Statistics of Non-real Zeros and Critical Points of Systems of Real Random Polynomials

Brian Macdonald

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1. Abstract

In this paper we study the expected density of non-real zeros of a system of real random polynomials in several variables and critical points of a real random polynomial in several variables. We use the Poincaré-Lelong formula to show that the expected density of non-real zeros of random polynomial systems with real coefficients rapidly approaches the expected density of non-real zeros in the complex coefficients case. We use the Kac-Rice formula to prove the analogous result for critical points of a real random polynomial in several variables.

2. Introduction

1. **Expected density of zeros.** Kac [Kac48] and Rice [Ric54] independently found the expected density of zeros of a random polynomial with real standard Gaussian coefficients. Bogomolny, Bohigas, and Leboeuf ([BBL92], [BBL96]) and Hannay [Han96] have results on the density of (and correlations between) complex zeros of random polynomials with complex Gaussian coefficients. Edelman and Kostlan [EK95] generalize the results for density of real (resp. complex) zeros to systems of functions in several variables when the coefficients are real (resp. complex) Gaussian random variables.

In one variable, Shepp and Vanderbei [SV95] and Prosen [Pro96] have studied non-real zeros of real polynomials. Shepp and Vanderbei extended Kac's formula for the expected density of zeros of polynomials in one real variable, in the case where the coefficients are standard real Gaussian coefficients, to include non-real zeros of those same polynomials. Prosen followed Hannay's approach to show that, as the degree of the polynomials goes to infinity, the expected density of non-real zeros of a random polynomial with real Gaussian coefficients approaches the expected density of non-real zeros of the random polynomials with the corresponding complex Gaussian coefficients (i.e. those complex coefficients with the same variance). Prosen's motivation came from quantum chaos, and he wanted the statistics of zeros of eigenstates of 1-dim chaotic systems.

In this paper we generalize Prosen's result on the density of non-real zeros of real random polynomials to random polynomial systems in several variables. Consider $h_{m,N} = (f_{1,N}, ..., f_{m,N}) : \mathbb{C}^m \to \mathbb{C}^m$, where $f_{q,N}$ is a polynomial of the form

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J,$$

where c_J^q is a real or complex random variable with associated measure $d\gamma$ for each q. We show that for these m independent functions in m variables, the expected density of non-real zeros in the real coefficients case rapidly approaches the expected density in the complex coefficients case as the degree of the polynomials gets large. In fact, we show that the convergence is exponential. More formally, let

$$d\gamma_{cx} = \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}, \textit{ and}$$
$$d\gamma_{real} = \delta_{\mathbb{R}^{D_N}} \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}.$$

where $D_N = \binom{N+m}{m}$ and $\delta_{\mathbb{R}^{D_N}}$ is the delta measure on $\mathbb{R}^{D_N} \subset \mathbb{C}^{D_N}$. Then we have the following result.

Theorem 1.1. *Let $K \subseteq \mathbb{C}^m \backslash \mathbb{R}^m$ be compact, let λ_z be a positive constant. Then*

$$E_{\gamma_{real}}(Z_{h_N(z)}) = E_{\gamma_{cx}}(Z_{h_N(z)}) + O(e^{-\lambda_z N}),$$

for all $z \in \mathbb{C}^m \backslash \mathbb{R}^m$.

Here $d\gamma_{cx}$ corresponds to the standard complex Gaussian coefficients case, where we are considering

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J,$$

where the c_J^q's are standard complex Gaussian random variables, and $d\gamma_{real}$ corresponds to the standard real Gaussian coefficients case, where we have

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J = \sum_{|J|=0}^{N} a_J^q \binom{N}{J}^{1/2} z^J,$$

where $c_J^q = a_J^q + i0$ is a standard real Gaussian random variable.

We prove this result using the Poincare-Lelong formula, which is similar to that which was used in [BSZ00a], but has the added complication that the coefficients are real. The proof uses 2-point Szego kernel asymptotics, which still applies to the polynomials with real coefficients because we are viewing them as functions of complex variables.

Shiffman and Zelditch [SZ99] and Bleher, Shiffman and Zelditch ([BSZ00a], [BSZ00b]) have generalized many results about random polynomials on $\mathbb{C}^m$ or $\mathbb{R}^m$ to complex manifolds, and they have several results relating to the statistics of zeros of holomorphic sections of powers of a line bundle over a complex manifold. In particular, in [BSZ00a] the authors use the Poincare-Lelong formula to find a formula for density of zeros and correlations between zeros.

2. **Expected density of critical points.** In [DSZ04], Douglas, Shiffman, and Zelditch look at the critical points of a holomorphic section of a line bundles over a complex manifold, motivated by applications in string theory. They use a generalized Kac-Rice formula to find statistics of these critical points, namely the density of zeros and correlations between zeros. In this paper, we also study critical points in the special case $\mathbb{C}^m$ and generalize the result in Theorem 1 for critical points in several variables. That is, we prove the following:

Theorem 2.2. *Let $K \subseteq \mathbb{C}^m \backslash \mathbb{R}^m$ be compact, let λ_z be a positive constant, and let h, γ_{cx}, and γ_{real} be defined as above. We have*

$$E_{\gamma_{real}}(C_{h_N(z)}) = E_{\gamma_{cx}}(C_{h_N(z)}) + O(e^{-\lambda_z N}),$$

for all $z \in \mathbb{C}^m \backslash \mathbb{R}^m$.

The formula for $E_{\gamma_{cx}}(C_{h_N(z)})$ is covered by [DSZ04]. Note that finding the critical points of h is equivalent to finding the statistics of zeros of the m partial derivatives of h. However, it is more difficult than the zeros case above in Theorem 1 because of the fact that the m partial derivatives are not independent random functions. This fact makes the Poincare-Lelong method much more difficult to apply. So we use the Kac-Rice formula to get a strong limit to further generalize Theorem 1 to critical points in several variables, and we also get an exact formula.

3. **Current and Future work.** The following is a list of current work and possibilities for future work.

(1) **Non-zero mean coefficients, and applications in engineering** - Rice's original motivation for studying zeros random polynomials was zero crossings of noisy signals [Ric54]. Schober and Gerstacker [SG02] discussed using results on random polynomials for the purposes of filter design. It would be interesting to me to use my background in electrical engineering to find other applications of random polynomials in electrical engineering, and to let problems in that and other engineering fields guide my future work. These problems would likely involve random polynomials whose coefficients are non-zero mean random variables. While some results are known for such polynomials in one variable, the case of polynomials in two or more variables with non-zero mean coefficients has not been well-studied.
(2) **Variance** - One could consider variance of the number of zeros within a region. Shiffman and Zelditch have done this in the complex manifolds case using asymptotics of Szego kernels (see [SZ06a], [SZ06b], [SZ07]). We have shown a result for variance analogous to the results regarding average zeros above, namely that the variance of the number of zeros in a subset U of $\mathbb{C}$ for a real random SO(2) polynomial approaches that of the complex random SU(2) polynomial as the degree of the polynomials goes to infinity.
(3) **Higher moments and asymptotic normality** - We also have shown an analagous result for higher moments, and hope to be able to show that the random variables (Z_f, ϕ) where ϕ compactly supported test function, are asymptotically normal. The complex coefficients case is found in [ST04].
(4) **Polynomials with coefficients of different variances or distributions** - Prosen's result was for Gaussian coefficients of arbitrary variances. One could generalize my results for arbitrary Gaussian coefficients or non-Guassian coefficients, e.g. [Mas75], [IZ97].

4. **Outline.** The paper is organized as follows:

3. Density of zeros in one variable

Consider the real random polynomial

$$f_N(z) = \sum_{\ell=0}^{N} \tilde{a}_\ell z^\ell,$$

where the $\tilde{a}_j$'s are *real* independent Gaussian random variable with mean 0 and variance $\binom{N}{\ell}$. Alternatively, one often writes

$$f_N(z) = \sum_{\ell=0}^{N} a_\ell \binom{N}{\ell}^{1/2} z^\ell,$$

where a_ℓ is a standard real Gaussian random variable. Instead, we choose to think of the random polynomial

$$f_N(z) = \sum_{\ell=0}^{N} c_\ell \binom{N}{\ell}^{1/2} z^\ell,$$

where c_ℓ is a more general complex random variable with associated measure $d\gamma$. We then consider two special cases

$$d\gamma_{cx} = \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{N+1},$$
$$d\gamma_{real} = \delta_S \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{N+1},$$

where δ_S is the delta function on $S \subset \mathbb{C}^{N+1}$, the set of points $c = a + ib \in \mathbb{C}^{N+1}$ where $b = 0 \in \mathbb{R}^{N+1}$. Here $d\gamma_{cx}$ corresponds to the standard complex Gaussian coefficients case, where we are considering

$$f_N(z) = \sum_{\ell=0}^{N} c_\ell \binom{N}{\ell}^{1/2} z^\ell,$$

where the c_ℓ's are standard complex Gaussian random variables, and $d\gamma_{real}$ corresponds to the standard real Gaussian coefficients case, where we have

$$f_N(z) = \sum_{\ell=0}^{N} c_\ell \binom{N}{\ell}^{1/2} z^\ell = \sum_{\ell=0}^{N} a_\ell \binom{N}{\ell}^{1/2} z^\ell,$$

where $c_\ell = a_\ell + i0$ is a standard real Gaussian random variable. We let $E(\cdot)$ denote expectation with respect to $d\gamma_{real}$ and $E_{\gamma_{cx}}(\cdot)$ denote expectation with respect to $d\gamma_{cx}$.

The goal of this section is to show the following results about the density of zeros of $f_N(z)$ using the Poincare-Lelong formula: we write

$$E(Z_{f_N}(z)) = E_{\gamma_{cx}}(Z_{f_N}(z)) + E_{2,N}(z),$$

where

$$E_{\gamma_{cx}}(Z_{f_N}(z)) = \frac{N}{\pi} \frac{1}{(1+|z|^2)^2}\, dx \wedge dy$$

and $E_{2,N}(z)$ is some "error term," and we show that

$$E_{2,N}(z) = \frac{\partial^2}{\partial z \partial \bar{z}} \log \left(1 + \sqrt{1 - \left| \frac{(1+z^2)^N}{(1+|z|^2)^N} \right|^2} \right) dx \wedge dy, z \in \mathbb{C},$$

and

$$E_{2,N}(z) = O(e^{-\lambda N}), z \in \mathbb{C} \backslash \mathbb{R}, \lambda > 0,$$

or, in other words,

$$\frac{1}{N} E(Z_{f_N}(z)) = \frac{1}{N} E_{\gamma_{cx}}(Z_{f_N}(z)) + O(e^{-\lambda N}), z \in \mathbb{C} \backslash \mathbb{R},$$

and uniformly on compact sets $K \subset \mathbb{C} \setminus \mathbb{R}$. We find the scaling limit of the second term to be

$$\lim_{N \to \infty} \frac{1}{N} E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left(1 + \sqrt{1 - \left| \frac{e^{z^2}}{e^{|z|^2}} \right|^2} \right) dx \wedge dy, z \in \mathbb{C} \backslash \mathbb{R}.$$

Setting $z = x + iy$, we can write

$$\lim_{N \to \infty} \frac{1}{N} E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{4\pi} \frac{\partial^2}{\partial y^2} \log \left(1 + \sqrt{1 - e^{-4y^2}} \right) dx \wedge dy, y \neq 0,$$

which after adding to $\frac{1}{N} E_{\gamma cx}(Z_{f(\frac{z}{\sqrt{N}})}) \to \frac{1}{\pi}$, we recover Prosen's result:

$$\lim_{N \to \infty} \frac{1}{N} E_{\gamma_{real}}(Z_{f(\frac{z}{\sqrt{N}})}) = \frac{1}{\pi} \frac{1 - (4y^2 + 1)e^{-4y^2}}{(1 - e^{-4y^2})^{3/2}}.$$

We also show that the error term goes to 0 weakly on compact sets $K \subset \mathbb{C}$:

$$\frac{1}{N} E_{2,N}(z) = O(N^{-1}), \text{ weakly on compact sets } K \subset \mathbb{C},$$

by which we mean that for any $\phi \in C^\infty(K)$,

$$\frac{1}{N} (E_{2,N}(z), \phi(z)) = \frac{1}{N} \int_K E_{2,N}(z)\, \phi(z)\, dz = O(N^{-1}).$$

Note that K could contain some points in $\mathbb{R}$, whereas the strong convergence result excludes points in $\mathbb{R}$.

The first result was given by Prosen as mentioned, but we do it here using a different method that we will be generalizing in later sections. The approach is similar to that described in [BSZ00a], where they find the limit of the pair correlations of zeros of random holomorphic sections of powers of a line bundle of a complex manifold. While we only deal with density of zeros in this section, the condition that the coefficients a_j are real causes the method in [BSZ00a] to be useful.

1. **Pointwise limit for $E(Z_{f_N})$.** We write $a = (a_0, ..., a_N)$ and

$$F_N = (\binom{N}{0}^{1/2} z^0, \binom{N}{1}^{1/2} z^1, ..., \binom{N}{N}^{1/2} z^N),$$

so that $f_N = a \cdot F_N$. By the Poincare-Lelong formula, the density of the zeros of f, $E(Z_{f_N})$, satisfies

$$E(Z_{f_N}) = E(\frac{i}{\pi}\partial\bar{\partial}\log|f|) = E(\frac{i}{\pi}\partial\bar{\partial}\log|a \cdot F|).$$

We write $F_N(z) = ||F_N(z)||u_N(z)$, where $u_N(z)$ is a unit vector. We have

$$\begin{aligned} E(Z_{f_N}) &= E(\frac{i}{\pi}\partial\bar{\partial}\log||F_N(z)||) + E(\frac{i}{\pi}\partial\bar{\partial}\log|a \cdot u_N(z)|) \\ &= E_{1,N}(z) + E_{2,N}(z) \end{aligned}$$

The first term is what we want. First, note that from [BSZ00a] we can see that $E_{1,N}(z) = E_{cx}(Z_{f_N}(z))$. Since F does not depend on a, we have

$$\begin{aligned} E_{1,N}(z) &= E(\frac{i}{\pi}\partial\bar{\partial}\log||F_N||) \\ &= \frac{i}{\pi}\int_{\mathbb{R}^{N+1}} \partial\bar{\partial}\log||F_N||d\mu(a) = \frac{i}{\pi}\partial\bar{\partial}\log||F_N|| = \frac{i}{2\pi}\partial\bar{\partial}\log||F_N||^2 \\ &= \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{\ell=0}^{N}\binom{N}{\ell}z^\ell\bar{z}^\ell \\ &= \frac{i}{2\pi}\partial\bar{\partial}\log(1+|z|^2)^N = \frac{i}{2\pi}\frac{\partial^2}{\partial\bar{z}\partial z}\log(1+|z|^2)^N dz \wedge d\bar{z} \\ &= \frac{i}{2\pi}\frac{\partial^2}{\partial\bar{z}\partial z}N\log(1+|z|^2)dz \wedge d\bar{z} = \frac{i}{2\pi}N\frac{\partial}{\partial\bar{z}}\frac{\bar{z}}{(1+|z|^2)}dz \wedge d\bar{z} \\ &= \frac{N}{2\pi}\frac{1}{(1+|z|^2)^2}\, i\, dz \wedge d\bar{z} = \frac{N}{2\pi}\frac{1}{(1+|z|^2)^2}\, i\,(-2i)dx \wedge dy \\ &= \frac{N}{\pi}\frac{1}{(1+|z|^2)^2}\ dx \wedge dy \end{aligned}$$

So we have

$$E(Z_f) = E_{\gamma_{cx}}(Z_f) + E_{2,N}(z)$$

and we just need to show

$$\begin{aligned} E_{2,N}(z) &= E(\frac{i}{\pi}\partial\bar{\partial}\log|a \cdot u_N(z)|) \\ &= \int_{\mathbb{R}^{N+1}}\frac{i}{\pi}\partial\bar{\partial}\log|a \cdot u_N(z)|d\mu(a) = O(e^{-\lambda N}). \end{aligned}$$

2. **Limit of the second term.** In the case that c_j is a standard complex Gaussian random variable, this second term is zero *for all* N (not just as $N \to \infty$). Because of the SU(2)-invariance of the standard complex Gaussian measure, one can perform a unitary change of variables so that u becomes $(1, 0, ..., 0)$ and the integral $\int_{\mathbb{C}^{N+1}} \partial\bar{\partial} \log |c \cdot u| d\mu(c)$ becomes a single integral that evaluates to 0:

$$\int_{\mathbb{C}^{N+1}} \partial\bar{\partial} \log |c \cdot (1, 0, ..., 0)| d\mu(c) = \int_{\mathbb{C}} \partial\bar{\partial} \log |c_0| d\mu(c_0) = 0.$$

In the case where a_j is real, the second term is not zero for all N. Because only real rotations can be performed, u can not be rotated to $(1, 0, ..., 0)$, giving a single integral. But we can still use the rotational invariance of real Gaussian measures to obtain a double integral over $\mathbb{R}^2$ which is a little more manageable than the integral over $\mathbb{R}^{N+1}$.

