair* $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ *is isomorphic, as a locally ringed space, to the completion* $\hat{X}$ *of a Noetherian scheme* X *along a closed subscheme* Y*. An affine (Noetherian) formal scheme* $\hat{X}$ *is obtained by completing a single Noetherian scheme* X *along a closed subscheme* Y*.*

If X is a Noetherian scheme and we complete X along itself, we get $\hat{X} = X$; thus the category of Noetherian formal schemes includes all Noetherian schemes. Note that if the Noetherian scheme X is completed along a subscheme Y, then the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is supported on Y. Choose an open set U in X and consider the ring $\mathcal{O}_{\mathfrak{X}}(U)$. By definition, $\mathcal{O}_{\mathfrak{X}}(U)$ is the completion $\varprojlim_{n\geq 1} \mathcal{O}_X(U)/\mathcal{I}_Y^n(U)$ where $\mathcal{I}_Y$ is the sheaf of ideals corresponding to the subscheme Y. The completed ring $\mathcal{O}_{\mathfrak{X}}(U)$ is therefore a Banach algebra.

Note that when V is an open set in X such that $V \subseteq X - Y$, we have $\mathcal{O}_{\mathfrak{X}}(V) = 0$. For any $N \geq 1$, let us set $GL_n(\mathcal{O}_{\mathfrak{X}}(V)) = 1$ and $GL(\mathcal{O}_{\mathfrak{X}}(V)) = 1$.

For each N, let $\mathbf{BGL}_N^+$ and $\mathbf{BGL}_N^{top}$ denote the sheaves on X asssociated to the presheaves $U \mapsto BGL_N(\mathcal{O}_{\mathfrak{X}}(U))^+$ and $U \mapsto BGL_N(\mathcal{O}_{\mathfrak{X}}(U)_*)$ respectively. Here $\mathcal{O}_{\mathfrak{X}}(U)_*$ refers to the simplicial ring $C^\infty(\Delta^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(U)$ obtained from the Banach algebra $\mathcal{O}_{\mathfrak{X}}(U)$ as in Section 5.2. Further, denote by $\mathbf{BGL}^+$ and $\mathbf{BGL}^{top}$ the sheaves associated to the presheaves

$$U \mapsto BGL^+(\mathcal{O}_{\mathfrak{X}}(U)) \qquad U \mapsto BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$$

respectively, for any open set U in X. Again, it is understood that if $U \subseteq X - Y$, then $GL(\mathcal{O}_{\mathfrak{X}}(U)_*) = 1$. We have:

Proposition 5.10. *(1)There is a global fibration of presheaves*

$$(U \mapsto BGL(\mathcal{O}_{\mathfrak{X}}(U))) \longrightarrow (U \mapsto BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$$

Here $(U \mapsto BGL(\mathcal{O}_{\mathfrak{X}}(U)))$ refers to the presheaf that associates the simplicial set $BGL(\mathcal{O}_{\mathfrak{X}}(U)))$ to every open set U in X and $(U \mapsto BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$ refers to the presheaf that associates the simplicial set $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ to every open set U of X.

(2) The morphism of sheaves

$$\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top}$$

is a local fibration.

Proof. (1) Let U and V be open sets in X such that $V \subseteq U$. When $V \nsubseteq X - Y$, we proceed as follows: The restriction map from $\mathcal{O}_{\mathfrak{X}}(U)$ to $\mathcal{O}_{\mathfrak{X}}(V)$ induces restrictions

$GL(\mathcal{O}_{\mathfrak{X}}(U)) \to GL(\mathcal{O}_{\mathfrak{X}}(V))$, $GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \to GL(\mathcal{O}_{\mathfrak{X}}(V)_*)$ and a pullback square

$$\begin{array}{ccc} GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \times_{GL(\mathcal{O}_{\mathfrak{X}}(V)_*)} GL(\mathscr{D}(V)) & \longrightarrow & GL(\mathcal{O}_{\mathfrak{X}}(V)) \\ \big\downarrow & & \big\downarrow \\ GL(\mathcal{O}_{\mathfrak{X}}(U)_*) & \longrightarrow & GL(\mathcal{O}_{\mathfrak{X}}(V)_*) \end{array} \tag{5.6.1}$$

Since $GL(\mathcal{O}_{\mathfrak{X}}(U)) \hookrightarrow GL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ is an injection, it follows that we must have an injection of $GL(\mathcal{O}_{\mathfrak{X}}(U))$ into the pullback:

$$GL(\mathcal{O}_{\mathfrak{X}}(U)) \hookrightarrow GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \times_{GL(\mathcal{O}_{\mathfrak{X}}(V)_*)} GL(\mathcal{O}_{\mathfrak{X}}(V)) \tag{5.6.2}$$

and hence induces a Kan fibration on classifying spaces. Applying the classifying space functor, we see that

$$BGL(\mathcal{O}_{\mathfrak{X}}(U)) \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U)_*) \times_{BGL(\mathcal{O}_{\mathfrak{X}}(V)_*)} BGL(\mathcal{O}_{\mathfrak{X}}(V)) \tag{5.6.3}$$

is a Kan fibration.

When $V \subseteq X - Y$ but $U \not\subseteq X - Y$, we have to check that

$$BGL(\mathcal{O}_{\mathfrak{X}}(U)) \to BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$$

is a fibration, which follows directly from (5.2.4). Finally, when $U \subseteq X - Y$, we must show that $* \to *$ is a fibration, which is trivial.

(2) Choose a point p in X. If $p \notin Y$, then the stalks of both sheaves at p are just $*$ and $* \to *$ is a fibration. If $p \in Y$, then we consider the local ring $\mathcal{O}_{\mathfrak{X},p} = \varinjlim_{U_p} \mathcal{O}_{\mathfrak{X}}(U_p)$ and the simplicial ring $\mathcal{O}_{\mathfrak{X},p*} = \varinjlim_{U_p} C^\infty(\Delta^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(U_p)$ where U_p varies over all open

sets in X containing p. Then, we have an injection

$$GL(\mathcal{O}_{\mathfrak{X},p}) \hookrightarrow GL(\mathcal{O}_{\mathfrak{X},p*}) \tag{5.6.4}$$

As in (5.2.4), this leads to a fibration

$$BGL(\mathcal{O}_{\mathfrak{X},p}) \longrightarrow BGL(\mathcal{O}_{\mathfrak{X},p*}) \tag{5.6.5}$$

Since $\pi_1(BGL(\mathcal{O}_{\mathfrak{X},p*}))$ is trivial, we can apply plus construction and still have a fibration

$$BGL(\mathcal{O}_{\mathfrak{X},p})^+ \longrightarrow BGL(\mathcal{O}_{\mathfrak{X},p*}) \tag{5.6.6}$$

Therefore, by definition, we have a local fibration of the sheafifications $\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top}$.

□

We shall now define the algebraic, topological and relative K theories of a formal scheme and show that they are related by a long exact sequence analogous to the one in Proposition 5.6.

Definition 5.11. *Let $\mathbf{Z}$ denote the constant sheaf on Y given by $\mathbb{Z}$. The algebraic K-theory of $\mathfrak{X}$ is defined to be*

$$K_n^{alg}(\mathfrak{X}) = H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^+)$$

Consider the sheafification $[\mathbf{BGL}^+, \mathbf{BGL}^{top}]$ of the presheaf which associates to an open set U in X the homotopy fibre $U \mapsto [\mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U)^+), \mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)]$. The homotopy fibre is identical to the presheaf $U \mapsto [BGL(\mathcal{O}_{\mathfrak{X}}(U)^+), BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)]$ We shall also denote this sheaf by $\mathbf{GL}^{rel+}$ and use $\mathbf{GL}^{rel}$ to denote the sheafification of the *presheaf which associates to an open set U in X the set* $\mapsto [BGL(\mathcal{O}_{\mathfrak{X}}(U)), BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)]$. *We define the topological and relative K theories of $\mathfrak{X}$ to be*

$$K_n^{top}(\mathfrak{X}) = H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^{top}) \qquad K_n^{rel}(\mathfrak{X}) = H^{-n}(X, \mathbf{GL}^{rel+}) \tag{5.6.7}$$

Proposition 5.12. *There is a long exact sequence of K-theory groups*

$$\ldots \longrightarrow K_n^{rel}(\mathfrak{X}) \to K_n^{alg}(\mathfrak{X}) \to K_n^{top}(\mathfrak{X}) \to K_{n-1}^{rel}(\mathfrak{X}) \longrightarrow \ldots \tag{5.6.8}$$

Proof. The proof is identical to that of Proposition 5.6.

$\square$

In the case of formal schemes, however, we will also show that the presheaves BGL, BGL^{top} and GL^{rel} are flasque.

Lemma 5.13. *Let R be a Noetherian integral domain and suppose that p is a prime ideal. Choose some $g \in R - p$ Then, the natural map from $\hat{R} = \varprojlim R/p^n$ to $\hat{R}_g = \varprojlim R_g/p_g^n$ is an injection.*

Proof. The elements of $\hat{R}$ are, by definition, sequences of the form $(r_1, r_2, ...)$ where, for positive integers $i \geq j$, $r_i \equiv r_j \pmod{p^j}$. Suppose that the sequence $(r_1, r_2, ...)$ maps to the zero sequence in $\hat{R}_g$. Then, for each r_i, we must have $r_i \in p_g^i$. Since R is an integral domain, this means that we have an $x_i \in p^i$ and a power g^{k_i} such that

$$g^{k_i} r_i = x_i \in p^i \text{ for each } i$$

Since p^i is a primary ideal, $r_i \notin p^i$ would imply that some power of g^{k_i} lies in p^i. This is impossible since $g \notin p$. Hence, each $r_i \in p^i$ and the map is injective. $\square$

Lemma 5.14. *Let X be a topological space and suppose that $F \to E \to B$ be a fibration sequence of simplicial presheaves on X, with E and B both flasque. Then the fibre F is also a flasque presheaf.*

Proof. Let $V \subset U$ be nonempty open sets in X. Consider the following fibre squares

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] & * \arrow[d] & F(V) \arrow[r] \arrow[d] & * \arrow[d] \\ E(U) \arrow[r] & B(U) & E(V) \arrow[r] & B(V) \end{tikzcd} \tag{5.6.9}$$

Since the restrictions $E(U) \to E(V)$ and $B(U) \to B(V)$ are Kan fibrations (E and B being flasque), we have to show that the morphism between their fibre products, i.e. the morphism $F(U) \to F(V)$ is also a Kan fibration. This is proved as follows: Recall that, by definition, a Kan fibration, is that which has a *right lifting property* with respect to monomorphisms of simplicial sets that are also weak equivalences (also called *trivial cofibrations*), i.e. $E' \to B'$ is a Kan fibration, if given any trivial cofibration $X \to Y$ of simplicial sets, and a commutative square of morphisms

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E' \arrow[d] \\ Y \arrow[r] & B' \end{tikzcd} \tag{5.6.10}$$

there exists a 'lift', i.e. a morphism $Y \to E'$ that fits into the commutative diagram.

Suppose, therefore that we have the following commutative digram, with a trivial cofibration $X \to Y$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F(U) \arrow[d] \\ Y \arrow[r] & F(V) \end{tikzcd} \tag{5.6.11}$$

Then, by composing with the maps $F(U) \longrightarrow E(U)$ and $F(U) \longrightarrow B(U)$, we get the commutative diagrams

$$\begin{array}{ccc} X & \longrightarrow & E(U) \\ \Big\downarrow & & \Big\downarrow \\ Y & \longrightarrow & E(V) \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & B(U) \\ \Big\downarrow & & \Big\downarrow \\ Y & \longrightarrow & B(V) \end{array} \tag{5.6.12}$$

Since the maps $E(U) \to E(V)$ and $B(U) \to B(V)$ are fibrations, we have 'lifts' in both the squares of (5.6.12). These lifts give maps from Y to $E(U)$ and $*$ that agree on $B(U)$. Since $F(U)$ is a pullback, we have a map $Y \to F(U)$ that makes the digram commutative. Hence, $F(U) \to F(V)$ is a Kan fibration.