Let $u = \operatorname{Re} u + i \operatorname{Im} u = (\operatorname{Re} u_1, ..., \operatorname{Re} u_N) + i(\operatorname{Im} u_1, ..., \operatorname{Im} u_N)$. Note that u, $\operatorname{Re} u$, and $\operatorname{Im} u$ depend on z and N but we frequently omit these arguments for convenience. Since we need to do real rotations, the real and imaginary parts of u must be rotated the same. (explain that better?). Therefore, as mentioned, we can not rotate u to (1, 0, ..., 0). However, we can rotate so that either the real part or the imaginary part of u is of the form $(r, 0, ..., 0)$, where $r = r_N(z)$ is some (non-zero) constant less than 1. So we choose to perform a (real) rotation of $a_0, a_1, ..., a_N$ so that

$$\tilde{u} = \operatorname{Re} \tilde{u} + i \operatorname{Im} \tilde{u} = (r, 0, ..., 0) + i(\operatorname{Im} \tilde{u_1}, ..., \operatorname{Im} \tilde{u_N}).$$

Then one can perform a rotation of the $a_1, ..., a_N$ variables so that $\operatorname{Re} u$ is unaffected and u becomes

$$(r_N(z), 0, ..., 0) + i(s_N(z), t_N(z), 0, ..., 0) \tag{2.1}$$

$$=(r_N(z) + is_N(z), it_N(z), 0, ..., 0). \tag{2.2}$$

Note that since u is a unit vector, and rotations preserve length, r, s, and t have the condition $r^2 + s^2 + t^2 = 1$. Note also that r, s, and t all depend on z and N but we frequently omit these. We are now concerned with the limit of the simpler integral,

$$\begin{aligned}
&\frac{i}{\pi} \int_{\mathbb{R}^{N+1}} \partial\bar{\partial} \log |(a_0, a_1, ..., a_N) \cdot (r + is, it, 0, ..., 0)| \, d\mu(a) \\
=&\frac{i}{\pi} \int_{\mathbb{R}^2} \partial\bar{\partial} \log |a_0(r + is) + a_1(it)| \, d\mu(a_0) d\mu(a_1) \\
=&\frac{i}{\pi} \partial\bar{\partial} \int_{\mathbb{R}^2} \log |a_0(r + is) + a_1(it)| \frac{1}{2\pi} e^{-(a_0^2 + a_1^2)/2} \, da_0 da_1.
\end{aligned}$$

3. **Formula for r.** First, we know that since $u(z) = \frac{F(z)}{||F(z)||}$, and since the length of $\operatorname{Re} u$ doesn't change from a rotation, we can write

$$r^2 = || \operatorname{Re} \tilde{u}||^2 = || \operatorname{Re} u||^2 = \frac{|| \operatorname{Re} F||^2}{||F||^2}.$$

Note that we are assuming $\operatorname{Im} z \neq 0$. We now state a fact that we will use repeatedly.

Fact 3.3. *For complex numbers z and w,*

(1) $(\operatorname{Re} z)^2 = \frac{1}{2}(|z|^2 + \operatorname{Re} z^2)$
(2) $\operatorname{Re} z \operatorname{Re} w = \frac{1}{2}(\operatorname{Re} zw + \operatorname{Re} z\bar{w})$
(3) $\operatorname{Re} z \operatorname{Im} w = \frac{1}{2}(\operatorname{Im} zw - \operatorname{Im} z\bar{w})$
(4) $\operatorname{Im} z \operatorname{Re} w = \frac{1}{2}(\operatorname{Im} zw + \operatorname{Im} z\bar{w})$
(5) $\operatorname{Im} z \operatorname{Im} w = \frac{1}{2}(\operatorname{Re} z\bar{w} - \operatorname{Re} zw)$

Proof. (1) Special case of (2) with $w = z$.
(2) Let $z = x + iy, w = u + iv$. Then

$$\begin{aligned}
&\operatorname{Re} zw + \operatorname{Re} z\bar{w} \\
=&\operatorname{Re}(x+iy)(u+iv) + \operatorname{Re}(x+iy)(u-iv) \\
=&\operatorname{Re}(xu + ixv + iyu - yv) + \operatorname{Re}(xu + ixv + iyu + yv) = 2xu.
\end{aligned}$$

(3) and (4) are proved similarly.

□

By the fact we have $(\operatorname{Re} z^\ell)^2 = \frac{1}{2}|z|^{2\ell} + \frac{1}{2}\operatorname{Re} z^{2\ell}$, which gives us

$$\begin{aligned}
[r_N(z)]^2 &= \frac{\sum_{\ell=0}^{N}\binom{N}{\ell}(\operatorname{Re} z^\ell)^2}{\sum_{\ell=0}^{N}\binom{N}{\ell}|z|^{2\ell}} = \frac{\sum_{\ell=0}^{N}\binom{N}{\ell}\left(\frac{1}{2}|z|^{2\ell} + \frac{1}{2}\operatorname{Re} z^{2\ell}\right)}{\sum_{\ell=0}^{N}\binom{N}{\ell}|z|^{2\ell}} \\
&= \frac{\sum_{\ell=0}^{N}\binom{N}{\ell}\frac{1}{2}|z|^{2\ell}}{\sum_{\ell=0}^{N}\binom{N}{\ell}|z|^{2\ell}} + \frac{\sum_{\ell=0}^{N}\binom{N}{\ell}\frac{1}{2}\operatorname{Re} z^{2\ell}}{\sum_{\ell=0}^{N}\binom{N}{\ell}|z|^{2\ell}} \\
&= \frac{1}{2} + \frac{1}{2}\operatorname{Re}\frac{\sum_{\ell=0}^{N}\binom{N}{\ell}z^{2\ell}}{\sum_{\ell=0}^{N}\binom{N}{\ell}|z|^{2\ell}} = \frac{1}{2} + \frac{1}{2}\operatorname{Re}\left(\frac{1+z^2}{1+|z|^2}\right)^N
\end{aligned}$$

4. **Formula for s.** Next, we have the relationship $\operatorname{Re}\tilde{u} \cdot \operatorname{Im}\tilde{u} = rs$, so since the angle between $\operatorname{Re} u$ and $\operatorname{Im} u$ doesn't change under a rotation, we have

$$s = \frac{\operatorname{Re}\tilde{u} \cdot \operatorname{Im}\tilde{u}}{r} = \frac{\operatorname{Re} u \cdot \operatorname{Im} u}{r} = \frac{\operatorname{Re}\frac{F}{||F||} \cdot \operatorname{Im}\frac{F}{||F||}}{r} = \frac{\operatorname{Re} F \cdot \operatorname{Im} F}{||F||^2} \cdot \frac{1}{r}$$

Using the identity $\operatorname{Im} w^2 = 2 \operatorname{Re} w \operatorname{Im} w$, for any complex number w, we can write

$$\begin{aligned}
\operatorname{Re} F \cdot \operatorname{Im} F &= \left(\operatorname{Re} \tbinom{N}{0}^{1/2} z^0, ..., \operatorname{Re} \tbinom{N}{N}^{1/2} z^N \right) \cdot \left(\operatorname{Im} \tbinom{N}{0}^{1/2} z^0, ..., \operatorname{Im} \tbinom{N}{N}^{1/2} z^N \right) \\
&= \sum_{\ell=0}^{N} \operatorname{Re} \tbinom{N}{\ell}^{1/2} z^\ell \operatorname{Im} \tbinom{N}{\ell}^{1/2} z^\ell = \sum_{\ell=0}^{N} \tbinom{N}{\ell} \tfrac{1}{2} \operatorname{Im} z^{2\ell} \\
&= \frac{1}{2} \operatorname{Im} \sum_{\ell=0}^{N} \tbinom{N}{\ell} z^{2\ell} = \tfrac{1}{2} \operatorname{Im} (1+z^2)^N
\end{aligned}$$

So we have

$$\begin{aligned}
s_N(z) &= \frac{\frac{1}{2} \operatorname{Im} (1+z^2)^N}{||F||^2} \cdot \frac{1}{r} = \frac{\frac{1}{2} \operatorname{Im} (1+z^2)^N}{(1+|z|^2)^N} \cdot \frac{1}{r} \\
&= \frac{1}{2} \operatorname{Im} \left(\frac{1+z^2}{1+|z|^2} \right)^N \cdot \frac{1}{r}
\end{aligned}$$

5. **Formula for t.** Since $r^2 + s^2 + t^2 = 1$, we have t easily:

$$[t_N(z)]^2 = 1 - [r_N(z)]^2 - [s_N(z)]^2.$$

6. **Limits of r and its derivatives.** We have

$$\begin{aligned}
\lim_{N\to\infty} [r_N(z)]^2 &= \lim_{N\to\infty} \frac{1}{2} + \frac{1}{2} \operatorname{Re} \left(\frac{1+z^2}{1+|z|^2} \right)^N \\
&= \frac{1}{2} + \frac{1}{2} \operatorname{Re} \lim_{N\to\infty} \left(\frac{1+z^2}{1+|z|^2} \right)^N
\end{aligned}$$

Since $\left| \frac{1+z^2}{1+|z|^2} \right| \leq 1$ by the triangle inequality. Equality holds only when $z \in \mathbb{R}$, so we have that

$$\begin{aligned}
&\left(\frac{1+z^2}{1+|z|^2} \right)^N = O(e^{-\lambda N}), z \in \mathbb{C} \backslash \mathbb{R} \\
&\left(\frac{1+z^2}{1+|z|^2} \right)^N \equiv 1 \text{ (for all N) }, z \in \mathbb{R}
\end{aligned}$$

which gives

$$\begin{aligned}
&[r_N(z)]^2 = \frac{1}{2} + O(e^{-\lambda N}), z \in \mathbb{C} \backslash \mathbb{R} \\
&[r_N(z)]^2 \equiv 1, z \in \mathbb{R}
\end{aligned}$$

As a consequence, $\frac{\partial}{\partial z} r_N(z)$, $\frac{\partial^2}{\partial z \partial \bar{z}} r_N(z) = O(e^{-\lambda N})$, for $z \in \mathbb{C} \backslash \mathbb{R}$:

$$\begin{aligned}
\frac{\partial}{\partial \bar z}\left(\frac{1+z^2}{1+|z|^2}\right)^N &= N\left(\frac{1+z^2}{1+|z|^2}\right)^{N-1}\psi(z) = O(e^{-\lambda N})\\
\frac{\partial^2}{\partial z\partial \bar z}\left(\frac{1+z^2}{1+|z|^2}\right)^N &= N(N-1)\left(\frac{1+z^2}{1+|z|^2}\right)^{N-2}[\psi(z)]^2 \text{ (***not quite } \psi^2)\\
&+ N\left(\frac{1+z^2}{1+|z|^2}\right)^{N-1}\frac{\partial}{\partial z}\psi(z)\\
&= O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}
\end{aligned}$$

where the $\psi(z)$ is the derivative of the term inside the parentheses with respect to $\bar z$. Indeed, all derivatives of r are $O(e^{-\lambda N})$. Also note that all derivatives converge uniformly on compact sets $K \subset \mathbb{C} \setminus \mathbb{R}$.

7. **Limits of s, t, and their derivatives.** Recall

$$s_N(z) = \frac{1}{2}\,\mathrm{Im}\left(\frac{1+z^2}{1+|z|^2}\right)^N \cdot \frac{1}{r_N(z)}.$$

Since

$$\begin{aligned}
\left(\frac{1+z^2}{1+|z|^2}\right)^N &= O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}\\
\frac{1}{r_N(z)} &= \frac{1}{\sqrt{1/2 + O(e^{-\lambda N})}}, z \in \mathbb{C}\backslash\mathbb{R}
\end{aligned}$$

we have that

$$s_N(z) = O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}$$

for all z such that $\mathrm{Im}\, z \neq 0$. As we had with r, the derivatives of s go to 0 exponentially fast as well when $z \in \mathbb{C}\backslash\mathbb{R}$. Since $r^2 + s^2 + t^2 = 1$, and $[r_N(z)]^2 = 1, z \in \mathbb{R}$, we have that

$$s_N(z) = 0, z \in \mathbb{R}$$

Since $r^2 + s^2 + t^2 = 1$ we have $t = \sqrt{\frac{1}{2}} + O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}$, and uniformly on compact sets $K \subset \mathbb{C}\backslash\mathbb{R}$, as well as $t \equiv 0, z \in \mathbb{R}$.