When we consider $V = \phi$, then $E(V) = B(V) = *$ and $E(U)$ and $B(U)$ are Kan complexes. Once again, from the fibre square

$$\begin{array}{ccc} F(U) & \longrightarrow & * \\ \Big\downarrow & & \Big\downarrow \\ E(U) & \longrightarrow & B(U) \end{array} \tag{5.6.13}$$

it follows that $F(U)$ is a Kan complex. This shows that the presheaf F is flasque. □

Proposition 5.15. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be the algebraic formal scheme obtained by completing a Noetherian integral scheme X along an irreducible integral subschmeme (of positive codimension) Y. The simplicial presheaves BGL, BGL^{top} and GL^{rel} are flasque in the category of simplicial sheaves, in other words, the maps $BGL \to *$, $BGL^{top} \to *$ and $GL^{rel} \to *$ are global fibrations of presheaves.*

Proof. By definition, a simplicial sheaf $\mathcal{F}$ is flasque if and only if

(1) $\mathcal{F}(U) \to \mathcal{F}(V)$ is a Kan fibration for each $V \subset U$ and,

(2) Each simplicial set $\mathcal{F}(U)$ is a Kan complex, i.e. $\mathcal{F}(U) \to *$ is a Kan fibration.

Let $V \subseteq U$ be two open sets in X. Since Y is an irreducible subscheme, Y is given locally by a prime ideal. Therefore, from Lemma 5.13 above, it follows that the map $\mathcal{O}_{\mathfrak{X}}(U) \to \mathcal{O}_{\mathfrak{X}}(V)$ is an injection. Therefore, the maps $GL(\mathcal{O}_{\mathfrak{X}}(U)) \to GL(\mathcal{O}_{\mathfrak{X}}(V))$ and

$$GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \to GL(\mathcal{O}_{\mathfrak{X}}(V)_*)$$

are also injections. The injections induce a Kan fibration of classifying spaces and hence,

$$BGL(\mathcal{O}_{\mathfrak{X}}(U)) \to BGL(\mathcal{O}_{\mathfrak{X}}(V)) \qquad BGL(\mathcal{O}_{\mathfrak{X}}(U)_*) \to BGL(\mathcal{O}_{\mathfrak{X}}(V)_*)$$

are Kan fibrations. Further, it is well known for any group G, the simplicial set BG is a Kan complex, which implies that the simplicial sets $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ and $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ are Kan complexes. This proves that both the presheaves BGL and BGL^{top} are flasque.

The fact that the presheaf $U \mapsto GL(O_{\mathfrak{X}}(U)_*)/GL(O_{\mathfrak{X}})$, which is the fibre of $BGL \longrightarrow BGL^{top}$, is flasque follows from Lemma 5.14. This finishes the proof. $\square$

Finally, using the same exact proof as that of Proposition 5.8 we have the following long exact sequences:

Proposition 5.16. *There is a commutative diagram of long exact sequences:*

$$\begin{array}{ccccc} \dots \to K^{rel}_{i+1}(\mathfrak{X}) & \longrightarrow & K^{alg}_{i+1}(\mathfrak{X}) & \longrightarrow & K^{top}_{i+1}(\mathfrak{X}) \to \dots \\ \downarrow{\scriptstyle ch^{rel}_i} & & \downarrow{\scriptstyle D_i} & & \downarrow{\scriptstyle ch^{top}_i} \\ \dots \to HC_i(\mathfrak{X}) & \longrightarrow & HH_{i+1}(\mathfrak{X}) & \longrightarrow & HC_{i+1}(\mathfrak{X}) \to \dots \end{array} \tag{5.6.14}$$

Proof. As in the proof of Proposition 5.8 we have to construct the maps D_i, ch^{top}_i and ch^{rel}_i, which is achieved by the same construction as in Section 5.5. The rest of this proof is also identical to that of Proposition 5.8. $\square$

6 Modular Hecke Algebras, Rankin Cohen Brackets and an Enriched Archimedean Complex

For a congruence subgroup $\Gamma = \Gamma(N)$ of $SL_2(\mathbb{Z})$, we can consider the modular curve $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$ where $\mathbb{H}^*$ denotes the union of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ with the rational points on the real line. It is well known that the modular curve $X(\Gamma)$ is a compact Riemann surface. For any compact Riemann surface X, Consani's complex K^{ijk} (see Chapter 4) becomes (for $i, j, k \in \mathbb{Z}$)

$$K^{i,j,k}(X) = \begin{cases} \bigoplus_{a\leq b, a+b=j+1, |a-b|\leq 2k-i} (\Omega_X^{a,b} + \Omega_X^{b,a})_{\mathbb{R}} \left(\frac{1+j-i}{2}\right) & \text{if } k \geq max\{0, i\} \\ 0 & \text{otherwise} \end{cases} \tag{6.0.15}$$

with differentials d' and d'' and total differential $d = d' + d''$ as defined in (4.1.2). In Theorem 2.8 we have recalled how the complex $K^{i,j,k}(X)$ can be used to compute the real Deligne cohomology of X. When we choose $X = X(N)$, i.e. the N-th modular curve $X(N) = \overline{\Gamma(N)\backslash\mathbb{H}}$ we shall consider the corresponding complexes $K^{***}(X(N))$. We tensor the terms $K^{***}(X(N))$ by modular forms of level $\Gamma(N)$ and appropriate weight and consider their direct limit K^{***} over all $N \geq 1$, which we refer to as the "enriched archimedean complex" K_{en}^{***}.

For any fixed congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, we consider the module of functions on $\Gamma\backslash G_2^+(\mathbb{Q})$ of finite support taking values in the enriched archimedean complex

K^{***} and satisfying a suitable "covariance condition" (see Definition 6.4). This module, which we denote by $\mathcal{B}^{i,j,k}(\Gamma)$, is our main object of study. In Proposition 6.8), we show that $\mathcal{B}^{i,j,k}(\Gamma)$ is a bimodule over an algebra $\mathcal{A}_T(\Gamma)$, which is a slight variant of Connes-Moscovici's modular Hecke algebra $\mathcal{A}(\Gamma)$(see Section 6.1 for definitions). Following [10], we know that the Hopf algebra $\mathcal{H}_1$ of "codimension one foliations" of Connes-Moscovici acts on the modular Hecke algebra $\mathcal{A}(\Gamma)$. We then show that the action of the same Hopf algebra $\mathcal{H}_1$ determines an action on the module $\mathcal{B}^{i,j,k}(\Gamma)$ and that the action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathcal{B}^{i,j,k}(\Gamma))$ is "flat" in a sense to be made precise in Definition 6.12.

We also consider a smaller Hopf algebra $\mathfrak{h}_1$ that is obtained from $\mathcal{H}_1$ by setting some of the generators of $\mathcal{H}_1$ equal to zero. We then define a "reduced version" of the algebra $\mathcal{A}_T(\Gamma)$ on which $\mathfrak{h}_1$ acts and denote this new algebra by $\mathcal{A}_T^r(\Gamma)$. We also have a counterpart of $\mathcal{B}^{i,j,k}(\Gamma)$ in this framework which we denote by $\mathcal{B}_r^{i,j,k}(\Gamma)$. Once again, we show that $\mathcal{B}_r^{i,j,k}(\Gamma)$ is a bimodule over $\mathcal{A}_T^r(\Gamma)$. The smaller Hopf algebra $\mathfrak{h}_1$ acts on $\mathcal{B}_r^{i,j,k}(\Gamma)$ and the action of $\mathfrak{h}_1$ on the system $(\mathcal{A}_T^r(\Gamma), \mathcal{B}_r^{i,j,k}(\Gamma))$ is "flat" (see Proposition 6.16).

Finally, in Section 6.3, we define Rankin Cohen brackets on the modules $\mathcal{B}^{i,j,k}(\Gamma)$. These Rankin Cohen pairings combine the Rankin Cohen pairings defined by Connes and Moscovici [9] with the pairings on the archimedean complex defined by Consani in [18].

6.1 Hecke Algebras and Hecke Correspondences

Throughout, we will assume $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgroup $\Gamma = \Gamma(N)$ for some integer $N \geq 1$. Recall that the congruence subgroup $\Gamma(N)$ is defined as

$$\Gamma(N) = \{\alpha \in SL_2(\mathbb{Z}) : \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\} \tag{6.1.1}$$

We will sometimes denote this group by Γ_M. For any $\alpha \in GL_2^+(\mathbb{Q})$, consider the double coset $\Gamma\alpha\Gamma$ as a subset of $GL_2^+(\mathbb{Q})$. Any left coset $\Gamma\beta$ of Γ in $SL_2(\mathbb{Z})$ that has nonempty intersection with $\Gamma\alpha\Gamma$ is contained in it. Hence, the double coset $\Gamma\alpha\Gamma$ may be written as a (disjoint) union of left cosets; in other words, there exist finitely many $\alpha_i \in GL_2^+(\mathbb{Q})$, $1 \leq i \leq k$, such that

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{k} \alpha_i\Gamma \tag{6.1.2}$$

Consider an orbit Γz for some $z \in \mathbb{H}$ for the action of Γ on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$. Then Γz is an element of the (compact) modular curve $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$, where $\mathbb{H}^*$ denotes the union $\mathbb{H} \cup \mathbb{Q}$ of the upper half plane $\mathbb{H}$ with the rational points on the real line. Then the coset $\Gamma\alpha\Gamma$ induces a "multi valued map" on $X(\Gamma)$ by mapping the orbit Γz to the set of orbits $\{\Gamma\alpha_i z\}$. This notion may be described precisely in terms of a correspondence, known as the Hecke correspondence.

For $\alpha \in GL_2^+(\mathbb{Q})$, consider, therefore, the group $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$. There is a natural surjection $p : \Gamma_\alpha\backslash\mathbb{H}^* = X(\Gamma_\alpha) \twoheadrightarrow X(\Gamma) = \Gamma\backslash\mathbb{H}^*$ and also a well defined map $\alpha : X(\Gamma_\alpha) \to X(\Gamma)$ defined by sending the orbit $\Gamma_\alpha z$ to $\Gamma\alpha z$. Hence, we obtain a diagram:

$$
\begin{array}{ccc} & X(\Gamma_\alpha) & \\ {}^{p}\swarrow & & \searrow^{\alpha} \\ X(\Gamma) & & X(\Gamma) \end{array} \tag{6.1.3}
$$

We recall that a correspondence from an algebraic scheme X to a scheme Y is a subscheme Z of $X \times Y$ that is dominant over X by means of the projection $p_1 : X \times Y \to X$ onto the first coordinate. If $f : X \to Y$ is any morphism of schemes, its graph $\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\} \subseteq X \times Y$ is a correspondence from X to Y. The set of correspondences from X to Y is denoted by $Cor(X, Y)$. In the diagram (6.1.3), the image of $X(\Gamma_\alpha)$ under the map

$$
X(\Gamma_\alpha) \xrightarrow{p\times\alpha} X(\Gamma) \times X(\Gamma)
$$

gives us a correspondence from $X(\Gamma)$ to itself. We refer to this construction as the Hecke correspondence $T(\Gamma\alpha\Gamma)$ or simply $T(\alpha)$. For $\beta \in GL_2^+(\mathbb{Q})$, we set

$$
T(\beta) \cdot T(\alpha) = \sum_{i=1}^{k} \Gamma\beta\alpha_i\Gamma \qquad\qquad \Gamma\alpha\Gamma = \bigcup_{i=1}^{k} \alpha_i\Gamma \tag{6.1.4}
$$

The $\mathbb{C}$-algebra generated by these operators $T(\alpha)$, $\alpha \in GL_2^+(\mathbb{Q})$ is the Hecke Algebra relative to the curve $X(\Gamma)$ and is denoted by $\mathcal{H}(\Gamma)$.

Remark 6.1. For three algebraic schemes X, Y and Z, we can compose $f \in Cor(X, Y)$ and $g \in Cor(Y, Z)$ to define the correspondence

$$
g \circ f = p_{XZ*}(p_{XY}^*(f) \cdot p_{YZ}^*(g)) \in Cor(X, Z) \tag{6.1.5}
$$

where $p_{XY} : X \times Y \times Z \to X \times Y$ denotes the natural projection (and similarly for

p_{YZ} and p_{XZ}) where $\cdot$ denotes the product of cycles in $X \times Y \times Z$. However, note that the product of cycles is defined only up to an equivalence. Hence the right hand side of (6.1.5) is defined only up to equivalence. It is therefore advantageous to use the explicit cycle given by (6.1.4) to define the product of correspondences $T(\beta) \cdot T(\alpha)$. The equivalence class of the explicit cycle given by (6.1.4) for the product $T(\beta) \cdot T(\alpha)$ agrees with the general definition for the product $T(\beta) \cdot T(\alpha)$ given by the method of (6.1.5).