8. **Switch to polar coordinates.** We now use Jensen's formula to evaluate the integral

$$\frac{i}{\pi}\partial\bar\partial \int_{\mathbb{R}^2} \log|a_0(r+is) + a_1(it)|\frac{1}{2\pi}e^{-(a_0^2+a_1^2)/2}\,da_0da_1.$$

First, we switch to polar coordinates, so the integral becomes

$$\frac{i}{2\pi^2}\partial\bar\partial \int_{\theta=0}^{2\pi}\int_{\rho=0}^{\infty} \log|(\rho\sin\theta)(r+is) + (\rho\cos\theta)(it)|e^{-\rho^2/2}\,\rho d\rho d\theta$$

We may factor out a ρ from the argument of the log and get

$$\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{\infty}(\log\rho+\log|(\sin\theta)(r+is)+(\cos\theta)(it)|)\,e^{-\rho^2/2}\,\rho d\rho d\theta$$

Since

$$\int_{\theta=0}^{2\pi}\int_{\rho=0}^{\infty}\log\rho e^{-\rho^2}\rho d\rho d\theta$$

doesn't depend on z, it gets killed by $\partial\bar{\partial}$, so we are left with

$$\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{\infty}\log|(\sin\theta)(r+is)+(\cos\theta)(it)|e^{-\rho^2/2}\,\rho d\rho d\theta$$

The log term doesn't depend on ρ, so we may pull that term outside the integral, and integrate with respect to ρ to get

$$\begin{aligned}
&\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(\sin\theta)(r+is)+(\cos\theta)(it)|\left[\int_{\rho=0}^{\infty}e^{-\rho^2/2}\,\rho\,d\rho\right]d\theta\\
=\;&\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(\sin\theta)(r+is)+(\cos\theta)(it)|\left[-e^{-\rho^2/2}\right]_0^{\infty}\,d\theta\\
=\;&\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(\sin\theta)(r+is)+(\cos\theta)(it)|\,[1]\,d\theta\\
=\;&\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(\sin\theta)(r+is)+(\cos\theta)(it)|\,d\theta
\end{aligned}$$

9. **Jensen's Formula.** Using the fact that $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ and $\sin\theta=\frac{1}{2i}(e^{i\theta}-e^{-i\theta})$, we can write

$$-\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log\frac{1}{2}|(e^{i\theta}-e^{-i\theta})(-i)(r+is)+(e^{i\theta}+e^{-i\theta})it|\,d\theta$$

We can bring out a $\log\frac{1}{2}$, and since $\partial\bar{\partial}\log\frac{1}{2}=0$, we have

$$-\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(e^{i\theta}-e^{-i\theta})(-ir+s)+(e^{i\theta}+e^{-i\theta})it|\,d\theta$$

We can factor out $e^{-i\theta}$, and since $\log|e^{-i\theta}|=0$, we get

$$\begin{aligned}
&-\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(e^{i2\theta}-1)(-ir+s)+(e^{i2\theta}+1)it|\,d\theta\\
=&-\frac{i}{2\pi^2}\partial\bar{\partial}\int_{\theta=0}^{2\pi}\log|(s+i(t-r))e^{i2\theta}+(-s+i(t+r))|\,d\theta
\end{aligned}$$

We can now use Jensen's formula to evaluate the inner integral. Recall that Jensen's formula states that, assuming $\phi(0) \neq 0$, and ϕ is non-zero on $\partial D(0,1)$, then

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log|\phi(e^{i\theta})|d\theta = \log|\phi(0)| + \sum_{\substack{\phi(w_j)=0\\ |w_j|<1}} \log\frac{1}{|w_j|}.$$

In our case, $\phi(w) = (s+i(t-r))w^2 + (-s+i(t+r))$ so that

$$\begin{aligned}\phi(e^{i\theta}) &= (s+i(t-r))e^{i2\theta} + (-s+i(t+r))\\ \phi(0) &= -s+i(t+r)\\ \phi(w_j) = 0 &\iff [w_j(z)]^2 = -\frac{-s+i(t+r)}{s+i(t-r)}\end{aligned}$$

Note that since $|\phi(0)|^2 = |-s+i(t+r)|^2 = s^2+(t+r)^2 = s^2+r^2+2rt+t^2 = 1+2rt$, and r and t are non-negative (by construction), $\phi(0) \neq 0$. We show that $|w_j(z)|^2 \geq 1$, for all z, implying that $|w_j(z)| \geq 1$ and that all the zeros $w_j(z)$ of ϕ are outside the unit disk for every z.

We have

$$|w_j(z)|^4 = \left|\frac{-s+i(t+r)}{s+i(t-r)}\right|^2 = \frac{s^2+t^2+2rt+r^2}{s^2+t^2-2rt+r^2} \geq 1$$

for $z \in \mathbb{C}$ since r and t are non-negative by construction. Note that the only time rt is zero is when $z \in \mathbb{R}$. In this case, $r_N(z) = 1$ and $s_N(z) = t_N(z) = 0$ for all N, and $|w_j(z)| = 1$.

10. **Exact formula for $E_{2,N}(z)$.** So since all of the zeros of ϕ are outside the unit disk, we have for $z \in \mathbb{C}$,

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log|\phi(e^{i\theta})|d\theta &= \log|\phi(0)| = \log|-s+i(t+r)|\\ &= \frac{1}{2}\log|-s+i(t+r)|^2 = \frac{1}{2}\log(1+2rt)\end{aligned}$$

or

$$\frac{1}{2\pi}\int \log|a_0(r+is)+a_1(it)|e^{-a_0^2-a_1^2}da_0da_1 = \frac{1}{2}\log(1+2rt),$$

and

$$\begin{aligned}E_{2,N}(z) = -\frac{i}{2\pi^2}\partial\bar{\partial}\int \log|\phi(e^{i\theta})|d\theta &= -\frac{i}{2\pi^2}\partial\bar{\partial}\,(2\pi)\,\frac{1}{2}\log(1+2rt)\\ &= -\frac{i}{2\pi}\partial\bar{\partial}\log(1+2rt)\\ &= \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\bar{z}}\log(1+2rt)\ dx\wedge dy, z \in \mathbb{C}\backslash\mathbb{R}.\end{aligned}$$

After some simplification of $2rt$ we have

$$E_{2,N}(z) = \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}} \log\left(1 + \sqrt{1 - \left|\frac{(1+z^2)^N}{(1+|z|^2)^N}\right|^2}\right)\ dx \wedge dy, z \in \mathbb{C}\backslash\mathbb{R}.$$

11. **Limit for $E_{2,N}(z)$.** Since we found that $r = r_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N}), t = t_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N})$, and $s = s_N(z) = O(e^{-\lambda N})$ and all derivatives (in particular, the first and second derivatives) of r, s, and t are $O(e^{-\lambda N})$ uniformly on compact sets $K \subset \mathbb{C}\backslash\mathbb{R}$, we can say that

$$\begin{aligned} E_{2,N}(z) &= E(\frac{i}{\pi}\partial\bar{\partial}\log|a \cdot u_N(z)|) \\ &= -\frac{i}{2\pi}\partial\bar{\partial}\log(1+2rt) = O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

and uniformly on compact sets $K \subset \mathbb{C}\backslash\mathbb{R}$, which is our desired result.

12. **Scaling limit for $E_{2,N}(z)$.** By the chain rule we have for any differentiable function $f(z)$

$$\frac{\partial^2}{\partial z \partial \bar{z}} f(z)\Big|_{\frac{z}{\sqrt{N}}} = N\frac{\partial^2}{\partial z \partial \bar{z}}\left[f(\frac{z}{\sqrt{N}})\right]$$

So we have

$$\begin{aligned} \frac{1}{N}E_{2,N}(\frac{z}{\sqrt{N}}) &= \frac{1}{N\pi}\frac{\partial^2}{\partial z \partial \bar{z}}\log\left[1 + 2r_N(z)t_N(z)\right]\Big|_{\frac{z}{\sqrt{N}}}\ dx \wedge dy \\ &= \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}\log\left[1 + 2r_N(\frac{z}{\sqrt{N}})t_N(\frac{z}{\sqrt{N}})\right]\ dx \wedge dy, z \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

and after some simplification we get

$$\frac{1}{N}E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}\log\left(1 + \sqrt{1 - \left|\frac{(1+(\frac{z}{\sqrt{N}})^2)^N}{(1+|\frac{z}{\sqrt{N}}|^2)^N}\right|^2}\right)\ dx \wedge dy, z \in \mathbb{C}\backslash\mathbb{R}.$$

We now take the limit and get

$$\lim_{N\to\infty}\frac{1}{N}E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}\log\left(1 + \sqrt{1 - \left|\frac{e^{z^2}}{e^{|z|^2}}\right|^2}\right)\ dx \wedge dy, z \in \mathbb{C}\backslash\mathbb{R}.$$

Setting $z = x + iy$, we can write

$$\lim_{N\to\infty}\frac{1}{N}E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{4\pi}\frac{\partial^2}{\partial y^2}\log\left(1 + \sqrt{1 - e^{-4y^2}}\right)\ dx \wedge dy, y \neq 0,$$

and after simplification and adding to $\frac{1}{N}E_{1,N}(\frac{z}{\sqrt{N}}) \to \frac{1}{\pi}$, we recover Prosen's result in [Pro96]:

$$\lim_{N\to\infty}\frac{1}{N}E_{\gamma_{real}}(Z_{f(\frac{z}{\sqrt{N}})}) = \frac{1}{\pi}\,\frac{1-(4y^2+1)e^{-4y^2}}{(1-e^{-4y^2})^{3/2}}.$$

13. **Weak limit for $E_{2,N}(z)$.** Let $K \subset \mathbb{C}$ be a compact set. Note that unlike before, we are including points on the real line. We now show that $\frac{1}{N}E_2$ goes to 0 weakly on K. More specifically, we show that for any $\phi \in C^\infty(K)$,

$$\frac{1}{N}(E_{2,N}(z),\phi(z)) = \frac{1}{N}\int_K E_{2,N}(z)\,\phi(z)\,dz = O(N^{-1}).$$

Recall that $E_{2,N}(z) = E(\frac{i}{\pi}\partial\bar{\partial}\log|a\cdot u_N(z)|)$ By the definition of the expectation of a distribution, we have

$$\begin{aligned}(E_{2,N}(z),\phi(z)) &= \left(E(\frac{i}{\pi}\partial\bar{\partial}\log|a\cdot u_N(z)|)\,,\,\phi(z)\right)\\ &= E\left(\frac{i}{\pi}\partial\bar{\partial}\log|a\cdot u_N(z)|\,,\,\phi(z)\right)\end{aligned}$$

By the definition of the derivative of a distribution, we have

$$E\left(\frac{i}{\pi}\partial\bar{\partial}\log|a\cdot u_N(z)|\,,\,\phi(z)\right) = E\left(\log|a\cdot u_N(z)|\,,\,\frac{i}{\pi}\partial\bar{\partial}\phi(z)\right)$$

By the definition of a distribution, we have that this term equals

$$E\left(\int_K \log|a\cdot u_N(z)|\,\frac{i}{\pi}\partial\bar{\partial}\phi(z)\right).$$

Recall that E denotes expectation with respect to the Gaussian measure $d\mu(a)$. We then have by definition of expected value that this equals

$$\int_{\mathbb{R}^{\mathbb{N}}}\left(\int_K \log|a\cdot u_N(z)|\,\frac{i}{\pi}\partial\bar{\partial}\phi(z)\right)d\mu(a).$$

Since the integrand is bounded, and since $\phi(z)$ does not depend on a, we can switch the order of the integrals and get

$$\int_K\left(\int_{\mathbb{R}^{\mathbb{N}}}\log|a\cdot u_N(z)|\,d\mu(a)\right)\frac{i}{\pi}\partial\bar{\partial}\phi(z).$$

Recall that by our calculation above we have that the inner integral is $\frac{1}{2}\log(1+2rt)$, so we have

$$\int_K\left(\int_{\mathbb{R}^{\mathbb{N}}}\log|a\cdot u_N(z)|\,d\mu(a)\right)\frac{i}{\pi}\partial\bar{\partial}\phi(z) \;=\int_K\frac{i}{2\pi}\log(1+2rt)\,\partial\bar{\partial}\phi(z)$$

Recall also that $r_N(z)$ and $t_N(z)$ are both non-negative by construction, and both are bounded by 1 since $r^2+s^2+t^2=1$. Both of these conditions are true even on the real line, where $r_N=0$ and $t_N=0$ for all N. This implies the crude estimate $1 \leq (1+2rt) \leq 3$, everywhere on $\mathbb{C}$ and, in particular, on K. Since $\phi \in C^\infty(K)$, we can write

$$\begin{aligned}\int_K \frac{i}{2\pi} \log(1+2rt)\, \partial\bar{\partial}\phi(z) &\leq \int_K C\frac{i}{\pi}\partial\bar{\partial}\phi(z), \\ &= C||\frac{i}{\pi}\partial\bar{\partial}\phi(z)||_{L^1(K)}\end{aligned}$$

where C is independent of N, K, and z, including z on the real line, and the L^1 norm $||\frac{i}{\pi}\partial\bar{\partial}\phi(z)||_{L^1(K)}$ depends only on K. So then we have that

$$(E_{2,N}(z), \phi(z)) \leq C_K$$

where C_K is a constant which depends only on K. We now have want we want:

$$\frac{1}{N}(E_{2,N}(z), \phi(z)) \leq \frac{1}{N}C_K = O(N^{-1}).$$

Note that when we consider compact sets K that include part of the real line, the weak limit is the only result we have. This is because the derivatives of r, s, t, and therefore E_2 blow up near the real line. When we find the weak limit and move the $\partial\bar{\partial}$ from the log term to the ϕ term as we did above, we avoid this problem: only the derivatives of r and t blow up near the real line, not the values of the functions themselves.

4. Density of critical points in one variable

In this section we study the density of critical points of

$$h_N(z) = \sum_{\ell=0}^{N} a_\ell \binom{N}{\ell}^{1/2} z^\ell,$$

where the a_j's are *real* independent standard Gaussian random variables. This corresponds to the zeros of

$$f_N(z) = \frac{\partial h}{\partial z} = \sum_{\ell=0}^{N} a_\ell \binom{N}{\ell}^{1/2} \frac{\partial}{\partial z} z^\ell.$$

The goal of this section is to show the following results about the density of critical using the Poincare-Lelong formula: we write

$$E(C_{f_N}(z)) = E_{\gamma_{cx}}(C_{f_N}(z)) + E_{2,N}(z).$$

(1) We show that

$$\frac{1}{N} E_{\gamma_{cx}}(C_{f_N}(z)) = \frac{1}{\pi}\left(\frac{1}{(1+|z|^2)^2} - \frac{2}{N(1+|z|^2)^2} + \frac{1}{(1+N|z|^2)^2}\right) dx \wedge dy,$$

so that

$$\frac{1}{N} E_{\gamma_{cx}}(C_{f_N}(z)) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\, dx \wedge dy + O(N^{-1}), z \neq 0$$
$$\frac{1}{N} E_{\gamma_{cx}}(C_{f_N}(z)) = \frac{2}{\pi}\, dx \wedge dy + O(N^{-1}), z = 0.$$

(2) We also show that

$$E_{2,N}(z) = \partial\bar\partial \log\left(1 + \sqrt{1 - \left|\frac{(N^2 z^2 + N)(1+z^2)^{N-2}}{(N^2|z|^2+N)(1+|z|^2)^{N-2}}\right|^2}\right)$$
$$= O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R}, \lambda > 0,$$

so that

$$E(C_{f_N}(z)) = E_{\gamma_{cx}}(C_{f_N}(z)) + O(e^{-\lambda N}), z \in \mathbb{C}\backslash\mathbb{R},$$

and uniformly on compact sets $K \subset \mathbb{C}\backslash\mathbb{R}$, giving

$$E(C_{f_N}(z)) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}\, dx \wedge dy + O(N^{-1}), z \in \mathbb{C}\backslash\mathbb{R}.$$

Note that we get the same limit $\frac{1}{\pi}\frac{1}{(1+|z|^2)^2}$ that we did for the density of zeros, but the rate of convergence $O(N^{-1})$ is slower than the rate $O(e^{-\lambda N})$ that we got for the density of zeros. We still have that $E(C_h)$ approaches $E_{\gamma_{cx}}$ exponentially fast, but $E_{\gamma_{cx}}$ is now only asymptotically equal to the limit $\frac{1}{\pi}\frac{1}{(1+|z|^2)^2}$.