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, and $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function, we define

$$f|_{2k}\gamma(z) = det(\gamma)^k f(z) j_\gamma(z)^k \qquad \text{where } j_\gamma(z) = (cz+d)^{-2} \tag{6.1.6}$$

Definition 6.2. *Let $\Gamma = \Gamma(N)$ be a congruence subgroup and let k be any nonnegative integer. Suppose that $f : \mathbb{H} \to \mathbb{C}$ is a function satisfying the following properties*

(1) f is holomorphic.

(2) $f|_{2k}(z)\gamma = f(z)$ for all $\gamma \in \Gamma$.

(3) Let $q = e^{2\pi i z/N}$ and define $f_\infty : \{q \in \mathbb{C} | 0 < |q| < 1\} \to \mathbb{C}$ by setting

$$f_\infty(q) = f(N \frac{\log q}{2\pi i}) \tag{6.1.7}$$

(well defined due to (2)). Then f_∞ can be continued holomorphically at $q = 0$.

We say that f is a modular form of weight $2k$ and level $\Gamma = \Gamma(N)$. Moreover, if $f_\infty(0) = 0$, we say that f is cuspidal.

The space of modular (resp. cuspidal) forms of level Γ and weight $2k$ is denoted by $\mathcal{M}_{2k}(\Gamma)$ (resp. $\mathcal{M}^0_{2k}(\Gamma)$) and we set

$$\mathcal{M}(\Gamma)=\bigoplus_{k\geq 0}\mathcal{M}_{2k}(\Gamma)\qquad \mathcal{M}^0(\Gamma)=\bigoplus_{k\geq 0}\mathcal{M}^0_{2k}(\Gamma) \tag{6.1.8}$$

If N' is a multiple of N, we have a morphism $\mathcal{M}(\Gamma(N))\to\mathcal{M}(\Gamma(N'))$ (resp. $\mathcal{M}^0(\Gamma(N))\to\mathcal{M}^0(\Gamma(N')))$. We define $\mathcal{M}$ (resp. $\mathcal{M}^0$) to be the direct limit

$$\mathcal{M}=\varinjlim \mathcal{M}(\Gamma(N))\qquad \mathcal{M}^0=\varinjlim \mathcal{M}^0(\Gamma(N)) \tag{6.1.9}$$

In [10], Connes and Moscovici have introduced the algebra $\mathcal{A}(\Gamma)$ of Hecke operator forms of level Γ along with a natural embedding $\mathcal{H}(\Gamma)\hookrightarrow\mathcal{A}(\Gamma)$ of the usual algebra of Hecke operators $\mathcal{H}(\Gamma)$.

Definition 6.3. *A Hecke operator form of level Γ is a function*

$$F:\Gamma\backslash GL_2^+(\mathbb{Q})\longrightarrow\mathcal{M}\qquad \Gamma\alpha\mapsto F_\alpha\in\mathcal{M} \tag{6.1.10}$$

with finite support and satisfying the covariance condition

$$F_\alpha|\gamma=F_{\alpha\gamma}\qquad \forall\ \alpha\in GL_2^+(\mathbb{Q})\qquad \gamma\in\Gamma \tag{6.1.11}$$

The Hecke operator form is said to be cuspidal if

$$F_\alpha\in\mathcal{M}^0\qquad \forall\ \alpha\in GL_2^+(\mathbb{Q}) \tag{6.1.12}$$

The Hecke operator forms of level Γ form an algebra $\mathcal{A}(\Gamma)$ under the product

$$(F^1*F^2)_\alpha=\sum_{\alpha_2\alpha_1=\alpha}F^1_{\alpha_1}\cdot F^2_{\alpha_2}|\alpha_1\qquad F^1,F^2\in\mathcal{A}(\Gamma) \tag{6.1.13}$$

where α_1, $\alpha_2 \in GL_2^+(\mathbb{Q})$. Now, $\mathcal{H}(\Gamma)$ may be thought of as the algebra of functions from the set of double cosets of Γ to $\mathbb{C}$ having finite support. Then $\mathcal{H}(\Gamma)$ embeds into $\mathcal{A}(\Gamma)$ as

$$j : \mathcal{H}(\Gamma) \hookrightarrow \mathcal{A}(\Gamma) \qquad j(h)_\alpha = h(\Gamma\alpha\Gamma) \qquad \alpha \in GL_2^+(\mathbb{Q}) \tag{6.1.14}$$

The cuspidal Hecke operators form an ideal in $\mathcal{A}(\Gamma)$, which we denote by $\mathcal{A}^0(\Gamma)$. Also, we note that $\mathcal{M}(\Gamma)$ has the structure of an $\mathcal{A}(\Gamma)$-module given by

$$F * f = \sum_{\alpha \in \Gamma \backslash GL_2^+(\mathbb{Q})} F_\alpha \cdot f|\alpha \tag{6.1.15}$$

for $F \in \mathcal{A}(\Gamma)$, $f \in \mathcal{M}(\Gamma)$. $\mathcal{M}^0(\Gamma)$ is a submodule of $\mathcal{M}(\Gamma)$ and the ideal $\mathcal{A}^0(\Gamma)$ takes all elements in $\mathcal{M}(\Gamma)$ to $\mathcal{M}^0(\Gamma)$. Finally, if N' is a multiple of N, there exist natural morphisms

$$f(N', N) : \mathcal{A}(\Gamma(N)) \longrightarrow \mathcal{A}(\Gamma(N')) \qquad f(M', M) : \mathcal{H}(\Gamma(N)) \longrightarrow \mathcal{H}(\Gamma(N')) \tag{6.1.16}$$

and we can form the direct limit algebras

$$\mathcal{A} = \varinjlim \mathcal{A}(\Gamma(N)) \qquad \mathcal{H} = \varinjlim \mathcal{H}(\Gamma(N)) \tag{6.1.17}$$

Finally, we recall the definition of the Hopf algebra $\mathcal{H}_1$ of Connes and Moscovici that acts on the algebra $\mathcal{A}(\Gamma)$. This Hopf algebra $\mathcal{H}_1$ belongs to the family of algebras $\{\mathcal{H}_n\}_{n\geq 1}$ defined by Connes and Moscovici in [12], where it is interpreted as the "Hopf algebra of codimension one foliations". The Hopf algebra $\mathcal{H}_1$ is the universal enveloping algebra of the Lie algebra $\mathfrak{L}_1$ with generators X, Y, δ_n, $n \geq 1$, satisfying the relations

$$[Y, X] = X \qquad [Y, \delta_n] = n\delta_n \qquad [X, \delta_n] = \delta_{n+1} \qquad [\delta_k, \delta_l] = 0 \quad \forall\, k, l \in \mathbb{N} \tag{6.1.18}$$

along with the coproducts

$$\begin{gathered} \Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \qquad \Delta(Y) = Y \otimes 1 + 1 \otimes Y \\ \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{gathered} \tag{6.1.19}$$

and the antipode

$$S(Y) = -Y \qquad S(X) = -X + \delta_1 Y \qquad S(\delta_1) = -\delta_1 \tag{6.1.20}$$

For any $f \in \mathcal{M}_k$, the operator X is defined as

$$X(f) = \frac{1}{2\pi i}\left(\frac{d}{dz} f - (1/6)\frac{d}{dz}(\log \Delta)\, Y(f)\right) \tag{6.1.21}$$

where Y is the grading operator $Y(f) = \frac{k}{2} f$ and $\Delta(z)$ is the modular discriminant

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} \qquad q = e^{2\pi i z} \tag{6.1.22}$$

which is a modular form of weight 12 and level $\Gamma(1) = SL_2(\mathbb{Z})$. It may be checked that X defines an operator $X : \mathcal{M}_k \to \mathcal{M}_{k+2}$. We set

$$\tilde{X}(f) = (2\pi i) \cdot X(f) \tag{6.1.23}$$

Moreover, the operator X (and hence $\tilde{X}$) determines a derivation, called the Ramanujan derivation. Now, given $F \in \mathcal{A}(\Gamma)$, $\mathcal{H}_1$ acts on $\mathcal{A}(\Gamma)$ as follows; for $\alpha \in G_2^+(\mathbb{Q})$

and $F \in \mathcal{A}(\Gamma)$;

$$X(F)_\alpha = X(F_\alpha) \qquad Y(F)_\alpha = Y(F_\alpha) \qquad \delta_1(F)_\alpha = \mu_\alpha \cdot F \tag{6.1.24}$$

where $\mu_\alpha = (1/12\pi i)\frac{d}{dz}\left(\log \frac{\Delta|\alpha}{\Delta}\right)$. Note that μ_α measures the difference

$$\mu_\alpha \cdot Y(f) = X(f) - X(f|_k\alpha^{-1})|_{k+2}\alpha \tag{6.1.25}$$

whence it follows directly that $\mu_\alpha = 0$ for all $\alpha \in SL_2(\mathbb{Z})$. For sake of convenience, set

$$\tilde{\mu}_\alpha = (2\pi i) \cdot \mu_\alpha \qquad \forall\ \alpha \in G_2^+(\mathbb{Q}) \tag{6.1.26}$$

Remark: Notice that while the operators X and Y can be directly defined on a modular form $f \in \mathcal{M}$, the operators δ_n, $n \geq 1$ cannot. The definition of δ_1 only makes sense when we apply it to a function from the set of cosets $\Gamma\backslash G_2^+(\mathbb{Q})$ to $\mathcal{M}$. Since $\mu_\alpha = 0$ for all $\alpha \in SL_2(\mathbb{Z})$, the operator δ_1 (and hence the operators δ_n, inductively defined by the relation $[X, \delta_n] = \delta_{n+1}$) vanishes whenever we use a coset representative $\Gamma\alpha$ with $\alpha \in SL_2(\mathbb{Z})$. It also follows that the algebra $\mathcal{A}(\Gamma)$ carries a Hopf action of the finitely generated Hopf algebra $\mathfrak{h}_1$, i.e. the universal enveloping algebra of the Lie Algebra with two generators X, Y, with relation $[Y, X] = X$ and coproducts derived from $\mathcal{H}_1$ by setting all $\delta_n = 0$.

6.2 The Enriched Archimedean complex with Hopf action

Let $X(N)$, $N \geq 1$ denote the N-th modular curve, which is the compactification of $\Gamma(N)\backslash\mathbb{H}$. Now define

$$K_N^{i,j,k} = \bigoplus_{l \geq i-j-1} \mathcal{M}_l(\Gamma_N) \otimes_{\mathbb{R}} \bigoplus_{a \leq b, a+b=j+1, |a-b| \leq 2k-i} (\Omega^{a,b}_{X(N)} + \Omega^{b,a}_{X(N)})_{\mathbb{R}} \left(\frac{1+j-i}{2}\right) \tag{6.2.1}$$

We define the two differentials, given $f \otimes \omega \in K_N^{i,j,k}$, we set

$$\begin{aligned} d' : K_N^{i,j,k} &\longrightarrow K_N^{i+1,j+1,k+1} \qquad (f \otimes \omega) \mapsto f \otimes d'(\omega) \\ d'' : K_N^{i,j,k} &\longrightarrow K_N^{i+1,j+1,k} \qquad (f \otimes \omega) \mapsto f \otimes d''(\omega) \end{aligned} \tag{6.2.2}$$

For any integers $N, N' \geq 1$, we have the projection maps $p : X(NN') \to X(N)$. We can define morphisms:

$$\mathcal{M}_l(\Gamma_N) \otimes (\Omega^{a,b}_{X(N)} + \Omega^{b,a}_{X(N)})_{\mathbb{R}} \left(\frac{1+j-i}{2}\right) \to \mathcal{M}_l(\Gamma_{NN'}) \otimes (\Omega^{a,b}_{X(NN')} + \Omega^{b,a}_{X(NN')})_{\mathbb{R}} \left(\frac{1+j-i}{2}\right) \tag{6.2.3}$$

by tensoring pullback maps p^* on differential forms with the inclusions $\mathcal{M}_l(\Gamma_N) \hookrightarrow \mathcal{M}_l(\Gamma_{NN'})$. Define $K_{en}^{i,j,k}$ to be the direct limit of this system

$$K_{en}^{i,j,k} = \varinjlim_{N} K_N^{i,j,k} \tag{6.2.4}$$

Definition 6.4. *For a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$, define $\mathcal{B}^{i,j,k}(\Gamma)$ to be the set of all functions of finite support*

$$F : \Gamma \backslash G_2^+(\mathbb{Q}) \longrightarrow K_{en}^{i,j,k} \tag{6.2.5}$$

satisfying the following covariance condition: If $F_\alpha = \sum_{l=1}^m f_l \otimes \omega_l \in K_{en}^{i,j,k}$*, then*

$$F_{\alpha\gamma} = \sum_{l=1}^{m} f_l|\gamma \otimes \omega_l \tag{6.2.6}$$

where F_α *denotes the function* F *evaluated on the coset* $\Gamma\alpha$ *for some* $\alpha \in G_2^+(\mathbb{Q})$*. Again, we can make* $\mathcal{B}^{i,j,k}(\Gamma)$ *into a complex by defining differentials*

$$\begin{aligned} d' : \mathcal{B}^{i,j,k}(\Gamma) \longrightarrow \mathcal{B}^{i+1,j+1,k+1}(\Gamma) \qquad d'(F)_\alpha = d'(F_\alpha) \\ d'' : \mathcal{B}^{i,j,k}(\Gamma) \longrightarrow \mathcal{B}^{i+1,j+1,k}(\Gamma) \qquad d''(F)_\alpha = d''(F_\alpha) \end{aligned} \tag{6.2.7}$$

Set $\mathbb{B}^{i,j}(\Gamma) = \oplus_k \mathcal{B}^{i,j,k}(\Gamma)$ *and* $\mathbb{B}^*(\Gamma) = \oplus_{i+j=*}\mathbb{B}^{i,j}(\Gamma)$*. This gives us a complex* $(\mathbb{B}^*(\Gamma), d' + d'')$*.*

For sake of simplicity, we shall frequently use the expression $f_\alpha \otimes \omega_\alpha$ to denote the sum $F_\alpha = \sum_{l=1}^k f_i \otimes \omega_i$.