(3) We find the following scaling limits:

$$\frac{1}{N}E_{\gamma_{cx}}(C_{f_N})(\frac{z}{\sqrt{N}}) = \frac{1}{N}E_{1,N}(\frac{z}{\sqrt{N}}) = 1 + \frac{1}{(1+|z|^2)^2} + O(N^{-1}), z \in \mathbb{C}$$

$$\lim_{N\to\infty}\frac{1}{N}E_{2,N}(\frac{z}{\sqrt{N}}) = \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\bar{z}}\log\left(1+\sqrt{1-\left|\frac{(1+z^2)e^{z^2}}{(1+|z|^2)e^{|z|^2}}\right|^2}\right)\ dx\wedge dy, z\in\mathbb{C}$$

(4) Finally, we show that the error term $E_{2,N}(z)$ goes to 0 weakly on compact sets $K \subset \mathbb{C}$ (including points in $\mathbb{R}$):

$$\frac{1}{N}E_{2,N}(z) = O(N^{-1}),\ \text{weakly on compact sets } K \subset \mathbb{C},$$

by which we mean that for any $\phi \in C^\infty(K)$,

$$\frac{1}{N}(E_{2,N}(z), \phi(z)) = \frac{1}{N}\int_{\mathbb{K}} E_{2,N}(z)\,\phi(z)\,dz = O(N^{-1}).$$

The proofs are very similar to the density of zeros in one variable case, so we will leave out some of the details.

1. **Application of the Poincare-Lelong formula.** We write $a = (a_0, ..., a_N)$ and

$$F = F_N(z) = \big(0, \binom{N}{1}^{1/2}\frac{\partial}{\partial z}z^1, \binom{N}{2}^{1/2}\frac{\partial}{\partial z}z^2, ..., \binom{N}{N}^{1/2}\frac{\partial}{\partial z}z^N\big),$$

so that $f_N = a\cdot F_N$. By the Poincare-Lelong formula, the density of the zeros of f, $E(Z_f)$, satisfies

$$\begin{aligned} E(C_{f_N}) = E(\frac{i}{2\pi}\partial\bar{\partial}\log|f_N|^2) &= E(\frac{i}{2\pi}\partial\bar{\partial}\log|a\cdot F_N|^2) \\ &= E(\frac{i}{2\pi}\partial\bar{\partial}\log||F_N(z)||^2) + E(\frac{i}{\pi}\partial\bar{\partial}\log|a\cdot u_N(z)|) \\ &= E_{1,N}(z) + E_{2,N}(z), \end{aligned}$$

where $u = u_N(z) = \frac{F_N(z)}{||F_N(z)||}$.

The first term is a known result. First, from [BSZ00a] and the section on density of zeros in one variable, we have that $E_{1,N}(z) = E_{\gamma_{cx}}(C_{f_N})$. Next, recall that $F_N(z) = \frac{\partial}{\partial z}H_N(z)$, so we have

$$||F_N||^2 = ||\frac{\partial}{\partial z}H_N(z)||^2 = \frac{\partial}{\partial z}H_N(z)\frac{\partial}{\partial\bar{z}}\overline{H(z)}$$

and since $H(z)$ is holomorphic, this equals

$$\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}H_N(z)\overline{H_N(z)} = \frac{\partial^2}{\partial z\partial\bar{z}}||H_N(z)||^2.$$

A quick calculation gives

$$||F_N||^2 = \frac{\partial^2}{\partial z\partial\bar{z}}||H_N(z)||^2 = N(1+|z|^2)^{N-2}(1+N|z|^2)$$

Using this formula, we get

$$\begin{aligned}\frac{1}{N}E_{1,N}(z) &= \frac{1}{N}E_{\gamma_{cx}}(C_{f_N})\\ &= \frac{1}{N}E(\frac{i}{2\pi}\partial\bar{\partial}\log||F_N(z)||^2) = \frac{i}{2\pi}\frac{1}{N}\left(\frac{N-2}{(1+|z|^2)^2}+\frac{N}{(1+N|z|^2)^2}\right)dz\wedge d\bar{z}\\ &= \frac{1}{\pi}\left(\frac{1}{(1+|z|^2)^2}-\frac{2}{N(1+|z|^2)^2}+\frac{1}{(1+N|z|^2)^2}\right)dx\wedge dy\end{aligned}$$

Note that

$$\frac{1}{\pi}\left(-\frac{2}{N(1+|z|^2)^2}+\frac{1}{(1+N|z|^2)^2}\right)dx\wedge dy = O(N^{-1}), z\neq 0$$

So we have that

$$\frac{1}{N}E_{1,N}(z) = \frac{1}{N}E_{\gamma_{cx}}(C_{f_N}) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}dx\wedge dy + O(N^{-1}), z\neq 0$$

We remark that when $z=0$,

$$\begin{aligned}\frac{1}{N}E_{1,N}(z) &= \frac{1}{\pi}\left(\frac{1}{(1+|z|^2)^2}-\frac{2}{N(1+|z|^2)^2}+\frac{1}{(1+N|z|^2)^2}\right)dx\wedge dy\\ &= \frac{1}{\pi}\left(2-\frac{2}{N(1+|z|^2)^2}\right)dx\wedge dy = \frac{1}{\pi}\left(2-\frac{2}{N}\right)dx\wedge dy\end{aligned}$$

and

$$\frac{1}{N}E_{1,N}(z) = \frac{2}{\pi}dx\wedge dy + O(N^{-1}), z=0$$

So we have so far that

$$E(C_{f_N}) = \frac{1}{\pi}\frac{1}{(1+|z|^2)^2}dx\wedge dy + O(N^{-1}) + E_{2,N}(z)$$

and it reamins to show that

$$E_{2,N}(z) = \frac{i}{\pi}E(\partial\bar{\partial}\log|a\cdot u_N(z)|) = O(e^{-\lambda N}), z\in\mathbb{C}\backslash\mathbb{R}.$$

2. **Scaling limit for $E_{\gamma_{cx}}(C_{f_N})$.** We have

$$\frac{1}{N}E_{\gamma_{cx}}(C_{f_N})(\frac{z}{\sqrt{N}}) = \frac{1}{N}E_{1,N}(\frac{z}{\sqrt{N}}) = 1 + \frac{1}{(1+|z|^2)^2} + O(N^{-1}), z\in\mathbb{C}.$$

3. **Pointwise Limit of $E_{2,N}(z)$.** By the definition of expected value,

$$E_{2,N}(z) = \frac{i}{\pi}\int_{\mathbb{R}^{N+1}} \partial\bar{\partial}\log|a\cdot u_N(z)|d\mu(a).$$

As in the density of zeros case, we can perform rotations of $a_0, ..., a_N$ and then $a_1, \ldots, a_N$ so that we have

$$\begin{aligned} E_{2,N}(z) =& \frac{i}{\pi}\int_{\mathbb{R}^{N+1}} \partial\bar{\partial}\log|(a_0, a_1, ..., a_N)\cdot(r_N(z)+is_N(z), it_N(z), 0, ..., 0)|\, d\mu(a) \\ =& \frac{i}{\pi}\int_{\mathbb{R}^2} \partial\bar{\partial}\log|a_0(r+is)+a_1(it)|\, d\mu(a_0)d\mu(a_1), \end{aligned}$$

which is of the same form as we had in the zeros case. The formulas $r = r_N(z), s = s_N(z)$, and $t = t_N(z)$ are different here in the critical points case than they were in the zeros case, but the formulas are similar and a simple fact will show that we still have the same asympotic results, namely, for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} [r_N(z)]^2 &= \frac{1}{2} + O(e^{-\lambda N}), \\ s_N(z) &= O(e^{-\lambda N}), \\ [t_N(z)]^2 &= \frac{1}{2} + O(e^{-\lambda N}), \end{aligned}$$

and for $z \in \mathbb{R}$,

$$\begin{aligned} [r_N(z)]^2 &\equiv 1 \text{ (for all } N) \ , \\ s_N(z) &\equiv 0 \text{ (for all } N) \ , \\ [t_N(z)]^2 &\equiv 0 \text{ (for all } N) \ . \end{aligned}$$

4. **Formula for $r_N(z)$.** Recall that have

$$[r_N(z)]^2 = \frac{|\operatorname{Re} F_N(z)|^2}{|F_N(z)|^2}.$$

Also, recall that $(\operatorname{Re} z)^2 = \frac{1}{2}|z|^2 + \frac{1}{2}z^2$, so that we have

$$(\operatorname{Re} z^{k-1})^2 = \frac{1}{2}|z|^{2(k-1)} + \frac{1}{2}\operatorname{Re} z^{2(k-1)}.$$

This fact gives

$$[r_N(z)]^2 = \frac{\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\operatorname{Re} \frac{\partial}{\partial z} z^\ell \right)^2}{\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial \bar{z}} \bar{z}^\ell \right)} = \frac{\sum_{\ell=0}^{N} \binom{N}{\ell} \left[\frac{1}{2} \left| \frac{\partial}{\partial z} z^\ell \right|^2 + \frac{1}{2} \operatorname{Re} \left(\frac{\partial}{\partial z} z^\ell \right)^2 \right]}{\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial \bar{z}} \bar{z}^\ell \right)}$$

$$= \frac{1}{2} + \frac{1}{2} \operatorname{Re} \frac{\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial z} z^\ell \right)}{\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial \bar{z}} \bar{z}^\ell \right)}.$$

We state a brief fact. It will be more general than what we need here, but the more general form will be used later when we do the several variables case.

Fact 4.4. *For complex numbers $z, w, \tilde{z}$, and $\tilde{w}$, we have the following:*

(1) $\sum_{\ell_1+\ell_2=0}^{N} \binom{N}{\ell_1,\ell_2} (\frac{\partial}{\partial z} z^{\ell_1} w^{\ell_2})(\frac{\partial}{\partial \tilde{z}} \tilde{z}^{\ell_1} \tilde{w}^{\ell_2}) = [N^2 z\tilde{z} + N(1 + w\tilde{w})](1 + z\tilde{z} + w\tilde{w})^{N-2}$
(2) $\sum_{\ell_1+\ell_2=0}^{N} \binom{N}{\ell_1,\ell_2} (\frac{\partial}{\partial w} z^{\ell_1} w^{\ell_2})(\frac{\partial}{\partial \tilde{w}} \tilde{z}^{\ell_1} \tilde{w}^{\ell_2}) = [N^2 w\tilde{w} + N(1 + z\tilde{z})](1 + z\tilde{z} + w\tilde{w})^{N-2}$
(3) $\sum_{\ell_1+\ell_2=0}^{N} \binom{N}{\ell_1,\ell_2} (\frac{\partial}{\partial z} z^{\ell_1} w^{\ell_2})(\frac{\partial}{\partial \tilde{w}} \tilde{z}^{\ell_1} \tilde{w}^{\ell_2}) = (N^2 - N)\tilde{z}w(1 + z\tilde{z} + w\tilde{w})^{N-2}$
(4) $\sum_{\ell_1+\ell_2=0}^{N} \binom{N}{\ell_1,\ell_2} (\frac{\partial}{\partial w} z^{\ell_1} w^{\ell_2})(\frac{\partial}{\partial \tilde{z}} \tilde{z}^{\ell_1} \tilde{w}^{\ell_2}) = (N^2 - N)z\tilde{w}(1 + z\tilde{z} + w\tilde{w})^{N-2}$

Proof. See Appendix.

We use (1) with $w = \tilde{w} = 0$, and we evaluate at $\tilde{z} = z$ to get

$$\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial z} z^\ell \right) = (N^2 z^2 + N)(1 + z^2)^{N-2}$$

and we use (1) with $w = \tilde{w} = 0$, and we evaluate at $\tilde{z} = \bar{z}$ to get

$$\sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial \bar{z}} \bar{z}^\ell \right) = (N^2 |z|^2 + N)(1 + |z|^2)^{N-2}$$

So we have

$$[r_N(z)]^2 = \frac{1}{2} + \frac{1}{2} \operatorname{Re} \frac{[N^2 z^2 + N](1 + z^2)^{N-2}}{[N^2 |z|^2 + N](1 + |z|^2)^{N-2}} = \frac{1}{2} + O(e^{-\lambda N}), \text{ for } z \in \mathbb{C} \backslash \mathbb{R},$$

and uniformly on compact sets $K \subset \mathbb{C} \backslash \mathbb{R}$, and

$$[r_N(z)]^2 \equiv 1, z \in \mathbb{R}.$$

5. **Formula for** $s_N(z)$**.** Next, recall that

$$s_N(z) = \frac{\operatorname{Re} u_N(z) \cdot \operatorname{Im} u_N(z)}{r_N(z)} = \frac{\operatorname{Re} F_N(z) \cdot \operatorname{Im} F_N(z)}{||F_N(z)||^2} \cdot \frac{1}{r_N(z)}$$

Using the identity $\operatorname{Im} w^2 = 2 \operatorname{Re} w \operatorname{Im} w$, for any complex number w (Fact ???), we can write

$$\begin{aligned}
\operatorname{Re} F_N(z) \cdot \operatorname{Im} F_N(z) &= \sum_{\ell=0}^{N} \operatorname{Re} \left[\binom{N}{\ell}^{1/2} \frac{\partial}{\partial z} z^\ell \right] \operatorname{Im} \left[\binom{N}{\ell}^{1/2} \frac{\partial}{\partial z} z^\ell \right] \\
&= \sum_{\ell=0}^{N} \binom{N}{\ell} \frac{1}{2} \operatorname{Im} \left(\frac{\partial}{\partial z} z^\ell \right)^2 \\
&= \frac{1}{2} \operatorname{Im} \sum_{\ell=0}^{N} \binom{N}{\ell} \left(\frac{\partial}{\partial z} z^\ell \right) \left(\frac{\partial}{\partial z} z^\ell \right) \\
&= \frac{1}{2} \operatorname{Im} \left[(N^2 z^2 + N)(1+z^2)^{N-2} \right],
\end{aligned}$$

using the Fact again. We have

$$\begin{aligned}
s_N(z) &= \frac{\frac{1}{2} \operatorname{Im} \left[(N^2 z^2 + N)(1+z^2)^{N-2} \right]}{||F||^2} \cdot \frac{1}{r_N(z)} \\
&= \frac{1}{2} \operatorname{Im} \left[\frac{(N^2 z^2 + N)(1+z^2)^{N-2}}{(N^2|z|^2 + N)(1+|z|^2)^{N-2}} \right] \cdot \frac{1}{r_N(z)} \\
&= \frac{1}{2} \operatorname{Im} \left[\frac{N^2 z^2 + N}{N^2|z|^2 + N} \left(\frac{1+z^2}{1+|z|^2} \right)^{N-2} \right] \cdot \frac{1}{r_N(z)} = O(e^{-\lambda N}), \text{ for } z \in \mathbb{C} \backslash \mathbb{R},
\end{aligned}$$

and

$$s_N(z) \equiv 0, z \in \mathbb{R}.$$

6. **Formula for t.** Since $r^2 + s^2 + t^2 = 1$, we have t easily:

$$[t_N(z)]^2 = 1 - [r_N(z)]^2 - [s_N(z)]^2 = \sqrt{\frac{1}{2}} + O(e^{-\lambda N}), z \in \mathbb{C} \backslash \mathbb{R},$$

and uniformly on compact sets $K \subset \mathbb{C} \backslash \mathbb{R}$, and

$$t \equiv 0, z \in \mathbb{R}.$$

Also, note that all derivatives of $r_N(z), s_N(z)$ and $t_N(z)$ are $O(e^{-\lambda N})$ and that all derivatives converge uniformly on compact sets $K \subset \mathbb{C} \backslash \mathbb{R}$.

7. **Exact Formula for $E_{2,N}(z)$.** Recall that

$$E_{2,N}(z) = \frac{i}{\pi} \int_{\mathbb{R}^2} \partial\bar{\partial} \log |a_0(r+is) + a_1(it)| \, d\mu(a_0) d\mu(a_1),$$

and note that it is of the same form as the formula for $E_{2,N}(z)$ in the zeros case. The only difference is the formulae for r, s, and t. If we can write our integral in polar coordinate and use Jensen's formula the same way as before, we get that

$$E_{2,N}(z) = -\frac{i}{2\pi} \partial\bar{\partial} \log(1+2rt) = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log(1+2rt) \, dx \wedge dy.$$

After further simplification of $2rt$, we have that

$$E_{2,N}(z) = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left(1 + \sqrt{1 - \left| \frac{(N^2 z^2 + N)(1+z^2)^{N-2}}{(N^2|z|^2 + N)(1+|z|^2)^{N-2}} \right|^2} \right) \, dx \wedge dy.$$

8. **Limit for $E_{2,N}(z)$.** Though we got different formulas for r, s, and t, we still had that $r = r_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N})$, $t = t_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N})$, and $s = s_N(z) = O(e^{-\lambda N})$ and all derivatives (in particular, the first and second derivatives) of r, s, and t are $O(e^{-\lambda N})$ on compact sets $K \subset \mathbb{C} \backslash \mathbb{R}$. So we can still say that

$$E_{2,N}(z) = -\frac{i}{2\pi} \partial\bar{\partial} \log(1+2rt) = O(e^{-\lambda N})$$

on compact sets $K \subset \mathbb{C} \backslash \mathbb{R}$, which is our desired result.

9. **Scaling limit for $E_{2,N}(z)$.** We have

$$\lim_{N \to \infty} \frac{1}{N} E_{2,N}\left(\frac{z}{\sqrt{N}}\right) = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \log \left(1 + \sqrt{1 - \left| \frac{(1+z^2)e^{z^2}}{(1+|z|^2)e^{|z|^2}} \right|^2} \right) \, dx \wedge dy, z \in \mathbb{C}$$

10. **Weak limit for $E_{2,N}(z)$.** Let $K \subset \mathbb{C}$ be a compact set. We now show that $\frac{1}{N} E_2$ goes to 0 weakly on K. More specifically, we show that for any $\phi \in C^\infty(K)$,

$$\frac{1}{N}(E_{2,N}(z), \phi(z)) = \frac{1}{N} \int_{\mathbb{K}} E_{2,N}(z) \, \phi(z) \, dz = O(N^{-1}).$$

We have shown that

$$\begin{aligned} (E_{2,N}(z), \phi(z)) &= \int_{\mathbb{K}} \left(\int_{\mathbb{R}^\mathbb{N}} \log |a \cdot u_N(z)| \, d\mu(a) \right) \frac{i}{\pi} \partial\bar{\partial} \phi(z) \, dz \\ &= \int_{\mathbb{K}} \left(\frac{1}{2} \log(1+2rt) \right) \left(\frac{i}{\pi} \partial\bar{\partial} \phi(z) \right) \, dz \end{aligned}$$

In the zeros case, we used the fact that $1 \leq (1+2rt) \leq 3$, everywhere on $\mathbb{C}$ and, in particular, on K. Here, r and t are different, but we still have that both are non-negative by construction, and both are bounded by 1 since $r^2+s^2+t^2=1$. So we still have the rough bound $1 \leq (1+2rt) \leq 3$, and we therefore have

$$\begin{aligned}\int_{\mathbb{K}} \left(\frac{1}{2}\log(1+2rt)\right)\left(\frac{i}{\pi}\partial\bar{\partial}\phi(z)\right)\,dz &\leq \int_{\mathbb{K}} C\frac{i}{\pi}\partial\bar{\partial}\phi(z)\,dz,\\ &= C||\frac{i}{\pi}\partial\bar{\partial}\phi(z)||_{L^1(K)} \leq C_K,\end{aligned}$$

where C_K is independent of z and N and depends only on K. We now have want we want:

$$\frac{1}{N}(E_{2,N}(z),\phi(z)) \leq \frac{1}{N}C_K = O(N^{-1}).$$

11. **Appendix-Proof of Fact.**

Proof. We prove just (1) and (3).

(1)

$$\sum_{\ell_1+\ell_2=0}^{N}\binom{N}{\ell_1,\ell_2}(\frac{\partial}{\partial z}z^{\ell_1}w^{\ell_2})(\frac{\partial}{\partial\tilde{z}}\tilde{z}^{\ell_1}\tilde{w}^{\ell_2})=\frac{\partial}{\partial z}\frac{\partial}{\partial\tilde{z}}\sum_{\ell_1+\ell_2=0}^{N}\binom{N}{\ell_1,\ell_2}z^{\ell_1}w^{\ell_2}\tilde{z}^{\ell_1}\tilde{w}^{\ell_2}$$
$$=\frac{\partial}{\partial z}\frac{\partial}{\partial\tilde{z}}(1+z\tilde{z}+w\tilde{w})^N=\frac{\partial}{\partial z}N(1+z\tilde{z}+w\tilde{w})^{N-1}z$$
$$=N(N-1)(1+z\tilde{z}+w\tilde{w})^{N-2}z\tilde{z}+N(1+z\tilde{z}+w\tilde{w})^{N-1}$$
$$=N^2(1+z\tilde{z}+w\tilde{w})^{N-2}z\tilde{z}-N(1+z\tilde{z}+w\tilde{w})^{N-2}z\tilde{z}+N(1+z\tilde{z}+w\tilde{w})^{N-2}(1+z\tilde{z}+w\tilde{w})$$
$$=N^2(1+z\tilde{z}+w\tilde{w})^{N-2}z\tilde{z}+N(1+z\tilde{z}+w\tilde{w})^{N-2}(1+w\tilde{w})$$
$$=[N^2z\tilde{z}+N(1+w\tilde{w})](1+z\tilde{z}+w\tilde{w})^{N-2}$$

(2) Similar to (1).

(3)

$$\sum_{\ell_1+\ell_2=0}^{N}\binom{N}{\ell_1,\ell_2}(\frac{\partial}{\partial z}z^{\ell_1}w^{\ell_2})(\frac{\partial}{\partial\tilde{w}}\tilde{z}^{\ell_1}\tilde{w}^{\ell_2})=\frac{\partial}{\partial z}\frac{\partial}{\partial\tilde{w}}\sum_{\ell_1+\ell_2=0}^{N}\binom{N}{\ell_1,\ell_2}z^{\ell_1}w^{\ell_2}\tilde{z}^{\ell_1}\tilde{w}^{\ell_2}$$
$$=\frac{\partial}{\partial z}\frac{\partial}{\partial\tilde{w}}(1+z\tilde{z}+w\tilde{w})^N=\frac{\partial}{\partial z}N(1+z\tilde{z}+w\tilde{w})^{N-1}w$$
$$=N(N-1)(1+z\tilde{z}+w\tilde{w})^{N-2}\tilde{z}w$$

(4) Similar to (3).

5. Density of zeros - m variables case

In this section we are concerned with the zeros of $h_{m,N} = (f_{1,N}, ..., f_{m,N}) : \mathbb{C}^m \to \mathbb{C}^m$, where $f_{q,N}$ is a polynomial of the form

$$f_{q,N}(z) = \sum_{|J|=0}^{N} a_J^q \binom{N}{J}^{1/2} z^J$$

where a_J^q is a *real* standard Gaussian random variable, and where we use the following multi-index notation:

$$\begin{aligned} z &= (z_1, ..., z_m) \\ |J| &= j_1 + \cdots + j_m \\ a_J^q &= a_{j_1 ... j_m}^q \in \mathbb{R} \\ \binom{N}{J} &= \binom{N}{j_1, ..., j_m} = \frac{N!}{(N - j_1 - ... - j_m)! j_1! \; ... \; j_m!} \\ z^J &= z_1^{j_1} ... z_m^{j_m}. \end{aligned}$$

Instead, we choose to think of the random polynomials

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J,$$

where c_J^q is a more general complex random variable with associated measure $d\gamma$ for each q. We then consider two special cases

$$\begin{aligned} d\gamma_{cx} &= \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}, D_N = \binom{N+m}{m} \\ d\gamma_{real} &= \delta_S \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}, \end{aligned}$$

where δ_S is the delta function on $S \subset \mathbb{C}^{D_N}$, the set of points $c = a + ib \in \mathbb{C}^{D_N}$ where $b = 0 \in \mathbb{R}^{D_N}$. Here $d\gamma_{cx}$ corresponds to the standard complex Gaussian coefficients case, where we are considering

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J,$$

where the c_J^q's are standard complex Gaussian random variables, and $d\gamma_{real}$ corresponds to the standard real Gaussian coefficients case, where we have

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J^q \binom{N}{J}^{1/2} z^J = \sum_{|J|=0}^{N} a_J^q \binom{N}{J}^{1/2} z^J,$$

where $c_J^q = a_J^q + i0$ is a standard real Gaussian random variable. We let $E(\cdot)$ denote expectation with respect to $d\gamma_{real}$ and $E_{\gamma_{cx}}(\cdot)$ denote expectation with respect to $d\gamma_{cx}$.

Let $K \subseteq \mathbb{C}^m \backslash \mathbb{R}^m$ be compact, and let λ be a positive constant. The goal of this section is to show the following results about the density of zeros of h_N using the Poincare-Lelong formula: we write

$$E(Z_{h_N})(z) = E_{\gamma_{cx}}(Z_{h_N})(z) + \tilde{E}_N(z) d\omega$$

where

$$E_{\gamma_{cx}}(Z_{h_N})(z) = \frac{mN^m}{\pi^m} \frac{1}{(1+||z||^2)^{m+1}} d\omega,$$

where

$$d\omega = dx_1 \wedge dy_1 \ldots \wedge dx_m \wedge dy_m,$$

$||z||^2 = |z_1|^2 + \cdots + |z_m|^2$, and $\tilde{E}_N(z)$ is some "error" term. We give an exact formula for $\tilde{E}_N(z)$, and show that it goes to zero rapidly, i.e.,

$$\tilde{E}_N(z) = O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K,$$

so that we have

$$E(Z_{h_N})(z) = E_{\gamma_{cx}}(Z_{h_N})(z) + O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K$$

and,

$$\frac{1}{N^k} E(Z_{h_N})(z) = \frac{k}{\pi^m} \frac{1}{(1+|z|^2)^{m+1}} d\omega + O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K.$$

In other words, at any point away from $\mathbb{R}^m$, the expected density of zeros in the real coefficients case approaches the expected density of zeros in the complex coefficients case as N gets large.

We also give a formula for the scaling limit of $\tilde{E}_N(z)$, which we denote $\tilde{E}_\infty(z)$, and show that as $|\operatorname{Im} z| \to \infty, \tilde{E}_\infty(z) \to 0$. In words, the scaled density of zeros in the real coefficients case approaches the scaled density of zeros in the complex coefficients case as you move far away from $\mathbb{R}^m$.

We follow the proof in the one and two variable case and begin by writing

$$a^q = (a^q_{0,\ldots,0}, \ldots, a^q_J, \ldots, a^q_{0,\ldots,0,N}) \in \mathbb{R}^{D_N}$$

$$F_{q,N}(z) = F_N(z) = (\binom{N}{0,\ldots,0}^{1/2}, \ldots, \binom{N}{J}^{1/2} z^J, \ldots, \binom{N}{0,\ldots,0,N}^{1/2} z_1^0 \ldots z_m^N) \in \mathbb{R}^{D_N}$$

where $D_N = \binom{N+m}{m}$, so that we can write $f_{q,N} = a^q \cdot F_{q,N} = a^q \cdot F_N$.

By the Poincare-Lelong formula, we have

$$\begin{aligned} E(Z_{f_1}(z) \times \ldots \times Z_{f_m}(z)) &= E(\frac{i}{2\pi} \partial\bar{\partial} \log|f_1|^2 \wedge \ldots \wedge \frac{i}{2\pi} \partial\bar{\partial} \log|f_m|^2) \\ &= E\Big[\left(\frac{i}{2\pi}\right)^m \Big(\partial\bar{\partial} \log|a^1 \cdot F|^2 \wedge \ldots \wedge \partial\bar{\partial} \log|a^m \cdot F|^2 \Big) \Big] \end{aligned}$$

which we can write more succinctly as

$$= \left(\frac{i}{2\pi}\right)^m E\Big(\bigwedge_{q=1}^{m} \partial\bar{\partial}\log\left|a^q\cdot F\right|^2\Big).$$

Writing $F = \frac{F}{||F||}||F||$ and $u = \frac{F}{||F||}$, we can write this as

$$\begin{aligned}
&\left(\frac{i}{2\pi}\right)^m E\Big(\bigwedge_{q=1}^{m} \partial\bar{\partial}\log\left|a^q\cdot\frac{F}{||F||}||F||\right|^2\Big) \\
=&\left(\frac{i}{2\pi}\right)^m E\Big[\bigwedge_{q=1}^{m} \left(\partial\bar{\partial}\log||F||^2 + \partial\bar{\partial}\log|a^q\cdot u|^2\right)\Big]
\end{aligned}$$

Since $\partial\bar{\partial}\log||F||^2 + \partial\bar{\partial}\log|a^q\cdot u|^2$ is independent of a^ℓ for $l \neq q$, then by a lemma in SZ [???] we can write this term as

$$\begin{aligned}
&\left(\frac{i}{2\pi}\right)^m \bigwedge_{q=1}^{m} E\Big[\partial\bar{\partial}\log||F||^2 + \partial\bar{\partial}\log|a^q\cdot u|^2\Big] \\
=&\left(\frac{i}{2\pi}\right)^m \bigwedge_{q=1}^{m} \Big(\partial\bar{\partial}\log||F||^2 + E\Big[\partial\bar{\partial}\log|a^q\cdot u|^2\Big]\Big).
\end{aligned}$$

At this point, we could find the large N limit; we have essentially reduced the m-variables case to the same calculation as the 1-variable case, namely showing that $E\Big[\partial\bar{\partial}\log|a^q\cdot u|^2\Big]$ is $O(e^{-\lambda N})$. If this term is indeed $O(e^{-\lambda N})$, then all but one term in the wedge product goes to zero exponentially fast. Since we want an exact formula for the density of zeros, we delay the proof of the large N limit, and we first work out the details of writing an exact formula more explicitly. From that formula, the large N limit and the scaling limit will follow easily.