Our next step is to define an algebra $\mathcal{A}_T(\Gamma)$ which is a variant of the modular hecke algebra $\mathcal{A}(\Gamma)$. We will show that $\mathbb{B}^*(\Gamma)$ is a module over $\mathcal{A}_T(\Gamma)$ and that $\mathcal{H}_1$ acts on both $\mathcal{A}_T(\Gamma)$ and $\mathbb{B}^*(\Gamma)$ and that the action is well behaved (or "flat") in a sense we will make precise in Definition 6.12. First, we consider the multiplicative semigroup consisting of all *non negative powers* of $(2\pi i)^{-1}$ in $\mathbb{C}$ and denote this semigroup by T_+. Then $\mathbb{R}[T_+]$ denotes the free $\mathbb{R}$-module over T_+, considered as a "semigroup ring" (See Remark 6.9 for an explanation of why we restrict ourselves to this semigroup).

Definition 6.5. *Let* $\Gamma \subset SL_2(\mathbb{Z})$ *be a congruence subgroup. Denote by* $\mathcal{A}_T(\Gamma)$ *the set of all functions of finite support*

$$F : \Gamma\backslash G_2^+(\mathbb{Q}) \longrightarrow \mathcal{M} \otimes_{\mathbb{R}} \mathbb{R}[T_+] \tag{6.2.8}$$

satisfying the following covariance condition: If $G_\alpha = \sum_{l=1}^m g_l \otimes \varepsilon_l$*, then, for any* $\gamma \in \Gamma$*,*

$$G_{\alpha\gamma} = \sum_{l=1}^{m} g_l|\gamma \otimes \varepsilon_l \tag{6.2.9}$$

For simplicity, we will forgo the summation signs and write the sum $G_\alpha = \sum_{l=1}^m g_l \otimes \varepsilon_l$ *simply as* $g_\alpha \otimes \varepsilon_\alpha$*. Also, we define the submodule* $\mathcal{A}_T^0(\Gamma)$ *of all functions in* $\mathcal{A}_T(\Gamma)$ *whose values lie in the cuspidal part* $\mathcal{M}^0 \otimes \mathbb{R}[T_+]$*.*

Proposition 6.6. *(1)* $\mathcal{A}_T(\Gamma)$ *is an associative algebra: Given* $F, F' \in \mathcal{A}_T(\Gamma)$*, with* $F_\rho = f_\rho \otimes \varepsilon_\rho$ *and* $G_\rho = f'_\rho \otimes \varepsilon'_\rho$ *for all* $\rho \in G_2^+(\mathbb{Q})$*, the product structure is given by:*

$$(F * F')_\alpha = \sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_\beta \cdot f'_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \tag{6.2.10}$$

(2) The Hopf Algebra $\mathcal{H}_1$ *acts on* $\mathcal{A}_T(\Gamma)$ *as follows:*

$$\begin{gathered} X(F)_\alpha = \tilde{X}(f_\alpha) \otimes (2\pi i)^{-1}\varepsilon_\alpha \qquad Y(F)_\alpha = Y(f_\alpha) \otimes \varepsilon_\alpha \\ \delta_1(F)_\alpha = \tilde{\mu}_\alpha \cdot f_\alpha \otimes (2\pi i)^{-1}\varepsilon_\alpha \end{gathered} \tag{6.2.11}$$

Moreover, the action of $\mathcal{H}_1$ *on* $\mathcal{A}_T(\Gamma)$ *is flat in the sense that, given* F*,*$F' \in \mathcal{A}_T(\Gamma)$ *and* $h \in \mathcal{H}_1$*, with* $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$*, we have*

$$h(F * F') = \sum h_{(1)}(F) * h_{(2)}(F') \tag{6.2.12}$$

Proof. (1) We consider the module $\mathcal{M} \otimes \mathbb{R}[T_+]$ and the following maps: given $f \in \mathcal{M}$, $\varepsilon \in \mathbb{R}[T_+]$ and $\rho \in G_2^+(\mathbb{Q})$, we let

$$\begin{aligned} \psi_{f\otimes\varepsilon} : \mathcal{M} \otimes \mathbb{R}[T_+] \longrightarrow \mathcal{M} \otimes \mathbb{R}[T_+] \qquad & f' \otimes \varepsilon' \mapsto f \cdot f' \otimes \varepsilon\varepsilon' \\ T_\rho : \mathcal{M} \otimes \mathbb{R}[T_+] \longrightarrow \mathcal{M} \otimes \mathbb{R}[T_+] \qquad & f' \otimes \varepsilon' \mapsto f'|\rho \otimes \varepsilon' \end{aligned} \tag{6.2.13}$$

We choose $\gamma \in \Gamma$, it now follows:

$$\begin{aligned}(F * F')_{\gamma\alpha} &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_\beta \cdot f'_{\gamma\alpha\beta^{-1}} | \beta) \otimes \varepsilon_\beta \varepsilon'_{\gamma\alpha\beta^{-1}} \\ &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \psi_{f_\beta \otimes \varepsilon} \circ T_\beta (f'_{\gamma\alpha\beta^{-1}} \otimes \varepsilon'_{\gamma\alpha\beta^{-1}}) \\ &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \psi_{f_\beta \otimes \varepsilon} \circ T_\beta (f'_{\alpha\beta^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}}) \\ &= (F * F')_\alpha\end{aligned} \tag{6.2.14}$$

For any $\beta \in G_2^+(\mathbb{Q})$, it follows that:

$$\begin{aligned}(f_{\gamma\beta} \cdot f'_{\alpha\beta^{-1}\gamma^{-1}} | \gamma\beta) \otimes \varepsilon_{\gamma\beta} \varepsilon'_{\alpha\beta^{-1}\gamma^{-1}} &= \psi_{f_{\gamma\beta} \otimes \varepsilon_{\gamma\beta}} \circ T_{\gamma\beta} (f'_{\alpha\beta^{-1}\gamma^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}\gamma^{-1}}) \\ &= \psi_{f_\beta \otimes \varepsilon_\beta} \circ T_{\gamma\beta} \circ T_{\gamma^{-1}} (f'_{\alpha\beta^{-1}} \otimes \varepsilon'_{\alpha\beta^{-1}}) \\ &= (f_\beta \cdot f'_{\alpha\beta^{-1}} | \beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}}\end{aligned} \tag{6.2.15}$$

Hence the expression for $(F * F')_\alpha$ is independent of the choice of coset representatives β. Finally, we check the covariance condition:

$$\begin{aligned}(F * F')_{\alpha\gamma} &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_\beta \cdot f'_{\alpha\gamma\beta^{-1}} | \beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\gamma\beta^{-1}} \\ &= \textstyle\sum_{\delta \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_{\delta\gamma} \cdot f'_{\alpha\delta^{-1}} | \delta\gamma) \otimes \varepsilon_{\delta\gamma} \varepsilon'_{\alpha\delta^{-1}} \qquad (\beta = \delta\gamma) \\ &= \textstyle\sum_{\delta \in \Gamma \backslash G_2^+(\mathbb{Q})} \psi_{f_{\delta\gamma} \otimes \varepsilon_{\delta\gamma}} \circ T_{\delta\gamma} (f'_{\alpha\delta^{-1}} \otimes \varepsilon'_{\alpha\delta^{-1}}) \\ &= \textstyle\sum_{\delta \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_\delta \cdot f'_{\alpha\delta^{-1}}) | \gamma \otimes \varepsilon_\delta \varepsilon'_{\alpha\delta^{-1}}\end{aligned} \tag{6.2.16}$$

(2) We check this on the generators. The Lie Algebra relations $[Y, X] = X$, $[Y, \delta_1] = \delta_1$ and $[\delta_k, \delta_l] = 0 \ \forall \ k, l \in \mathbb{N}$ may be checked directly as in Proposition 6.10 below. We

check the coproduct relations. Choose F, $F' \in \mathcal{A}_T(\Gamma)$. Then

$$\begin{aligned}
X(F*F')_\alpha &= \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})} \tilde{X}(f_\beta\cdot f'_{\alpha\beta^{-1}}|\beta)\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&= \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})} \tilde{X}(f_\beta)\cdot f'_{\alpha\beta^{-1}}|\beta\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&+ \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})} f_\beta\cdot\tilde{X}(f'_{\alpha\beta^{-1}}|\beta)\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&= (X(F)*F')_\alpha + \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})}(f_\beta\cdot\tilde{X}(f'_{\alpha\beta^{-1}})|\beta)\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&\qquad\qquad - \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})}(f_\beta(\tilde{\mu}_{\beta^{-1}}\cdot Y(f'_{\alpha\beta^{-1}}))|\beta)\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&= (X(F)*F')_\alpha + (F*X(F'))_\alpha \\
&+ \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})}(\tilde{\mu}_\beta\cdot f_\beta)\cdot Y(f'_{\alpha\beta^{-1}})|\beta\otimes(2\pi i)^{-1}\varepsilon_\beta\varepsilon'_{\alpha\beta^{-1}} \\
&\qquad\qquad\qquad (\text{as } \tilde{\mu}_{\beta^{-1}\beta}=0=\tilde{\mu}_{\beta^{-1}}|\beta+\tilde{\mu}_\beta) \\
&= (X(F)*F')_\alpha + (F*X(F'))_\alpha + (\delta_1(F)*Y(F'))_\alpha
\end{aligned} \tag{6.2.17}$$

It can be easily checked that both Y and δ_1 are derivations on the algebra $\mathcal{A}_T(\Gamma)$ and hence the action of $\mathcal{H}_1$ on $\mathcal{A}_T(\Gamma)$ is flat.

□

Corollary 6.7. *(1) $\mathcal{A}_T^0(\Gamma)$ is an ideal in $\mathcal{A}_T(\Gamma)$, which we shall refer to as the cuspidal ideal.*

(2) The cuspidal ideal $\mathcal{A}_T^0(\Gamma)$ is invariant under the action of $\mathcal{H}_1$.

Proof. (1) follows directly from the definition of the product. From [10], we know that $\tilde{X}$ preserves cuspidal modular forms. From the expression of the action of X on $\mathcal{A}_T(\Gamma)$, it is clear that the $\mathcal{H}_1$ preserves $A_T^0(\Gamma)$.

□

Proposition 6.8. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let $G\in\mathcal{A}(\Gamma)$, $F\in\mathbb{B}^*(\Gamma)$. Let $G_\alpha = g_\alpha\otimes\varepsilon_\alpha$ for any $\alpha\in G_2^+(\mathbb{Q})$. Let $F\in\mathcal{B}^{i,j,k}(\Gamma)$ for some*

$i, j, k \in \mathbb{Z}$ and let $F_\alpha = f_\alpha \otimes \omega_\alpha$. Then, we have a module action of $\mathcal{A}(\Gamma)$ on $\mathbb{B}^*(\Gamma)$ as

$$(G * F)_\alpha = \sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (G_\beta \cdot f_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \omega_{\alpha\beta^{-1}} \tag{6.2.18}$$

where the right hand side of (6.2.18) *belongs to the sum $\mathcal{B}^{i,j,k}(\Gamma) \oplus \mathcal{B}^{i+2,j,k+1}(\Gamma) \oplus \mathcal{B}^{i+4,j,k+2}(\Gamma) \oplus \ldots$. (It is understood that if the summand $\mathcal{B}^{i+2l,j,k+l}(\Gamma)$ vanish, the corresponding term on the right hand side is taken to be zero).*

Proof. We consider the tensor product $\mathcal{M} \otimes \Omega_T$, where Ω_T is the direct limit of the sum of all twisted real differentials, i.e.

$$\Omega_T = \varinjlim_{N} \, (\Omega^{a,b}_{X(N)} + \Omega^{b,a}_{X(N)})_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}[T_+] \tag{6.2.19}$$

For any $\alpha \in G_2^+(\mathbb{Q})$ and any function $g \otimes \varepsilon \in \mathcal{M} \otimes \mathbb{R}[T_+]$, we define functions

$$\begin{aligned} &\psi_{g\otimes\varepsilon} : \mathcal{M} \otimes \Omega_T \to \mathcal{M} \otimes \Omega_T \qquad && f \otimes \omega \mapsto g \cdot f \otimes \varepsilon\omega \\ &T_\alpha : \mathcal{M} \otimes \Omega_T \to \mathcal{M} \otimes \Omega_T \qquad && f \otimes \omega \mapsto f|\alpha \otimes \omega \end{aligned} \tag{6.2.20}$$

To prove that the module action is well defined, we check that, for $\gamma \in \Gamma$,

$$\begin{aligned} (G * F)_{\gamma\alpha} &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (g_\beta \cdot f_{\gamma\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \omega_{\gamma\alpha\beta^{-1}} \\ &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \psi_{g_\beta \otimes \varepsilon_\beta} \circ T_\beta (f_{\gamma\alpha\beta^{-1}} \otimes \omega_{\gamma\alpha\beta^{-1}}) \\ &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \psi_{g_\beta \otimes \varepsilon_\beta} \circ T_\beta (f_{\alpha\beta^{-1}} \otimes \omega_{\alpha\beta^{-1}}) \\ &= \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} g_\beta \cdot f_{\alpha\beta^{-1}}|\beta \otimes \varepsilon_\beta \omega_{\alpha\beta^{-1}} = (G * F)_\alpha \end{aligned} \tag{6.2.21}$$

This action is also independent of the choice of coset representatives β, i.e.

$$\begin{aligned}(g_{\gamma\beta}\cdot f_{\alpha\beta^{-1}\gamma^{-1}}|\gamma\beta)\otimes\varepsilon_{\gamma\beta}\omega_{\alpha\beta^{-1}\gamma^{-1}} &= (g_{\beta}\cdot f_{\alpha\beta^{-1}\gamma^{-1}}|\gamma\beta)\otimes\varepsilon_{\beta}\omega_{\alpha\beta^{-1}\gamma^{-1}}\\ &= \psi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\gamma\beta}(f_{\alpha\beta^{-1}\gamma^{-1}}\otimes\omega_{\alpha\beta^{-1}\gamma^{-1}})\\ &= \psi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\gamma\beta}\circ T_{\gamma^{-1}}(f_{\alpha\beta^{-1}}\otimes\omega_{\alpha\beta^{-1}})\\ &= \psi_{g_\beta\otimes\varepsilon_\beta}\circ T_{\beta}(f_{\alpha\beta^{-1}}\otimes\omega_{\alpha\beta^{-1}})\\ &= g_\beta\cdot f_{\alpha\beta^{-1}}|\beta\otimes\varepsilon_\beta\omega_{\alpha\beta^{-1}}\end{aligned} \tag{6.2.22}$$

Finally, we check the covariance condition, for $\gamma\in\Gamma$,

$$\begin{aligned}(G*F)_{\alpha\gamma} &= \textstyle\sum_{\beta\in\Gamma\backslash G_2^+(\mathbb{Q})}(g_\beta\cdot f_{\alpha\gamma\beta^{-1}}|\beta\otimes\varepsilon_\beta\omega_{\alpha\gamma\beta^{-1}})\\ &= \textstyle\sum_{\delta\in\Gamma\backslash G_2^+(\mathbb{Q})}(g_{\delta\gamma}\cdot f_{\alpha\delta^{-1}}|\delta\gamma\otimes\varepsilon_{\delta\gamma}\omega_{\alpha\delta^{-1}}) \qquad (\beta=\delta\gamma)\\ &= \textstyle\sum_{\delta\in\Gamma\backslash G_2^+(\mathbb{Q})}\psi_{g_{\delta\gamma}\otimes\varepsilon_{\delta\gamma}}(f_{\alpha\delta^{-1}}|\delta\gamma\otimes\omega_{\alpha\delta^{-1}})\\ &= \textstyle\sum_{\delta\in\Gamma\backslash G_2^+(\mathbb{Q})}(g_\delta\cdot f_{\alpha\delta^{-1}}|\delta)|\gamma\otimes\omega_{\alpha\delta^{-1}}\end{aligned} \tag{6.2.23}$$

Note that the product lies entirely in the direct sum $\mathcal{B}^{i,j,k}(\Gamma)\oplus\mathcal{B}^{i+2,j,k+1}(\Gamma)\oplus\dots$, i.e. the direct sum is infinite in one direction only. This is because multiplication by an element of $\mathcal{A}_T(\Gamma)$ cannot increase the twist. □

Remark 6.9. *By looking at the proof of Proposition 6.8, we can see why the semigroup T_+ consisting of all non-negative powers of $(2\pi i)^{-1}$ in Definition 6.5 instead of the full cyclic group of all powers of $(2\pi i)^{-1}$. If we were to allow multiplication by negative powers of $(2\pi i)^{-1}$, say, for instance, by $(2\pi i)$, we could define a map $\mathcal{B}^{i,j,k}(\Gamma)\longrightarrow\mathcal{B}^{i-2,j,k-1}(\Gamma)$. Unfortunately, however, such a map does not commute with differentials d' and d''. Hence, we exclude negative powers of $(2\pi i)^{-1}$ and consider only the semigroup T_+.*

Now, we will define an action of the Lie Algebra $\mathfrak{L}_1$ as defined in Section 6.1, on the terms $\mathbb{B}^{i,j}(\Gamma)$. This action extends to an action of the universal enveloping algebra on $\mathbb{B}^{i,j}(\Gamma)$.

Proposition 6.10. $\mathbb{B}^*(\Gamma)$ *is a module over the universal enveloping algebra of the Lie algebra* $\mathfrak{L}_1$.

Proof. Let $F \in \mathcal{B}^{i,j,k}(\Gamma)$ and $\alpha \in G_2^+(\mathbb{Q})$. We define the following operators

$$\begin{aligned} X : \mathcal{B}^{i,j,k}(\Gamma) \to \mathcal{B}^{i+2,j,k+1}(\Gamma) \quad & X(F)_\alpha = \tilde{X}(f_\alpha) \otimes (2\pi i)^{-1}\omega_\alpha \\ Y : \mathcal{B}^{i,j,k}(\Gamma) \longrightarrow \mathcal{B}^{i,j,k}(\Gamma) \qquad & Y(F)_\alpha = Y(f_\alpha) \otimes \omega_\alpha \\ \delta_1 : \mathcal{B}^{i,j,k}(\Gamma) \longrightarrow \mathcal{B}^{i+2,j,k+1}(\Gamma) \qquad & \delta_1(F)_\alpha = \tilde{\mu}_\alpha \cdot f_\alpha \otimes (2\pi i)^{-1}\omega_\alpha \end{aligned} \tag{6.2.24}$$

where $F_\alpha = f_\alpha \otimes \omega_\alpha$. We obtain

$$\begin{aligned} YX(F)_\alpha &= Y(\tilde{X}(f_\alpha)) \otimes (2\pi i)^{-1}\omega_\alpha \\ XY(F)_\alpha &= \tilde{X}(Y(f_\alpha)) \otimes (2\pi i)^{-1}\omega_\alpha \end{aligned} \tag{6.2.25}$$

and hence it follows that $[Y, X] = X$. Similarly, we can check that $[Y, \delta_1] = \delta_1$. The action of the operators δ_n for $n > 1$ is determined by the relation $[X, \delta_n] = \delta_{n+1}$. Note that the relation

$$\delta_n(F_\alpha) = \sum \tilde{X}^{n-1}(\tilde{\mu}_\alpha) \cdot f_\alpha \otimes (2\pi i)^{-n}\omega_\alpha \in \mathcal{B}^{i+2n,j,k+n}(\Gamma) \tag{6.2.26}$$

holds for $n = 1$. If (6.2.26) holds for n, then

$$\begin{aligned} \delta_{n+1}(F)_\alpha &= X\delta_n(F)_\alpha - \delta_n X(F)_\alpha \\ &= X(\tilde{X}^{n-1}(\tilde{\mu}_\alpha) \cdot f_\alpha \otimes (2\pi i)^{-n}\omega_\alpha) - (\tilde{X}^{n-1}(\tilde{\mu}_\alpha) \cdot \tilde{X}(f_\alpha) \otimes (2\pi i)^{-n-1}\omega_\alpha) \\ &= \tilde{X}^n(\tilde{\mu}_\alpha) \cdot f_\alpha \otimes (2\pi i)^{-n-1}\omega_\alpha \end{aligned} \tag{6.2.27}$$

which proves the result for all n, by induction. From the expression, it is now obvious that $[\delta_k, \delta_l] = 0$ for all $k, l \in \mathbb{N}$. Hence $\mathbb{B}^*(\Gamma)$ is a module over the universal enveloping algebra of $\mathfrak{L}_1$. $\square$

Remark Note that the operator X of Proposition 6.10 is a "composite" of the monodromy operator N on the archimedean complex and the Ramanujan derivation X on modular forms. The definiton of the operator Y reflects both the grading operator on modular forms and $-\Phi$ on the archimedean complex (Φ being the Frobenius), which is defined to be $\Phi(x) = \left(\frac{1+j-i}{2}\right)$ for $x \in K^{i,j,k}$, where the $K^{i,j,k}$'s are the terms of the original archimedean complex we worked with in Chapter 4. One can check that the operators Φ and N on Consani's complex satisfy the relation $[-\Phi, N] = N$, which leads to the comparison with the commutator relation $[Y, X] = X$ for operators X and Y on modular forms as explained above.

Corollary 6.11. *The action of the operators X, Y and δ_n, $n \geq 1$ commutes with the differentials d' and d''.*

Proof. Take $F \in \mathcal{B}^{i,j,k}(\Gamma)$ with $F_\alpha = f_\alpha \otimes \omega_\alpha$ for $\alpha \in G_2^+(\mathbb{Q})$. Then

$$(d'X(F))_\alpha = d'(X(F)_\alpha) = \tilde{X}(f_\alpha) \otimes (2\pi i)^{-1} d'(\omega_\alpha) = X(d'(F))_\alpha \in \mathcal{B}^{i+2,j,k+1}(\Gamma) \tag{6.2.28}$$

and the same commutation relation holds for Y and δ_n, $n \geq 1$.

$\square$

Definition 6.12. *Let M be a module over an algebra A. Suppose that $\mathcal{H}$ is a Hopf algebra acting on both A and M. Then the action of $\mathcal{H}$ on A is said to be flat if*

$$h(a_1 a_2) = \sum h_{(1)}(a_1) h_{(2)}(a_2) \qquad \Delta(h) = \sum h_{(1)} \otimes h_{(2)} \qquad \forall\ h \in \mathcal{H}, a_1, a_2 \in A$$

The action of $\mathcal{H}$ on the system (A, M) is said to be flat if

$$h(am) = \sum h_{(1)}(a)h_{(2)}(m) \qquad \Delta(h) = \sum h_{(1)} \otimes h_{(2)} \qquad \forall\ h \in \mathcal{H}, a \in A, m \in M \tag{6.2.29}$$

In Proposition 6.10 we have already shown that the Hopf algebra $\mathcal{H}_1$ acts on $\mathbb{B}^*(\Gamma)$. We know from Proposition 6.6 that the Hopf algebra $\mathcal{H}_1$ has a flat action on the Hecke algebra $\mathcal{A}_T(\Gamma)$ and we proved in Proposition 6.8 that $\mathbb{B}^*(\Gamma)$ is a module over $\mathcal{A}_T(\Gamma)$. We will now show that the action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$ is flat, following Definition 6.12.