We write

$$\left(\frac{i}{2\pi}\right)^m E\Big[\bigwedge_{q=1}^{m} \left(\partial\bar{\partial}\log||F||^2 + \partial\bar{\partial}\log|a^q\cdot u|^2\right)\Big] = (E_{1,N}(z) + E_{2,N}(z) + ... + E_{2^m,N}(z))d\omega$$

where

$$E_{1,N}(z)d\omega := \left(\frac{i}{2\pi}\right)^m E\Big(\partial\bar\partial \log ||F||^2 \wedge \partial\bar\partial \log ||F||^2 \wedge ... \wedge \partial\bar\partial \log ||F||^2\Big)$$

$$E_{2,N}(z)d\omega := \left(\frac{i}{2\pi}\right)^m E\Big(\partial\bar\partial \log |a^1 \cdot u| \wedge \partial\bar\partial \log ||F||^2 \wedge ... \wedge \partial\bar\partial \log ||F||^2\Big)$$

$$\vdots$$

$$E_{2^m,N}(z)d\omega := \left(\frac{i}{2\pi}\right)^m E\Big(\partial\bar\partial \log |a^1 \cdot u| \wedge \partial\bar\partial \log |a^2 \cdot u| \wedge ... \wedge \partial\bar\partial \log |a^m \cdot u|\Big)$$

We look at these 2^m terms and we claim that only the first term is non-zero in the limit. The first term is known:

$$E_{1,N}(z)d\omega = E(Z_{h_N}) = \left(\frac{i}{2\pi}\right)^m E\Big(\partial\bar\partial \log ||F||^2 \wedge \partial\bar\partial \log ||F||^2 \wedge ... \wedge \partial\bar\partial \log ||F||^2\Big)$$
$$= \frac{mN^m}{\pi^m} \frac{1}{(1+|z_1|^2 + ... + |z_m|^2)^{m+1}} d\omega = \frac{mN^m}{\pi^m} \frac{1}{(1+||z||^2)^{m+1}} d\omega$$

1. **The remaining terms.** We know show that the remaining terms $E_2, ..., E_{2^m}$ are $O(e^{-\lambda N})$. Consider the i-th term, E_q. This term is of the form

$$E_{q,N}(z)d\omega = E(\partial\bar\partial\phi_1^q \wedge ... \wedge \partial\bar\partial\phi_m^q)$$

where $\phi_{l,N}^q(z)$ is either $\log ||F_N(z)||$ or $\log |a^\ell \cdot u_N(z)|$ for each ℓ. For example, for $E_{2,N}(z)$ we have $\phi_1^2 = \log |a^1 \cdot u|$ and $\phi_\ell^2 = \log ||F_N(z)||$ for $1 < \ell \leq k$. Writing out the wedge product we get

$$E_{q,N}(z) = E\left[\sum_{\sigma,\tau} (-1)^{\sigma+\tau} \left(\frac{\partial^2}{\partial z_{\sigma(1)}\partial \bar z_{\tau(1)}}\phi_1^q\right) \cdots \left(\frac{\partial^2}{\partial z_{\sigma(m)}\partial \bar z_{\tau(m)}}\phi_m^q\right)\right]$$

where the sum is over all permutations σ and τ of $\{1, 2, ..., m\}$, and where $(-1)^\sigma$ denotes the sign associated to the permutation σ. Since the sum is finite, we can write

$$E_{q,N}(z) = \sum_{\sigma,\tau} (-1)^{\sigma+\tau} E\left[\left(\frac{\partial^2}{\partial z_{\sigma(1)}\partial \bar z_{\tau(1)}}\phi_1^q\right) \cdots \left(\frac{\partial^2}{\partial z_{\sigma(m)}\partial \bar z_{\tau(m)}}\phi_m^q\right)\right]$$

or

$$E_{q,N}(z) = \sum_{\sigma,\tau} (-1)^{\sigma+\tau} E_{q,N}^{\sigma,\tau}(z)$$
$$\text{where} \quad E_{q,N}^{\sigma,\tau}(z) := E\left[\prod_{\ell=1}^m \left(\frac{\partial^2}{\partial z_{\sigma(\ell)}\partial \bar z_{\tau(\ell)}}\phi_\ell^q(z)\right)\right]$$

To simplify notation even more, let

$$D_\ell = \frac{\partial^2}{\partial z_{\sigma(\ell)} \partial \bar{z}_{\tau(\ell)}}$$

so that we have

$$E_{q,N}^{\sigma,\tau}(z) = E\left[\prod_{\ell=1}^{m} D_\ell \phi_\ell^q(z)\right]$$

Now, note that $\phi_\ell^q(z)$ does not depend on all of $a^1, ..., a^m$, but only depends *at most* on a^ℓ. (If $\phi_\ell^q(z) = \log ||F_N(z)||$, then it doesn't depend on a^ℓ either.) So because $\phi_\ell^q(z)$ is independent of $a^{\ell'}$ for all $\ell' \neq \ell$, we will write this integral over $\mathbb{R}^{D_N} \times \cdots \times \mathbb{R}^{D_N}$ as a product of integrals over $\mathbb{R}^{D_N}$. To do this, we first let

$$L_q = \{\ell : \phi_\ell^q \text{ is of the form } \log ||F_N(z)||\} \subset \{1, \ldots, m\}$$

and

$$L_q' = \{\ell : \phi_\ell^q \text{ is of the form } \log |a^\ell \cdot u_N(z)|\} = \{1, \ldots, m\} \backslash L.$$

We can now we split the product to get

$$\begin{aligned} E_{q,N}^{\sigma,\tau}(z) &= E\left[\left(\prod_{\ell\in L} D_\ell \phi_\ell^q(z)\right)\left(\prod_{\ell\in L'} D_\ell \phi_\ell^q(z)\right)\right] \\ &= E\left[\left(\prod_{\ell\in L} D_\ell \log ||F_N(z)||\right)\left(\prod_{\ell\in L'} D_\ell \log |a^\ell \cdot u_N(z)|\right)\right] \end{aligned}$$

By the definition of expected value, we have

$$E_{q,N}^{\sigma,\tau}(z) = \int_{\mathbb{R}^{D_N}} \left[\left(\prod_{\ell\in L} D_\ell \log ||F_N(z)||\right)\left(\prod_{\ell\in L'} D_\ell \log |a^\ell \cdot u_N(z)|\right)\right] d\mu(a^1)\cdots d\mu(a^m),$$

Note that the first product is independent of a^ℓ for all $\ell \in L'$, and the second product is independent of all $\ell \in L$, so we can write $E_{q,N}^{\sigma,\tau}(z)$ as

$$\left[\int_{\mathbb{R}^{|L|D_N}} \prod_{\ell\in L} D_\ell \log ||F_N(z)|| d\mu(a^\ell)\right]\left[\int_{\mathbb{R}^{|L'|D_N}} \prod_{\ell\in L'} D_\ell \log |a^\ell \cdot u_N(z)| \ d\mu(a^\ell)\right]$$

The first product is also independent of a^ℓ for all $\ell \in L$, and since $\int_{\mathbb{R}^{D_N}} d\mu(a^\ell) = 1$, we have for the first integral

$$\prod_{\ell\in L} D_\ell \log ||F_N(z)||$$

Even more, the ℓ-th factor in the second product depends only on ℓ and is therefore independent of all $\ell' \in L'$ not equal to ℓ. So the integral of this product becomes a product of the integrals:

$$\prod_{\ell\in L'} \int_{\mathbb{R}^{D_N}} D_\ell \log |a^\ell \cdot u_N(z)| \ d\mu(a^\ell)$$

and we can switch the derivatives and the integral to get

$$\prod_{\ell \in L'} D_\ell \int_{\mathbb{R}^{D_N}} \log |a^\ell \cdot u_N(z)| \; d\mu(a^\ell).$$

Putting everything together we have

$$E^{\sigma,\tau}_{q,N}(z) = \left[\prod_{\ell \in L} D_\ell \log ||F_N(z)||\right] \left[\prod_{\ell \in L'} D_\ell \int_{\mathbb{R}^{D_N}} \log |a^\ell \cdot u_N(z)| \; d\mu(a^\ell)\right]$$

Now, consider the integral

$$\int_{\mathbb{R}^{D_N}} \log |a^\ell \cdot u_N(z)| \; d\mu(a^\ell).$$

Note that the form of the integral is the same as what we got in the one variable case, so we proceed in a similar manner. We rotate $(a_{0\cdots 0}, ..., a_{0\cdots 0N})$ and then $(a_{10\cdots 0}, ..., a_{0\cdots 0N})$ so that the integral becomes

$$\int_{\mathbb{R}^{D_N}} \log |a^\ell \cdot (r + is, it, 0, ..., 0)| \, d\mu(a^\ell) = \int_{\mathbb{R}^2} \log |a_{0\cdots 0}(r + is) + a_{10\cdots 0} it| \, d\mu(a_{0\cdots 0}) \, d\mu(a_{10\cdots 0})$$

where $r = r_N(z), s = s_N(z)$, and $t = t_N(z)$.

We have similar formulae and large N limits for r, s, and t:

$$[r_N(z)]^2 = \frac{1}{2} + \frac{1}{2} \operatorname{Re} \left(\frac{1 + z_1^2 + \cdots + z_m^2}{1 + |z_1|^2 + \cdots + |z_m|^2} \right)^N = \frac{1}{2} + O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K$$

$$s_N(z) = \frac{1}{2} \operatorname{Im} \left(\frac{1 + z_1^2 + \cdots + z_m^2}{1 + |z_1|^2 + \cdots + |z_m|^2} \right)^N \frac{1}{r_N(z)} = O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K$$

$$[t_N(z)]^2 = 1 - [r_N(z)]^2 - [s_N(z)]^2 = \frac{1}{2} + O(e^{-\lambda N}), z \in \mathbb{C}^m \backslash \mathbb{R}^m, \text{ and uniformly on } K.$$

Also, all derivatives of r, s and t are $O(e^{-\lambda N})$ uniformly on K.

2. **Exact formula for $\tilde{E}_N(z)$.** By the calculation we did before, we can write

$$\int_{\mathbb{R}^2} \log |a_{0\cdots 0}(r + is) + a_{10\cdots 0} it| \, d\mu(a_{0\cdots 0}) \, d\mu(a_{10\cdots 0}) = \frac{1}{2} \log(1 + 2rt),$$

which gives us

$$E^{\sigma,\tau}_{q,N}(z) = \left[\prod_{\ell \in L} D_\ell \log ||F_N(z)||\right] \left[\prod_{\ell \in L'} D_\ell \frac{1}{2} \log(1 + 2rt)\right]$$

Further simplification gives

$$E^{\sigma,\tau}_{q,N}(z) = \left[\prod_{\ell \in L} D_\ell \frac{1}{2} \log(1 + ||z||^2)^N\right] \left[\prod_{\ell \in L'} D_\ell \frac{1}{2} \log \left(1 + \sqrt{1 - \left|\frac{(1 + z \cdot z)^N}{(1 + ||z||^2)^N}\right|^2}\right)\right]$$

where we use the notation $z \cdot z = z_1^2 + \cdots + z_m^2$, $||z||^2 = |z_1|^2 + \cdots + |z_m|^2$. Finally,

$$E_{q,N}(z) = \sum_{\sigma,\tau} (-1)^{\sigma+\tau} E_{q,N}^{\sigma,\tau}(z)$$

and

$$\tilde{E}_N(z) = \sum_{q=2}^{2^m} E_{q,N}(z).$$

3. **Large N limit for $\tilde{E}_N(z)$.** All derivatives of $\log ||F||$ are bounded. Next, $r = r_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N})$, $t = t_N(z) = \sqrt{\frac{1}{2}} + O(e^{-\lambda N})$, and $s = s_N(z) = O(e^{-\lambda N})$ and all derivatives (in particular, the first and second derivatives) of r, s, and t are $O(e^{-\lambda N})$ on $\mathbb{C}^m \backslash \mathbb{R}^m$. So we can say that all second derivatives of $\log(1 + 2rt)$ are $O(e^{-\lambda N})$ on $\mathbb{C}^m \backslash \mathbb{R}^m$. This means that

$$E_{q,N}^{\sigma,\tau}(z) = O(e^{-\lambda N}), z \in K.$$

Since this is true for each i, σ, and τ, we have

$$\tilde{E}_N(z) = \sum_{q=2}^{2^m} \sum_{\sigma,\tau} (-1)^{\sigma+\tau} E_{q,N}^{\sigma,\tau}(z) = O(e^{-\lambda N}), z \in K.$$

4. **Scaling limit for $\tilde{E}_N(z)$.** We have

$$\frac{1}{N^k} E_{q,N}^{\sigma,\tau}(\frac{z}{\sqrt{N}}) \to E_{q,\infty}^{\sigma,\tau}(z) := \left[\prod_{\ell \in L} D_\ell ||z||^2\right] \left[\prod_{\ell \in L'} D_\ell \frac{1}{2} \log\left(1 + \sqrt{1 - \left|\frac{e^{z\cdot z}}{e^{||z||^2}}\right|^2}\right)\right]$$

where we use the notation $z \cdot z = z_1^2 + \cdots + z_m^2$, $||z||^2 = |z_1|^2 + \cdots + |z_m|^2$. If we write $z = x + iy$, then $z \cdot z = |x|^2 + 2i(x \cdot y) + |y|^2$. Since $|e^{2i(x\cdot y)}| = 1$, we can write the second product as

$$\prod_{\ell \in L'} D_\ell \frac{1}{2} \log\left(1 + \sqrt{1 - e^{-4|y|^2}}\right)$$

Since the first product is bounded (it is either 1 or 0 for each ℓ, depending on $\sigma(\ell)$ and $\tau(\ell)$), and the second product goes to zero exponentially fast as $|y| \to \infty$, we have

$$E_{q,\infty}^{\sigma,\tau}(z) \to 0, \text{ as } |y| \to \infty.$$

Since this is true for each i, σ, and τ we have

$$\tilde{E}_\infty(z) = \sum_{q=2}^{2^m} \sum_{\sigma,\tau} (-1)^{\sigma+\tau} E_{q,\infty}^{\sigma,\tau}(z) \to 0, \text{ as } |y| \to \infty.$$

5. **Exact formula and Scaling limit for the 2 variable case.** When $m = 2$, the formulas for the density of zeros and the scaling limit of the density of zeros are simple enough to be written as a sum of 3 terms, and we work out the details now. We have $h_N(z,w) = (f_N(z,w), g_N(z,w)) : \mathbb{C}^2 \to \mathbb{C}^2$, where

$$f_N(z,w) = \sum_{j+k=0}^{N} a_{jk} \binom{N}{j,k}^{1/2} z^j w^k$$
$$g_N(z,w) = \sum_{i+l=0}^{N} b_{il} \binom{N}{i,l}^{1/2} z^i w^l$$

We write

$$a = (a_{00}, a_{10}, a_{20}, ..., a_{N0}, a_{01}, a_{11}, ..., a_{0N}) \in \mathbb{R}^{N(N+1)/2}$$
$$b = (b_{00}, b_{10}, b_{20}, ..., b_{N0}, b_{01}, b_{11}, ..., b_{0N}) \in \mathbb{R}^{N(N+1)/2}$$
$$F_N(z,w) = G_N(z,w) = \Big(\binom{N}{0,0}^{1/2} z^0 w^0, \binom{N}{1,0}^{1/2} z^1 w^0, ..., \binom{N}{0,N}^{1/2} z^0 w^N\Big) \in \mathbb{R}^{N(N+1)/2}$$

so that $f = a \cdot F$ and $g = b \cdot G = b \cdot F$. After some algebra (see Appendix), we have for the first term, $E_{1,N}(z,w)$,

$$E_{1,N}(z,w) = -N^2 \frac{1}{2\pi^2} \frac{1}{(1+|z|^2+|w|^2)^3}$$
$$= N^2 \frac{2}{\pi^2} \frac{1}{(1+|z|^2+|w|^2)^3}$$