Proposition 6.13. *The action of $\mathcal{H}_1$ on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$ is flat.*

Proof. Choose $F \in \mathcal{B}^{i,j,k}(\Gamma)$ and let $G \in \mathcal{A}_T(\Gamma)$. Let $F_\alpha = f_\alpha \otimes \omega_\alpha$ and $G_\alpha = g_\alpha \otimes \varepsilon_\alpha$ for $\alpha \in G_2^+(\mathbb{Q})$. By definition,

$$X(G * F)_\alpha = \sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \tilde{X}(g_\beta \cdot f_{\alpha\beta^{-1}}|\beta) \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} \in \bigoplus_{l=1}^{\infty} \mathcal{B}^{i+2l,j,k+l}(\Gamma) \tag{6.2.30}$$

Since $\tilde{X}$ is a derivation on $\mathcal{M}$, the right hand side of (6.2.30) is equal to

$$\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} \tilde{X}(g_\beta) \cdot f_{\alpha\beta^{-1}}|\beta \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} + \sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (g_\beta \cdot \tilde{X}(F_{\alpha\beta^{-1}}|\beta)) \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} \tag{6.2.31}$$

$$\begin{aligned}
&= (X(G) * F)_\alpha + \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (g_\beta \cdot \tilde{X}(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} \\
&\qquad\qquad - \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} (g_\beta \cdot (\tilde{\mu}_{\beta^{-1}} \cdot Y(f_{\alpha\beta^{-1}}))|\beta) \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} \\
&= (X(G) * F)_\alpha + (G * X(F))_\alpha \\
&\quad + \textstyle\sum_{\beta \in \Gamma \backslash G_2^+(\mathbb{Q})} ((\tilde{\mu}_\beta \cdot g_\beta) \cdot Y(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-1} \varepsilon_\beta \omega_{\alpha\beta^{-1}} \\
&= (X(G) * F)_\alpha + (G * X(F))_\alpha + (\delta_1(G) * Y(F))_\alpha
\end{aligned} \tag{6.2.32}$$

The result follows easily for the coproducts $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ and $\Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1$.

$\square$

Corollary 6.14. *(1) $\mathcal{A}_T(\Gamma)$ also acts on the right of $\mathbb{B}^*(\Gamma)$ as follows; For $G \in \mathcal{A}_T(\Gamma)$ and $F \in \mathcal{B}^{i,j,k}$ with $G_\rho = g_\rho \otimes \varepsilon_\rho$, $F_\rho = f_\rho \otimes \omega_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$,*

$$(F * G)_\alpha = \sum_{\alpha \in \Gamma \backslash G_2^+(\mathbb{Q})} (f_\beta \cdot g_{\alpha\beta^{-1}} | \beta) \otimes \omega_\beta \varepsilon_{\alpha\beta^{-1}} \tag{6.2.33}$$

(2) The right module structure of $\mathbb{B}^*(\Gamma)$ over $\mathcal{A}_T(\Gamma)$ is also flat with respect to the action of $\mathcal{H}_1$, i.e. for any $h \in \mathcal{H}_1$, $F \in \mathbb{B}^*(\Gamma)$ and $G \in \mathcal{A}_T(\Gamma)$, we have

$$h(F * G) = \sum h_{(1)}(F) * h_{(2)}(G) \qquad \textit{if} \qquad \Delta(h) = \sum h_{(1)} \otimes h_{(2)} \tag{6.2.34}$$

Proof. It can be checked directly that both expressions are well defined and that the action of the Hopf algebra is flat. $\square$

It follows that $\mathbb{B}^*(\Gamma)$ is actually a bimodule over $\mathcal{A}_T(\Gamma)$. We will use this fact in the next section.

Finally, we show that, by "restricting" the product on $\mathcal{A}_T(\Gamma)$ as defined in Proposition 6.6, we can define an action of the smaller Hopf algebra $\mathfrak{h}_1$ (i.e. the universal enveloping algebra of the Lie algebra $\mathfrak{l}_1$ with two generators X, Y satisfying $[Y, X] = X$, and with coproducts $\Delta(X) = 1 \otimes X + X \otimes 1$ and $\Delta(Y) = 1 \otimes Y + Y \otimes 1$) on $\mathcal{A}_T(\Gamma)$. Further, we can make $\mathbb{B}^*(\Gamma)$ into a module over $\mathcal{A}_T(\Gamma)$ in a way such that $\mathfrak{h}_1$ has a flat action on the system $(\mathcal{A}_T(\Gamma), \mathbb{B}^*(\Gamma))$.

Proposition 6.15. *(1) Let F,$F' \in \mathcal{A}_T(\Gamma)$. Suppose that $F_\alpha = f_\alpha \otimes \varepsilon_\alpha$, $F'_\alpha = f'_\alpha \otimes \varepsilon'_\alpha$, for any $\alpha \in G_2^+(\mathbb{Q})$. Then $\mathcal{A}_T(\Gamma)$ becomes an algebra under the product*

$$(F * F)_\alpha = \sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} (f_\beta \cdot f'_{\alpha\beta^{-1}}|\beta) \otimes \varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \tag{6.2.35}$$

Whenever we use the product of (6.2.35)*, we will refer to the algebra $\mathcal{A}_T(\Gamma)$ as $\mathcal{A}_T^r(\Gamma)$.*

(2) We define the operators X and Y on $\mathcal{A}_T^r(\Gamma)$ as

$$X(F)_\alpha = \tilde{X}(f_\alpha) \otimes (2\pi i)^{-1}\varepsilon_\alpha \qquad Y(F)_\alpha = Y(f_\alpha) \otimes \varepsilon_\alpha \tag{6.2.36}$$

Then both X and Y are derivations on $\mathcal{A}_T^r(\Gamma)$ and the above defines a flat action of the Hopf algebra $\mathfrak{h}_1$ on $\mathcal{A}_T^r(\Gamma)$.

Proof. (1) follows in the exact same manner as the proof of Proposition 6.6(1). To prove (2), we note that $X(f'_{\alpha\beta^{-1}}|\beta) = X(f'_{\alpha\beta^{-1}})|\beta$ because $\mu_{\beta^{-1}} = 0$ (as $\beta \in SL_2(\mathbb{Z})$). Hence

$$\begin{aligned} X(F * F')_\alpha &= \textstyle\sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \tilde{X}(f_\beta \cdot f'_{\alpha\beta^{-1}}|\beta) \otimes (2\pi i)^{-1}\varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\ &= \textstyle\sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} \tilde{X}(f_\beta) \cdot f'_{\alpha\beta^{-1}} \otimes (2\pi i)^{-1}\varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\ &\quad + \textstyle\sum_{\beta \in \Gamma \backslash SL_2(\mathbb{Z})} f_\beta \cdot \tilde{X}(f_{\alpha\beta^{-1}})|\beta \otimes (2\pi i)^{-1}\varepsilon_\beta \varepsilon'_{\alpha\beta^{-1}} \\ &= (X(F) * F')_\alpha + (F * X(F'))_\alpha \end{aligned} \tag{6.2.37}$$

We can also check directly that Y is a derivation.

□

Whenever we use the product of (6.2.35), we will refer to the algebra $\mathcal{A}_T(\Gamma)$ as $\mathcal{A}_T^r(\Gamma)$.

Proposition 6.16. *(1) Let $G \in \mathcal{A}_T^r(\Gamma)$ and $F \in \mathbb{B}^*(\Gamma)$. Let $G_\alpha = g_\alpha \otimes \varepsilon_\alpha$ and*

$F_\alpha = f_\alpha \otimes \omega_\alpha$ for $\alpha \in G_2^+(\mathbb{Q})$. We set

$$(G * F)_\alpha = \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} g_\beta \cdot f_{\alpha\beta^{-1}}|\beta \otimes \varepsilon_\beta \omega_{\alpha\beta^{-1}} \tag{6.2.38}$$

This makes $\mathbb{B}^*(\Gamma)$ a module over $\mathcal{A}_T^r(\Gamma)$. With the module action of (6.2.38)*, we will refer to $\mathbb{B}^*(\Gamma)$ as $\mathbb{B}_r^*(\Gamma)$.*

(2) For $F \in \mathcal{B}_r^{i,j,k}(\Gamma) \subset \mathbb{B}_r^*(\Gamma)$, define operators X and Y as

$$X(F)_\alpha = \tilde{X}(f_\alpha) \otimes (2\pi i)^{-1}\omega_\alpha \qquad Y(F)_\alpha = Y(f_\alpha) \otimes \omega_\alpha \tag{6.2.39}$$

with the right hand side lying in the direct sum $\mathcal{B}_r^{i,j,k}(\Gamma) \oplus \mathcal{B}_r^{i+2,j,k+1}(\Gamma) \oplus \dots$. This defines a flat action of $\mathfrak{h}_1$ on the system $(\mathcal{A}_T^r(\Gamma), \mathbb{B}_r^*(\Gamma))$.

Proof. The proof of (1) is analogous to that of Proposition 6.15(1). (2) also follows similarly; using again the fact that $\tilde{X}(f'_{\alpha\beta^{-1}}|\beta) = \tilde{X}(f'_{\alpha\beta^{-1}})|\beta$, since $\beta \in SL_2(\mathbb{Z})$.

$\square$

With the module action of (6.2.38), we will refer to $\mathbb{B}^*(\Gamma)$ as $\mathbb{B}_r^*(\Gamma)$.

Remark We recall that the universal elliptic curve E, with a map $E \to \mathbb{H}$ to the upper half plane, is defined by letting the fibre over $z \in \mathbb{H}$ be the elliptic curve $\mathbb{C}/\Lambda_z$, Λ_z being the lattice $\{\mathbb{Z} + \mathbb{Z}z\}$. When $\Gamma = \Gamma(N)$ is a congruence subgroup of $SL_2(\mathbb{Z})$ with $N \geq 3$, it is known that Γ acts freely on E and one deduces a morphism $E(\Gamma) \to Y(\Gamma) = \Gamma\backslash\mathbb{H}$ The universal elliptic curve $E(\Gamma)$ of level Γ can be compactified to define $\mathcal{E}(\Gamma)$ which maps to the modular curve $X(\Gamma)$. The results of this section, as well as those of the next, may be carried over to the context of the curves $\mathcal{E}(\Gamma) \to X(\Gamma)$, by pulling back the line bundle of modular forms from $X(\Gamma)$ to $\mathcal{E}(\Gamma)$ and by working with the module of relative differentials $\Omega^{*,*}_{\mathcal{E}(\Gamma)/X(\Gamma)}$.

6.3 Extension of Rankin Cohen brackets

Our goal in this section is to define "Rankin Cohen brackets" RC_n of any order $n \geq 1$ on $\mathbb{B}^*(\Gamma)$:

$$RC_n : \mathbb{B}^*(\Gamma) \otimes \mathbb{B}^*(\Gamma) \longrightarrow \mathcal{A}_T(\Gamma)(1) \tag{6.3.1}$$

where $\mathcal{A}_T(\Gamma)(1) = A_T(\Gamma)(2\pi i)$. We start with the definition of the first Rankin Cohen bracket. If f and g are modular forms of weight k and l respectively, the first Rankin Cohen bracket is defined as

$$RC_1(f,g) = X(f)Y(g) - Y(f)X(g) \tag{6.3.2}$$

In [10], Connes and Moscovici have shown that the extension of the first Rankin Cohen bracket to the modular Hecke Algebra $\mathcal{A}(\Gamma)$ is defined by the generator of the transverse fundamental class $[F] \in HC^2(\mathcal{H}_1)$, which is a class in Hopf cyclic cohomology. For the definition and properties of Hopf cyclic cohomology, see [11] or the Appendix to [10]. Here the class F is given by

$$F = X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y \tag{6.3.3}$$

We set $\delta_2' = \delta_2 - (1/2)\delta_1^2$ in the Hopf Algebra $\mathcal{H}_1$. We will now describe the action of δ_2' on the algebra $\mathcal{A}_T(\Gamma)$ and on $\mathcal{B}^*(\Gamma)$ more explicitly. For this, we set $\omega_4 = -\frac{10}{(2\pi i)^4} G_4$, where G_4 is the classical Eisenstein series

$$G_4(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^4} \tag{6.3.4}$$

We can think of ω_4 as an element of $\mathcal{A}_T(\Gamma)$, by choosing a constant function mapping all cosets in $\Gamma\backslash G_2^+(\mathbb{Q})$ to $(2\pi i)\omega_4 \otimes (2\pi i)^{-1}$.

Lemma 6.17. *(1) The action of δ_2' on $\mathcal{A}_T(\Gamma)$ is an inner derivation, implemented by $\omega_4 \in \mathcal{A}_T(\Gamma)$, i.e.*

$$\delta_2'(G) = [G, \omega_4] \qquad \textit{for any } G \in \mathcal{A}_T(\Gamma) \tag{6.3.5}$$

(2) The action of δ_2' on $\mathbb{B}^*(\Gamma)$ is also implemented by ω_4, i.e. given $F \in \mathbb{B}^*(\Gamma)$ we have

$$\delta_2'(F) = [F, \omega_4] \tag{6.3.6}$$

(The right hand side of (6.3.6) *is well defined because $\mathbb{B}^*(\Gamma)$ is a bimodule over $\mathcal{A}_T(\Gamma)$.)*

(3) The action of δ_2' on $\mathcal{A}_T^r(\Gamma)$ is zero.

Proof. (3) follows directly from the fact that all δ_n, $n \geq 0$ act as 0 on $\mathcal{A}_T^r(\Gamma)$. Now, from the proof of Proposition 10 of [10], it follows that, for any $\gamma \in G_2^+(\mathbb{Q})$

$$\left(\tilde{X}(\tilde{\mu}_\gamma) - \frac{\tilde{\mu}_\gamma^2}{2}\right) \cdot f = 2\pi i(\omega_4 \cdot f - f \cdot \omega_4|\gamma) \tag{6.3.7}$$

Also, from the expression for δ_n given in the proof of Proposition 6.10, it follows that, for $F \in \mathbb{B}^*(\Gamma)$ with $F_\rho = f_\rho \otimes \omega_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$;

$$\delta_2'(F)_\alpha = \delta_2(F)_\alpha - (1/2)\delta_1^2(F)_\alpha = (\tilde{X}(\tilde{\mu}_\alpha) \cdot f_\alpha - \frac{\tilde{\mu}_\alpha^2}{2} \cdot f_\alpha) \otimes (2\pi i)^{-2}\omega_\alpha \tag{6.3.8}$$

Then both (1) and (2) follow from (6.3.8) exactly as in Proposition 10 of [10].

□

In [18], Consani has defined a pairing on the terms of the archimedean complex taking values in $\mathbb{R}(1)$. We will now generalize this pairing using Rankin Cohen brackets, and this pairing will, in fact, take values in the twisted module $\mathcal{A}_T(\Gamma)(1)$. We will also define the pairing in the simpler case of $\mathbb{B}_r^*(\Gamma)$, with the pairing taking values in $\mathcal{A}_T^r(\Gamma)(1)$.

Set, for $m \in \mathbb{Z}$,

$$\epsilon(m) = (-1)^{\frac{m(m+1)}{2}} \tag{6.3.9}$$

and, for a differential form ω of type (a,b), we set

$$C(\omega) = (\sqrt{-1})^{a-b} \tag{6.3.10}$$

For F_1, F_2 in $\mathcal{A}(\Gamma)$, the first Rankin Cohen bracket is given by

$$RC_1(F_1,F_2) = X(F_1) * Y(F_2) - Y(F_1) * X(F_2) - \delta_1 Y(F_1) * Y(F_2) \tag{6.3.11}$$

This leads us to the introduction of a pairing

$$\mathcal{RC}_{1,0} : \mathcal{B}_r^{-i-2,-j,k-1}(\Gamma) \otimes \mathcal{B}_r^{i,j,k+i}(\Gamma) \longrightarrow \mathcal{A}_T^r(\Gamma)(1) \tag{6.3.12}$$

which is defined as follows. Let $F \in \mathcal{B}_r^{-i-2,-j,k-1}(\Gamma)$ and $F' \in \mathcal{B}_r^{i,j,k+i}(\Gamma)$ with $F_\rho = f_\rho \otimes \omega_\rho$ and $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$. Then, for any $\alpha \in G_2^+(\mathbb{Q})$,

$$\begin{aligned}
&\mathcal{RC}_{1,0}(F,F')_\alpha \\
&= \epsilon(1-j)(-1)^k \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(\tilde{X}(f_\beta)Y(f'_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\
&-\epsilon(1-j)(-1)^{k-1} \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(Y(f_\beta)\tilde{X}(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\
&-\epsilon(1-j)(-1)^k \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(\tilde{\mu}_\beta \cdot Y(f_\beta) \cdot (Y(f'_{\alpha\beta^{-1}})|\beta)) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}
\end{aligned} \tag{6.3.13}$$

The integral in the expression above is well defined on the direct limit since the integral of a top dimensional differential form is left unchanged by pullback maps.

We can also define a pairing:

$$\mathcal{RC}_{1,1} : \mathcal{B}_r^{-i,-j,k}(\Gamma) \otimes \mathcal{B}_r^{i-2,j,k+i-1}(\Gamma) \longrightarrow \mathcal{A}_T^r(\Gamma)(1) \tag{6.3.14}$$

as follows: Given $F \in \mathcal{B}_r^{-i,-j,k}(\Gamma)$ and $F' \in \mathcal{B}_r^{i-2,j,k+i-1}(\Gamma)$, we have

$$\begin{aligned}
&\mathcal{RC}_{1,1}(F, F')_\alpha \\
&= \epsilon(1-j)(-1)^{k+1} \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(\tilde{X}(f_\beta)Y(f'_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\
&-\epsilon(1-j)(-1)^{k} \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(Y(f_\beta)\tilde{X}(f_{\alpha\beta^{-1}})|\beta) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \\
&-\epsilon(1-j)(-1)^{k+1} \textstyle\sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})}(\tilde{\mu}_\beta \cdot Y(f_\beta) \cdot (Y(f'_{\alpha\beta^{-1}})|\beta)) \otimes (2\pi i)^{-2} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}}
\end{aligned} \tag{6.3.15}$$

Extending by zero, we have a first Rankin Cohen bracket

$$\mathcal{RC}_1 : (\mathcal{B}_r^{-i-2,-j,k-1}(\Gamma) \oplus \mathcal{B}_r^{-i,-j,k}(\Gamma)) \otimes (\mathcal{B}_r^{i-2,j,k+i-1}(\Gamma) \oplus \mathcal{B}_r^{i,j,k+i}(\Gamma)) \longrightarrow \mathcal{A}_T^r(\Gamma)(1)$$

In general, for the n-th Rankin Cohen bracket, we will have $n+1$ distinct pairings

$$\mathcal{RC}_{n,p} : \mathcal{B}_r^{-i-2(n-p),-j,k-(n-p)}(\Gamma) \otimes \mathcal{B}_r^{i-2p,j,k+i-p}(\Gamma) \longrightarrow \mathcal{A}_T^r(\Gamma)(1) \qquad p = 0, 1, 2, ..., n \tag{6.3.16}$$

and we will extend by zero to define

$$\mathcal{RC}_n : \bigoplus_{p=0}^{n} \mathcal{B}_r^{-i-2p,-j,k-p}(\Gamma) \otimes \bigoplus_{p=0}^{n} \mathcal{B}_r^{i-2p,j,k+i-p}(\Gamma) \longrightarrow \mathcal{A}_T^r(\Gamma)(1) \tag{6.3.17}$$

We also have Rankin Cohen brackets

$$RC_n : \bigoplus_{p=0}^{n} \mathcal{B}^{-i-2p,-j,k-p}(\Gamma) \otimes \bigoplus_{p=0}^{n} \mathcal{B}^{i-2p,j,k+i-p}(\Gamma) \longrightarrow \mathcal{A}_T(\Gamma)(1) \tag{6.3.18}$$

First we consider the case of $\mathbb{B}_r^*(\Gamma)$. In the previous section, we have shown that

$\mathbb{B}_r^*(\Gamma)$ is a bimodule over $\mathcal{A}_T^r(\Gamma)$ and that there is a flat action of $\mathfrak{h}_1$ on the system $(\mathcal{A}_T^r(\Gamma), \mathbb{B}_r^*(\Gamma))$. Hence, following [9, (1.5)], the n-th Rankin Cohen bracket is defined as follows: Let $p \in \{0,1,2,...,n\}$, and let $F \in \mathcal{B}_r^{-i-2(n-p),-j,k-(n-p)}(\Gamma)$, $F' \in \mathcal{B}_r^{i-2p,j,k+i-p}(\Gamma)$, and let $F_\rho = f_\rho \otimes \omega_\rho$, $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$. Then, for any $\alpha \in G_2^+(\mathbb{Q})$,

$$\mathcal{RC}_{n,p}(F,F')_\alpha = \sum_{\beta\in\Gamma\backslash SL_2(\mathbb{Z})} \sum_{l=0}^{n} \epsilon(1-j)(-1)^{k-n+p+l}$$
$$\left(\left(\frac{S(\tilde{X})^l}{l!}(2Y+l)_{n-l}(f_\beta)\right)\cdot\left(\frac{\tilde{X}^{n-l}}{(n-l)!}(2Y+n-l)_l(f'_{\alpha\beta^{-1}})\right)|\beta\right)\otimes(2\pi i)^{-n-1}\int\omega_\beta\wedge C\omega'_{\alpha\beta^{-1}} \tag{6.3.19}$$

where S is the antipode of the Hopf algebra and $(Y+K)_L = (Y+K)(Y+K-1)...(Y+K-L+1)$. Extending by zeroes, we can define the Rankin Cohen brackets $\mathcal{RC}_n$ for all n.

Proposition 6.18. *Choose $u \in \mathcal{A}_T^r(\Gamma)$ such that u is invertible and suppose that*

$$X(u) = 0 \qquad Y(u) = 0 \qquad \delta_1(u^{-1}\delta_1(u)) = 0 \qquad \delta_2'(u) = 0$$

1. *For any n and any elements F, $F' \in \mathbb{B}_r^*(\Gamma)$, we have*

$$\begin{gathered}\mathcal{RC}_n(Fu,F') = \mathcal{RC}_n(F,uF')\\ \mathcal{RC}_n(uF,F') = u\mathcal{RC}_n(F,F') \qquad \mathcal{RC}_n(F,F'u) = \mathcal{RC}_n(F,F')u\end{gathered} \tag{6.3.20}$$

2. *Let α_u be the inner automorphism defined by $u \in \mathcal{A}_T^r(\Gamma)$; then*

$$\mathcal{RC}_n(\alpha_u(F),\alpha_u(F')) = \alpha_u(\mathcal{RC}_n(F,F')) \tag{6.3.21}$$

Proof. Note that since u is an invertible element of $\mathcal{A}_T^r(\Gamma)$, for any $\rho \in G_2^+(\mathbb{Q})$, we must have $u_\rho = g_\rho \otimes 1$, because $\mathbb{R}[T_+]$ contains only non negative powers of $(2\pi i)^{-1}$.

Hence, if we choose $F \in \mathcal{B}_r^{i,j,k}(\Gamma)$, both $u * F$ and $\alpha_u(F)$ lie in the same $\mathcal{B}_r^{i,j,k}(\Gamma)$. The result now follows exactly as in the proof of [9, Lemma 2].

□

We now proceed to the general case, where we consider the action of the larger Hopf algebra $\mathcal{H}_1$ on $\mathcal{A}_T(\Gamma)$. For any $a \in \mathcal{A}_T(\Gamma)$, let us denote by L_a (resp. R_a) the operator of left (resp. right) multiplication by a on $\mathbb{B}^*(\Gamma)$ and, by abuse of notation, also on $\mathcal{A}_T(\Gamma)$ itself. Recall that we can think of ω_4 as an element of $\mathcal{A}_T(\Gamma)$, by choosing a constant function mapping all cosets in $\Gamma\backslash G_2^+(\mathbb{Q})$ to $(2\pi i)\omega_4 \otimes (2\pi i)^{-1}$. Also, define an operator N on both $\mathcal{A}_T(\Gamma)$ and $\mathbb{B}^*(\Gamma)$ as: Given $F \in \mathcal{A}_T(\Gamma)$ and $G \in \mathbb{B}^*(\Gamma)$ with $F_\rho = f_\rho \otimes \varepsilon_\rho$ and $G_\rho = g_\rho \otimes \omega_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$, set

$$N(G)_\rho = g_\rho \otimes (2\pi i)^{-1}\omega_\rho \qquad N(F)_\rho = f_\rho \otimes (2\pi i)^{-1}\varepsilon_\rho \tag{6.3.22}$$

Note that if $G \in \mathbb{B}^{i,j,k}(\Gamma)$, $N(G) \in \mathcal{B}^{i+2,j,k+1}(\Gamma)$ and $N(G)$ is understood to be zero if $\mathcal{B}^{i+2,j,k+1}(\Gamma) = 0$.