Next, note that since a and b are identically distributed,

$$E_{2,N}(z,w)d\omega = E\Big(\partial\bar{\partial}\log||F||^2 \wedge \partial\bar{\partial}\log|b\cdot u|^2\Big) = E\Big(\partial\bar{\partial}\log||F||^2 \wedge \partial\bar{\partial}\log|a\cdot u|^2\Big)$$

and since the $\partial\bar{\partial}$-terms are all 2-forms, we have that

$$E\Big(\partial\bar{\partial}\log||F||^2 \wedge \partial\bar{\partial}\log|a\cdot u|^2\Big) = E\Big(\partial\bar{\partial}\log|a\cdot u|^2 \wedge \partial\bar{\partial}\log||F||^2\Big) = E_{3,N}(z,w)d\omega,$$

which means

$$E_{2,N}(z,w) = E_{3,N}(z,w)$$

Writing $z_1 = z$ and $z_2 = w$, we have

$$E_{2,N}(z_1,z_2) = E_{3,N}(z_1,z_2) = \sum_{\sigma,\tau} \frac{\partial^2}{\partial z_{\sigma(1)}\partial \bar{z}_{\tau(1)}} \frac{1}{2}\log(1+2rt) \frac{\partial^2}{\partial z_{\sigma(2)}\partial \bar{z}_{\tau(2)}} \log||F||^2$$
$$E_{4,N}(z_1,z_2) = \sum_{\sigma,\tau} \frac{\partial^2}{\partial z_{\sigma(1)}\partial \bar{z}_{\tau(1)}} \frac{1}{2}\log(1+2rt) \frac{\partial^2}{\partial z_{\sigma(2)}\partial \bar{z}_{\tau(2)}} \frac{1}{2}\log(1+2rt)$$

and

$$\tilde{E}_N(z_1,z_2) = E_{2,N}(z_1,z_2) + E_{3,N}(z_1,z_2) + E_{4,N}(z_1,z_2)$$
$$= \sum_{\sigma,\tau} \frac{\partial^2}{\partial z_{\sigma(1)}\partial \bar{z}_{\tau(1)}} \left(\frac{1}{2}\log(1+2rt)\right) \frac{\partial^2}{\partial z_{\sigma(2)}\partial \bar{z}_{\tau(2)}} \left(2\log||F||^2 + \frac{1}{2}\log(1+2rt)\right).$$

We now write out the sum to get

$$\begin{aligned} &E_{2,N}(z,w) + E_{3,N}(z,w) + E_{4,N}(z,w) \\ &\quad = \tfrac{\partial^2}{\partial z\partial \bar{z}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial w\partial \bar{w}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \\ &\quad - \tfrac{\partial^2}{\partial z\partial \bar{w}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial w\partial \bar{z}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \\ &\quad - \tfrac{\partial^2}{\partial w\partial \bar{z}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial z\partial \bar{w}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \\ &\quad + \tfrac{\partial^2}{\partial w\partial \bar{w}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial z\partial \bar{z}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \end{aligned}$$

Since

$$\frac{\partial^2}{\partial w\partial \bar{z}}\psi(z,w) = \frac{\partial^2}{\partial \bar{z}\partial w}\psi(z,w) = \overline{\frac{\partial^2}{\partial z\partial \bar{w}}\psi(z,w)}$$

for any real function $\psi(z,w)$, and since

$$\frac{\partial^2}{\partial z\partial \bar{w}}\psi(z,w) + \overline{\frac{\partial^2}{\partial z\partial \bar{w}}\psi(z,w)} = 2\,\mathrm{Re}\,\frac{\partial^2}{\partial z\partial \bar{w}}\psi(z,w),$$

we have

$$\begin{aligned} E_{2,N}(z) + E_{3,N}(z) + E_{4,N}(z) &= \tfrac{\partial^2}{\partial z\partial \bar{z}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial w\partial \bar{w}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \\ &\quad -2\,\mathrm{Re}\,\tfrac{\partial^2}{\partial z\partial \bar{w}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial w\partial \bar{z}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right) \\ &\quad + \tfrac{\partial^2}{\partial w\partial \bar{w}} \left(\tfrac{1}{2}\log(1+2rt)\right) \tfrac{\partial^2}{\partial z\partial \bar{z}} \left(2\log||F||^2 + \tfrac{1}{2}\log(1+2rt)\right). \end{aligned}$$

where

$$||F||^2 = (1+|z|^2+|w|^2)^N$$
$$2rt = \sqrt{1 - \frac{(1+z^2+w^2)^N}{(1+|z|^2+|w|^2)^N}}$$

This simplified formula is still messy, but it can at least be written in a couple lines, and is useful enough to plug into Maple, for example, and get a plot.

6. **Scaling Limit for** $\tilde{E}_N(z,w)$**.** Writing $z = x+iy$ and $w = u+iv$, we have

$$\frac{1}{N^2}E_{2,N}(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}) + E_{3,N}(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}) + E_{4,N}(\frac{z}{\sqrt{N}},\frac{w}{\sqrt{N}}) \to$$

$$\tfrac{\partial^2}{\partial z\partial\bar{z}}\left(\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right)\tfrac{\partial^2}{\partial w\partial\bar{w}}\left(2\log(|z|^2+|w|^2)+\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right)$$

$$-2\,\mathrm{Re}\,\tfrac{\partial^2}{\partial z\partial\bar{w}}\left(\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right)\tfrac{\partial^2}{\partial w\partial\bar{z}}\left(2\log(|z|^2+|w|^2)+\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right)$$

$$+\tfrac{\partial^2}{\partial w\partial\bar{w}}\left(\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right)\tfrac{\partial^2}{\partial z\partial\bar{z}}\left(2\log(|z|^2+|w|^2)+\tfrac{1}{2}\log(1+\sqrt{1-e^{-4(y^2+v^2)}})\right).$$

7. **Appendix - Calculation of** $E_{1,N}(z,w)$**.**

$$E_{1,N}(z,w)d\omega = \partial\bar{\partial}\log||F||^2 \wedge \partial\bar{\partial}\log||F||^2$$

$$= \left[2\left(\frac{\partial^2}{\partial z\partial\bar{z}}\log||F||^2\right)\left(\frac{\partial^2}{\partial w\partial\bar{w}}\log||F||^2\right) - 2\left(\frac{\partial^2}{\partial z\partial\bar{w}}\log||F||^2\right)\left(\frac{\partial^2}{\partial w\partial\bar{z}}\log||F||^2\right)\right]d\omega$$

$$=2N^2\Big[\left(\frac{1+|z|^2}{(1+|z|^2+|w|^2)^2}\right)\left(\frac{1+|w|^2}{(1+|z|^2+|w|^2)^2}\right) - \left(\frac{-w\bar{z}}{(1+|z|^2+|w|^2)^2}\right)\left(\frac{-z\bar{w}}{(1+|z|^2+|w|^2)^2}\right)\Big]d\omega$$

$$=2N^2\frac{(1+|z|^2)(1+|w|^2)-|zw|^2}{(1+|z|^2+|w|^2)^4}d\omega$$

$$=2N^2\frac{(1+|w|^2+|z|^2+|zw|^2)-|zw|^2}{(1+|z|^2+|w|^2)^4}d\omega$$

$$=2N^2\frac{1+|w|^2+|z|^2}{(1+|z|^2+|w|^2)^3}d\omega = 2N^2\frac{1}{(1+|z|^2+|w|^2)^2}d\omega$$

after some algebra. So we have

$$-\frac{1}{4\pi^2}E\Big(\partial\bar{\partial}\log||F||^2\wedge\partial\bar{\partial}\log||F||^2\Big) = -\frac{2N^2}{4\pi^2}E\Big(\frac{1}{(1+|z|^2+|w|^2)^3}d\omega\Big)$$

$$= -\frac{N^2}{2\pi^2}\int\left[\frac{1}{(1+|z|^2+|w|^2)^3}d\omega\right]\,d\mu(a)d\mu(b)$$

and since the integrand is independent of a and b, and since $d\omega = dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$, this equals

$$\begin{aligned}
&-\frac{N^2}{2\pi^2}\frac{1}{(1+|z|^2+|w|^2)^3}\,dz \wedge d\bar{z} \wedge dw \wedge d\bar{w} \\
=&-\frac{N^2}{2\pi^2}\frac{1}{(1+|z|^2+|w|^2)^3}\,(-2i)dx \wedge dy \wedge (-2i)du \wedge dv \\
=&\frac{2N^2}{\pi^2}\frac{1}{(1+|z|^2+|w|^2)^3}\,dx \wedge dy \wedge du \wedge dv
\end{aligned}$$

6. Density of Critical Points in m variables case using Kac-Rice method

We first consider $h_{m,N} : \mathbb{C}^m \to \mathbb{C}$ where $h_{m,N}$ is a complex random polynomial of the form

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J$$

where the c_J's are *complex* random variables, with associated measure $d\gamma$ and where we use the following multi-index notation:

$$\begin{aligned} z &= (z_1, ..., z_m) \\ |J| &= j_1 + \cdots + j_m \\ c_J &= c_{j_1 \dots j_m} \in \mathbb{C} \\ \binom{N}{J} &= \binom{N}{j_1, ..., j_m} = \frac{N!}{(N - j_1 - ... - j_m)! j_1! \; ... \; j_m!} \\ z^J &= z_1^{j_1} ... z_m^{j_m}. \end{aligned}$$

We consider two special cases

$$\begin{aligned} d\gamma_{cx} &= \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}, D_N = \binom{N+m}{m} \\ d\gamma_{real} &= \delta_S \frac{1}{\pi^N} e^{-|c|^2} dc, c \in \mathbb{C}^{D_N}, \end{aligned}$$

where δ_S is the delta function on $S \subset \mathbb{C}^{D_N}$, the set of points $c = a + ib \in \mathbb{C}^{D_N}$ where $b = 0 \in \mathbb{R}^{D_N}$. Here $d\gamma_{cx}$ corresponds to the standard complex Gaussian coefficients case, where we are considering

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J,$$

where the c_J's are standard complex Gaussian random variables, and $d\gamma_{real}$ corresponds to the standard real Gaussian coefficients case, where we have

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J = \sum_{|J|=0}^{N} a_J \binom{N}{J}^{1/2} z^J,$$

where $c_J = a_J + i0$ is a standard real Gaussian random variable. We let $E_{\gamma_{real}}(\cdot)$ denote expectation with respect to $d\gamma_{real}$ and $E_{\gamma_{cx}}(\cdot)$ denote expectation with respect to $d\gamma_{cx}$.

The goal of this section is to show the following result about the density of critical points of h_N using the Kac-Rice formula:

Theorem 0.5. *Let $K \subseteq \mathbb{C}^m \backslash \mathbb{R}^m$ be compact, let λ be a positive constant, and let h, γ_{cx}, and γ_{real} be defined as above. We have*

$$E_{\gamma_{real}}(C_{h_N})(z) = E_{\gamma_{cx}}(C_{h_N})(z) + O(e^{-\lambda N}),$$

for all $z \in \mathbb{C}^m \backslash \mathbb{R}^m$, and uniformly on K.

In other words, at any point away from $\mathbb{R}^m$, the expected density of critical points in the real coefficients case rapidly approaches the expected density of critical points in the complex coefficients case as N gets large.

Instead of studying the critical points of this random polynomial h, we could equivalently study the zeros of $(f_{1,N}, ..., f_{m,N}) : \mathbb{C}^m \to \mathbb{C}^m$, where $f_{q,N}$ is a *complex* polynomial of the form

$$f_{q,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} \frac{\partial}{\partial z_q} z^J, \qquad 1 \le q \le m.$$

We'll consider $f_{q,N}(z)$ as a function from $\mathbb{R}^{2m}$ to $\mathbb{R}^{2m}$, use the fact that $C_h = Z_{f_1 \cdots f_m} = Z_{f_1^r \cdots f_m^r f_1^i \cdots f_m^i}$, where $f_q = f_q^r + i f_q^i$, and find $E_\gamma(Z_{f_1^r \cdots f_m^r f_1^i \cdots f_m^i})$. Consider $x = (f_1^r, \dots f_m^r, f_1^i, \dots, f_m^i)$. Let ξ be the matrix of derivatives of the function

$$(x_1, \dots x_m, y_1, \dots, y_m) \to x.$$

We can write

$$\xi = \begin{pmatrix} \left(\dfrac{\partial f_q^r}{\partial x_{q'}}\right)_{1 \le q, q' \le m} & \left(\dfrac{\partial f_q^r}{\partial y_{q'}}\right)_{1 \le q, q' \le m} \\ \\ \left(\dfrac{\partial f_q^i}{\partial x_{q'}}\right)_{1 \le q, q' \le m} & \left(\dfrac{\partial f_q^i}{\partial y_{q'}}\right)_{1 \le q, q' \le m} \end{pmatrix}$$

Noting the Cauchy-Riemann equations hold, and that $\dfrac{\partial f_q^r}{\partial x_{q'}} = \dfrac{\partial f_{q'}^r}{\partial x_q}$ and $\dfrac{\partial f_q^i}{\partial x_{q'}} = \dfrac{\partial f_{q'}^i}{\partial x_q}$ we can choose a new basis and write ξ as a vector

$$\hat{\xi} = [\xi]_{\mathcal{B}} = \left(\left(\frac{\partial f_q^r}{\partial x_{q'}}\right)_{q \le q'}, \left(\frac{\partial f_q^i}{\partial x_{q'}}\right)_{q \le q'} \right) \in \mathbb{R}^{2d_m},$$

where $d_m = m(m+1)/2$. Note also that because of Cauchy Riemann equations, $\det \xi$ is positive, and $\sqrt{\det \xi \xi^T} = \det \xi$. By the Kac-Rice formula, we have

$$E_\gamma(Z_{f_1^r \cdots f_m^r f_1^i \cdots f_m^i}) = \int_{\mathbb{R}^{2d_m}} \sqrt{\det(\xi \xi^T)} D_\gamma(0, \hat{\xi}; z) d\hat{\xi} = \int_{\mathbb{R}^{2d_m}} \det \xi \; D_\gamma(0, \hat{\xi}; z) d\hat{\xi}$$

where $D_\gamma(x, \hat{\xi}; z)$ is the Gaussian density in $2m + 2d_m$ variables given by

$$D_\gamma(x, \hat{\xi}; z) = \frac{1}{\pi^{m+d_m} \sqrt{\det \Delta_\gamma}} e^{-\frac{1}{2} \langle \Delta_\gamma^{-1} \binom{x}{\hat{\xi}}, \binom{x}{\hat{\xi}} \rangle}$$

and where Δ_γ, the covariance matrix of $\begin{pmatrix} x \\ \hat{\xi} \end{pmatrix}$, is given in block form by

$$\begin{aligned}
\Delta_\gamma &= \begin{pmatrix} A_\gamma & B_\gamma \\ B_\gamma^T & C_\gamma \end{pmatrix} && ((2m+2d_m)\times(2m+2d_m) \text{ matrix}) \\
A_\gamma &= (E_\gamma(x_q \bar{x}_{q'}))_{q,q'} = A_\gamma^T && (2m \times 2m \text{ matrix}) \\
B_\gamma &= \left(E_\gamma(x_q \bar{\hat{\xi}}_j)\right)_{q,j} && (2m \times 2d_m \text{ matrix}) \\
C_\gamma &= \left(E_\gamma(\hat{\xi}_j \bar{\hat{\xi}}_{j'})\right)_{j,j'} = C_\gamma^T && (2d_m \times 2d_m \text{ matrix}).
\end{aligned}$$