Define the sequences $\{A_n\}_{n\geq -1}$, $\{B_n\}_{n\geq 0}$ as:

$A_{-1} = 0 \quad A_0 = 1$	$B_0 = 1 \quad B_1 = X$
$A_{n+1} = S(X)A_n - nR_{\omega_4}\left(Y - \frac{n-1}{2}\right)A_{n-1}N$	$B_{n+1} = XB_n - nL_{\omega_4}\left(Y - \frac{n-1}{2}\right)B_{n-1}N$

These sequences differ from those of [9] in the use of the extra operator N. The operator N makes sure that applying either A_n or B_n to an element of $\mathbb{B}^*(\Gamma)$ reduces the twist by n. With this adjustment, we can, once again, define Rankin Cohen brackets,

$$RC_{n,p} : \mathcal{B}^{-i-2(n-p),-j,k-(n-p)}(\Gamma) \otimes \mathcal{B}^{i-2p,j,k+i-p}(\Gamma) \longrightarrow \mathcal{A}_T(\Gamma)(1) \qquad p = 0, 1, 2, ..., n \tag{6.3.23}$$

by the formula: Given $F \in \mathcal{B}^{-i-2(n-p),-j,k-(n-p)}(\Gamma)$, $F' \in \mathcal{B}^{i-2p,j,k+i-p}(\Gamma)$, and let $F_\rho = f_\rho \otimes \omega_\rho$, $F'_\rho = f'_\rho \otimes \omega'_\rho$ for any $\rho \in G_2^+(\mathbb{Q})$. Also, suppose that, for any $0 \leq l \leq n$,

$$\begin{aligned} \left(\tfrac{A_l}{l!}(2Y+l)_{n-l}\right)(F)_\rho &= g_\rho \otimes (2\pi i)^{-l}\omega_\rho \\ \left(\tfrac{B_{n-l}}{(n-l)!}(2Y+n-l)_l\right)(F')_\rho &= g'_\rho \otimes (2\pi i)^{-n+l}\omega'_\rho \end{aligned} \tag{6.3.24}$$

Then, for any $\alpha \in G_2^+(\mathbb{Q})$,

$$RC_{n,p}(F,F')_\alpha = \sum_{\beta \in \Gamma\backslash SL_2(\mathbb{Z})} \sum_{l=0}^{n} \epsilon(1-j)(-1)^{k-n+p+l} g_\beta \cdot g'_{\alpha\beta^{-1}}|\beta \otimes (2\pi i)^{-n-1} \int \omega_\beta \wedge C\omega'_{\alpha\beta^{-1}} \tag{6.3.25}$$

Extending by zeroes, we can define the full Rankin Cohen brackets:

$$RC_n : \bigoplus_{p=0}^{n} \mathcal{B}^{-i-2p,-j,k-p}(\Gamma) \otimes \bigoplus_{p=0}^{n} \mathcal{B}^{i-2p,j,k+i-p}(\Gamma) \longrightarrow \mathcal{A}_T(\Gamma)(1) \tag{6.3.26}$$

Now, as in [9, Lemma 2], we can prove:

Proposition 6.19. *Let the Hopf algebra $\mathcal{H}_1$ act on $\mathbb{B}^*(\Gamma)$ as above, such that δ'_2 is implemented by the inner derivation corresponding to $\omega_4 \in \mathcal{A}_T(\Gamma)$. Further, choose an invertible $u \in \mathcal{A}_T(\Gamma)$ and set $\mu = u^{-1}\delta_1(u)$. Then*

$$X(u) = 0 \qquad Y(u) \qquad \delta_n(u) = 0 \quad n \in \mathbb{N} \tag{6.3.27}$$

1. *For any n and any elements F, $F' \in \mathbb{B}^*(\Gamma)$, we have*

$$\begin{gathered} RC_n(Fu, F') = RC_n(F, uF') \\ RC_n(uF, F') = uRC_n(F,F') \qquad RC_n(F, F'u) = RC_n(F,F')u \end{gathered} \tag{6.3.28}$$

2. Let α_u be the inner automorphism defined by $u \in \mathcal{A}_T(\Gamma)$; then

$$RC_n(\alpha_u(F), \alpha_u(F')) = \alpha_u(RC_n(F, F')) \tag{6.3.29}$$

Proof. As explained before in the proof of Proposition 6.18, u being invertible, if $F \in \mathcal{B}^{i,j,k}(\Gamma)$, both $u * F$ and $\alpha_u(F)$ lie in $\mathcal{B}^{i,j,k}(\Gamma)$. The result now follows exactly as in the proof of [9, Lemma 7]. □

References

[1] Berrick, A. J: The plus-construction and fibrations. *Quart. J. Math. Oxford Ser. (2)* **33** (1982), no. 130, 149–157.

[2] Bloch, S.; Gillet, H.; Soulé, C.; Algebraic cycles on degenerate fibers. Arithmetic geometry (Cortona, 1994), 45–69, *Sympos. Math.*, **XXXVII** Cambridge Univ. Press, Cambridge, 1997.

[3] Block, Jonathan L: Cyclic homology of filtered algebras. *K-Theory* **1** (1987), no. 5, 515–518.

[4] Brown, Kenneth S; Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.* **186** (1974), 419–458.

[5] Brown, Kenneth S.; Gersten, Stephen M; Algebraic K-theory as generalized sheaf cohomology. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. *Lecture Notes in Math.*, **341** Springer, Berlin 1973.

[6] Brylinski, Jean-Luc: A differential complex for Poisson manifolds. *J. Differential Geom.* **28** (1988), no. 1, 93–114.

[7] Brylinski, Jean-Luc: Some examples of Hochschild and cyclic homology. Algebraic groups Utrecht 1986, 33–72, *Lecture Notes in Math.*, **1271**, Springer, Berlin, 1987.

[8] Burghelea, Dan; Vigué-Poirrier: Cyclic homology of commutative algebras I. Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986), 51–72, *Lecture Notes in Math.* **1318** Springer, Berlin, 1988.

[9] Connes, Alain; Moscovici, Henri: Rankin-Cohen brackets and the Hopf algebra of transverse geometry. *Mosc. Math. J.* **4** (2004), no. 1, 111–130, 311.

[10] Connes, Alain; Moscovici, Henri: Modular Hecke algebras and their Hopf symmetry. *Mosc. Math. J.* **4** (2004), no. 1, 67–109, 310.

[11] Connes, Alain; Moscovici, Henri: Cyclic cohomology and Hopf algebras. *Lett. Math. Phys.* **48** (1999), no. 1, 97–108.

[12] Connes, A.; Moscovici, H : Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.* **198** (1998), no. 1, 199–246.

[13] Connes, Alain; Karoubi, Max: Caractère multiplicatif d'un module de Fredholm. *K-Theory* **2** (1988), no. 3, 431–463.

[14] Connes, Alain: Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.* No. **62**, (1985), 257–360.

[15] Consani, Caterina; Marcolli, Matilde: Archimedean cohomology revisited. arXiv:math.AG/0407480.

[16] Consani, Caterina; Marcolli, Matilde: Noncommutative geometry, dynamics, and ∞-adic Arakelov geometry, *Selecta Math. (N.S.)* **10** (2004), no. 2, 167–251.

[17] Consani, Caterina; The local monodromy as a generalized algebraic correspondence. With an appendix by Spencer Bloch. *Doc. Math.* **4** (1999), 65–108.

[18] Consani, C: Double complexes and Euler L-factors, *Compositio Math.* **111** (1998), no. 3, 323–358.

[19] Consani, C: Double Complexes and Euler L-factors on Degenerations of Algebraic Varieties. PhD. Thesis. University of Chicago. 1996.

[20] Cuntz, Joachim; Quillen, Daniel: Cyclic homology and nonsingularity. *J. Amer. Math. Soc.* **8** (1995), no. 2, 373–442.

[21] Deligne, Pierre: Théorie de Hodge. III. *Inst. Hautes Études Sci.* Publ. Math. No. **44** (1974), 5–77.

[22] Deligne, P: "Comparison avec la théorie transcedente, Exp XIV" in Groupes de Monodromie en Géométrie Algébrique (SGA 7 II). *Lecture Notes in Math.* **340**, Springer-Verlag, Berlin-Heidelberg, 1973.

[23] Deligne, Pierre: Théorie de Hodge. II. *Inst. Hautes Études Sci.* Publ. Math. No. **40** (1971), 5–57.

[24] Dors, A. George; A Fréchet algebra example. *Proc. Amer. Math. Soc.* **64**(1977), no. 1, 62–64.

[25] Esnault, Hélène; Viehweg, Eckart: Deligne-Beĭlinson cohomology. Beĭlinson's conjectures on special values of L-functions, 43–91, *Perspect. Math.,* **4**, Academic Press, Boston, MA, 1988.

[26] Goodwillie, Thomas G.: Cyclic homology, derivations, and the free loopspace. *Topology* **24** (1985), no. 2, 187–215.

[27] Griffiths, Phillip; Harris, Joseph: Principles of algebraic geometry. Reprint of the 1978 original. *Wiley Classics Library.* John Wiley & Sons, Inc., New York, 1994.

[28] Guillén, F.; Navarro Aznar, V. : Sur le théorème local des cycles invariants. Duke Math. J. 61 (1990), no. 1, 133–155.

[29] Hartshorne, Robin: Algebraic geometry. *Graduate Texts in Mathematics*, No. **52**. Springer-Verlag, New York-Heidelberg, 1977.

[30] Karoubi, Max: Homologie cyclique et K-théorie. *Astérisque* No. **149** (1987).

[31] Loday, Jean-Louis: Cyclic Homology, *Grundlehren der Mathematischen Wissenschaften* **301** Springer-Verlag, Berlin, 1998.

[32] Loday, Jean-Louis: Cyclic homology, a survey. Geometric and algebraic topology, 281–303, *Banach Center Publ.* **18** PWN, Warsaw, 1986.

[33] Navarro Aznar, V; Sur la théorie de Hodge-Deligne. *Invent. Math.* **90** (1987), no. 1, 11–76.

[34] Quillen, Daniel: Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. *Lecture Notes in Math.*, **341** Springer, Berlin 1973.

[35] Quillen, Daniel G: Homotopical algebra. *Lecture Notes in Mathematics,* No. **43** Springer-Verlag, Berlin-New York 1967.

[36] Soulé, C : Opérations en K-théorie algébrique. *Canad. J. Math.* **37** (1985), no. 3, 488–550.

[37] Steenbrink, Joseph: Limits of Hodge structures. *Invent. Math.* **31** (1975/76), no. 3, 229–257.

[38] Steenbrink, Joseph: *PhD. Thesis.* Universiteit van Amsterdam 1974.

[39] Stevens, Glenn: Arithmetic on modular curves. *Progress in Mathematics,* **20.** Birkhäuser Boston, Inc., Boston, MA, 1982.

[40] Voisin, Claire: Hodge theory and complex algebraic geometry. I. Translated from the French original by Leila Schneps. *Cambridge Studies in Advanced Mathematics,* **76.** Cambridge University Press, Cambridge, 2002.

[41] Weibel, Charles: The Hodge filtration and cyclic homology. *K-Theory* **12** (1997), no. 2, 145–164.

[42] Weibel, Charles: Cyclic homology for schemes. *Proc. Amer. Math. Soc.* **124** (1996), no. 6, 1655–1662.

[43] Weibel, Charles A; An introduction to homological algebra. *Cambridge Studies in Advanced Mathematics*, **38**. Cambridge University Press, Cambridge, 1994.

[44] Weibel, Charles A; Geller, Susan C: Étale descent for Hochschild and cyclic homology. *Comment. Math. Helv.* **66** (1991), no. 3, 368–388.

[45] Wodzicki, Mariusz: Excision in cyclic homology and in rational algebraic K-theory. *Ann. of Math.* **129** (1989), no. 3, 591–639.

[46] Wodzicki, Mariusz: Cyclic homology of differential operators. *Duke Math. J.* **54** (1987), no. 2, 641–647.

Vita

Date of Birth: Sept 8, 1984. Place: Ranchi, India.

Thesis defended: March, 2009.