where $1 \le q, q' \le 2m, 1 \le j, j' \le 2d_m$. Note that we can also write A, B, and C in block form as

$$A_\gamma = \begin{pmatrix} \left(E_\gamma(f_q^r f_{q'}^r)\right)_{q,q'} & \left(E_\gamma(f_q^r f_{q'}^i)\right)_{q,q'} \\ \\ \left(E_\gamma(f_q^i f_{q'}^r)\right)_{q,q'} & \left(E_\gamma(f_q^i f_{q'}^i)\right)_{q,q'} \end{pmatrix}$$

$$B_\gamma = \begin{pmatrix} \left(E_\gamma\left(f_q^r \frac{\partial f_{q'}^r}{\partial x_{p'}}\right)\right)_{q,\,q',p'} & \left(E_\gamma\left(f_q^r \frac{\partial f_{q'}^i}{\partial x_{p'}}\right)\right)_{q,\,q',p'} \\ \\ \left(E_\gamma\left(f_q^i \frac{\partial f_{q'}^r}{\partial x_{p'}}\right)\right)_{q,\,q',p'} & \left(E_\gamma\left(f_q^i \frac{\partial f_{q'}^i}{\partial x_{p'}}\right)\right)_{q,\,q',p'} \end{pmatrix}$$

$$C_\gamma = \begin{pmatrix} \left[E_\gamma\left(\frac{\partial f_q^r}{\partial x_p}\frac{\partial f_{q'}^r}{\partial x_{p'}}\right)\right]_{q,p,q',p'} & \left[E_\gamma\left(\frac{\partial f_q^r}{\partial x_p}\frac{\partial f_{q'}^i}{\partial x_{p'}}\right)\right]_{q,p,q',p'} \\ \\ \left[E_\gamma\left(\frac{\partial f_q^i}{\partial x_p}\frac{\partial f_{q'}^r}{\partial x_{p'}}\right)\right]_{q,p,q',p'} & \left[E_\gamma\left(\frac{\partial f_q^i}{\partial x_p}\frac{\partial f_{q'}^i}{\partial x_{p'}}\right)\right]_{q,p,q',p'} \end{pmatrix}$$

where $1 \le q \le p \le m$, and $1 \le q' \le p' \le m$.

Now, using the fact that for $D_\gamma(0, \hat{\xi}; z)$ only the lower right block of Δ_γ^{-1} matters, we can write

$$\begin{aligned}
D_\gamma(0, \hat{\xi}; z) &= \frac{1}{\pi^{m+d_m}\sqrt{\det \Delta_\gamma}} \exp\left(-\frac{1}{2}\langle \Delta_\gamma^{-1}\tbinom{0}{\hat{\xi}}, \tbinom{0}{\hat{\xi}}\rangle\right) \\
&= \frac{1}{\pi^m\sqrt{\det A_\gamma}} \frac{1}{\pi^{d_m}\sqrt{\det \Lambda_\gamma}} \exp\left(-\frac{1}{2}\langle \Lambda_\gamma^{-1}\hat{\xi}, \hat{\xi}\rangle\right)
\end{aligned}$$

where Λ_γ^{-1} is the lower right block of Δ_γ^{-1} and is given by

$$\Lambda_\gamma = C_\gamma - B_\gamma^T A_\gamma^{-1} B_\gamma,$$

We have also used the fact that

$$\det \Delta_\gamma = \det A_\gamma \det \Lambda_\gamma.$$

So now we have that

$$\begin{aligned} E_\gamma(C_h) = E_\gamma(Z_{f_1\cdots f_m}) &= \frac{1}{\pi^m\sqrt{\det A_\gamma}}\int_{\mathbb{R}^{d_m}}(\det\xi)\frac{1}{\pi^{d_m}\sqrt{\det\Lambda_\gamma}}\exp\left(-\frac{1}{2}\langle\Lambda_\gamma^{-1}\hat{\xi},\hat{\xi}\rangle\right)d\hat{\xi} \\ &= \frac{1}{\pi^m\sqrt{\det A_\gamma}}E_{\Lambda_\gamma}(\det\xi), \end{aligned}$$

since $\det\xi \geq 0$. We now want to evaluate $E_{\Lambda_\gamma}(\det\xi)$ using the Wick formula, which states that if $X_1,\ldots,X_{2m}$ are jointly Gaussian random variables, then

$$E(\prod_{q=1}^{2m}X_q) = \sum\prod_{q=1}^{m}E(X_{i_q}X_{j_q})$$

where the sum is over partition of $\{1,...,2m\}$ into disjoint pairs $\{i_q,j_q\}$.

First we write

$$\begin{aligned} E_{\Lambda_\gamma}(\det\xi) = E_{\Lambda_\gamma}\left(\sum_{\sigma\in S_{2m}}sgn(\sigma)\prod_{q=1}^{2m}\xi_{q,\sigma(q)}\right) &= \sum_{\sigma\in S_{2m}}sgn(\sigma)E_{\Lambda_\gamma}\left(\prod_{q=1}^{2m}\xi_{q,\sigma(q)}\right) \\ = \sum_{\sigma\in S_{2m}}sgn(\sigma)\sum\prod_{q=1}^{m}E_{\Lambda_\gamma}\left(\xi_{i_q,\sigma(i_q)}\xi_{j_q,\sigma(j_q)}\right) \end{aligned}$$

where σ is a permutation, and where the second sum is over partitions of $\{1,...,2m\}$ into disjoint pairs $\{i_q,j_q\}$.

Note that terms of the form

$$E_{\Lambda_\gamma}\left(\xi_{i_q,\sigma(i_q)}\xi_{j_q,\tau(j_q)}\right)$$

are actually entries of Λ_γ. So we have written $E_{\Lambda_\gamma}(\det\xi)$ as a sum of products of entries in Λ_γ. More specifically, we have that $E_{\Lambda_\gamma}(\det\xi) = \phi(\Lambda_\gamma)$, where $\phi(\Lambda_\gamma)$ is a homogeneous polynomial in the entries of Λ_γ.

1. **Two special cases.** Suppose now that we have the measures

$$d\gamma_{cx} = \frac{1}{\pi^{D_N}}e^{-|c|^2}dc, c\in\mathbb{C}^{D_N}, D_N = \binom{N+m}{m}$$

$$d\gamma_{real} = \delta_S\frac{1}{\pi^{D_N}}e^{-|c|^2}dc, c\in\mathbb{C}^{D_N}, D_N = \binom{N+m}{m}$$

where δ_S is the delta function on $S \subset \mathbb{C}^{D_N}$, the set of points $c = a + ib \in \mathbb{C}^{D_N}$ where $b = 0 \in \mathbb{R}^{D_N}$. Here $d\gamma_{cx}$ corresponds to the standard complex Gaussian coefficients case, where we are considering

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J$$

where the c_J's are standard complex Gaussian random variables, and $d\gamma_{real}$ corresponds to the standard real Gaussian coefficients case, where we have

$$h_{m,N}(z) = \sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} z^J = \sum_{|J|=0}^{N} a_J \binom{N}{J}^{1/2} z^J$$

where $c_J = a_j + i0$ is a standard real Gaussian random variable.

We now state three important lemmas.

Lemma 1.6. *Let K be a compact set in $\mathbb{C}^m \setminus \mathbb{R}^m$. The following are true for all q, q', p, p' and for some $\lambda > 0$:*

(1) $E_{\gamma_{cx}}(f_q f_{q'}) = 0$
(2) $\dfrac{E_{\gamma_{real}}(f_q f_{q'})}{(1 + z \cdot \bar{z})^N} = O(e^{-\lambda N})$, uniformly on K
(3) $E_{\gamma_{cx}}\left(f_q \dfrac{\partial f_{q'}}{\partial z_{p'}}\right) = 0$
(4) $\dfrac{E_{\gamma_{real}}\left(f_q \frac{\partial f_{q'}}{\partial z_{p'}}\right)}{(1 + z \cdot \bar{z})^N} = O(e^{-\lambda N})$, uniformly on K
(5) $E_{\gamma_{cx}}\left(\dfrac{\partial f_q}{\partial z_p} \dfrac{\partial f_{q'}}{\partial z_{p'}}\right) = 0$
(6) $\dfrac{E_{\gamma_{real}}\left(\frac{\partial f_q}{\partial z_p} \frac{\partial f_{q'}}{\partial z_{p'}}\right)}{(1 + z \cdot \bar{z})^N} = O(e^{-\lambda N})$, uniformly on K

Proof. We prove just (1) and (2), the rest are proved similarly. For (1) we have

$$\begin{aligned}
&E_{\gamma_{cx}}(f_q f_{q'}) \\
&= E_{\gamma_{cx}} \left[\left(\sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} \frac{\partial}{\partial z_q} z^J \right) \left(\sum_{|K|=0}^{N} c_K \binom{N}{K}^{1/2} \frac{\partial}{\partial z_{q'}} z^K \right) \right] \\
&= E_{\gamma_{cx}} \left(\sum_{|J|=0}^{N} \sum_{|K|=0}^{N} c_J c_K \binom{N}{J}^{1/2} \binom{N}{K}^{1/2} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^K \right) \\
&= \sum_{|J|=0}^{N} \sum_{|K|=0}^{N} E_{\gamma_{cx}}(c_J c_K) \binom{N}{J}^{1/2} \binom{N}{K}^{1/2} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^K = 0
\end{aligned}$$

since $E(c_J c_K) = 0$ for all J, K. (Note that $E(c_J \bar{c_K}) = 1$ when $J = K$, so $E_{\gamma_{cx}}(f_q \bar{f}_{q'}) \neq 0$).

Similarly, for (2) we have

$$\begin{aligned}
\frac{E_{\gamma_{real}}(f_q f_{q'})}{(1 + z \cdot \bar{z})^N} &= \frac{1}{(1 + z \cdot \bar{z})^N} \sum_{|J|=0}^{N} \sum_{|K|=0}^{N} E_{\gamma_{real}}(c_J c_K) \binom{N}{J}^{1/2} \binom{N}{K}^{1/2} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^K \\
&= \frac{1}{(1 + z \cdot \bar{z})^N} \sum_{|J|=0}^{N} \sum_{|K|=0}^{N} E_{\gamma_{real}}(a_J a_K) \binom{N}{J}^{1/2} \binom{N}{K}^{1/2} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^K \\
&= \frac{1}{(1 + z \cdot \bar{z})^N} \sum_{|J|=0}^{N} \binom{N}{J} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^J
\end{aligned}$$

since $E_{\gamma_{real}}(a_J a_K) = 1$, when $J = K$, and is zero otherwise. We can then write

$$\begin{aligned}
\frac{E_{\gamma_{real}}(f_q f_{q'})}{(1 + z \cdot \bar{z})^N} &= \frac{1}{(1 + z \cdot \bar{z})^N} \sum_{|J|=0}^{N} \binom{N}{J} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial z_{q'}} z^J = \frac{1}{(1 + z \cdot \bar{z})^N} \sum_{|J|=0}^{N} \binom{N}{J} \frac{\partial}{\partial z_q} z^J \frac{\partial}{\partial \tilde{z}_{q'}} \tilde{z}^J \Big|_{\tilde{z}=z} \\
&= \frac{1}{(1 + z \cdot \bar{z})^N} \frac{\partial}{\partial z_q} \frac{\partial}{\partial \tilde{z}_{q'}} \sum_{|J|=0}^{N} \binom{N}{J} (z\tilde{z})^J \Big|_{\tilde{z}=z} = \frac{1}{(1 + z \cdot \bar{z})^N} \frac{\partial}{\partial z_q} \frac{\partial}{\partial \tilde{z}_{q'}} (1 + z \cdot \tilde{z})^N \Big|_{\tilde{z}=z} \\
&= \frac{1}{(1 + z \cdot \bar{z})^N} N(N-1) \tilde{z}_q z_{q'} (1 + z \cdot \tilde{z})^{N-2} \Big|_{\tilde{z}=z} = N(N-1) z_q z_{q'} \frac{(1 + z \cdot z)^{N-2}}{(1 + z \cdot \bar{z})^N} \\
&= \frac{N(N-1) z_q z_{q'}}{(1 + z \cdot \bar{z})^2} \left(\frac{1 + z \cdot z}{1 + z \cdot \bar{z}} \right)^{N-2}
\end{aligned}$$

Since $|1 + z \cdot z| < |1 + z \cdot \bar{z}| = 1 + z \cdot \bar{z}$, for all $z \in \mathbb{C}^m \setminus \mathbb{R}^m$, we have that $\left| \frac{1 + z \cdot z}{1 + z \cdot \bar{z}} \right| < 1$ which implies that $\left(\frac{1 + z \cdot z}{1 + z \cdot \bar{z}} \right)^{N-2} = O(e^{-\lambda N}), z \in \mathbb{C}^m \setminus \mathbb{R}^m$, and that

$$\frac{E_{\gamma_{real}}(f_q f_{q'})}{(1 + z \cdot \bar{z})^N} = O(e^{-\lambda N}), \text{ uniformly for all } z \in K \subset \mathbb{C}^m \setminus \mathbb{R}^m.$$

where K is compact. □

Lemma 1.7. *Let K be a compact set in $\mathbb{C}^m \setminus \mathbb{R}^m$. Using the results of the previous lemma, we have for all q, q', p, p':*

(1) $E_{\gamma_{cx}}(f_q^r f_{q'}^i) = 0$

(2) $\dfrac{E_{\gamma_{real}}(f_q^r f_{q'}^i)}{(1 + z \cdot \bar{z})^N} = O(e^{-\lambda N})$, uniformly on K

(3) $E_{\gamma_{cx}} \left(f_q^r \dfrac{\partial f_{q'}^i}{\partial x_{p'}} \right) = 0$

(4) $\dfrac{E_{\gamma_{real}}\left(f_q^r \frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N} = O(e^{-\lambda N})$, uniformly on K

(5) $E_{\gamma_{cx}}\left(\dfrac{\partial f_q^r}{\partial x_p}\dfrac{\partial f_{q'}^i}{\partial x_{p'}}\right) = 0$

(6) $\dfrac{E_{\gamma_{real}}\left(\frac{\partial f_q^r}{\partial x_p}\frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N} = O(e^{-\lambda N})$, uniformly on K

Proof. We again prove just (1) and (2). Using $f_q^r = \frac{1}{2}(f_q + \bar{f}_q), f_q^i = \frac{1}{2i}(f_q - \bar{f}_q)$, we can get that

$$E_{\gamma_{cx}}(f_q^r f_{q'}^i) = E_{\gamma_{cx}}(f_q f_{q'} - f_q \bar{f}_{q'} + \bar{f}_q f_{q'} - \bar{f}_q \bar{f}_{q'}) = 0$$
$$\frac{E_{\gamma_{real}}(f_q^r f_{q'}^i)}{(1+z\cdot\bar{z})^N} = \frac{E_{\gamma_{real}}(f_q f_{q'} - f_q \bar{f}_{q'} + \bar{f}_q f_{q'} - \bar{f}_q \bar{f}_{q'})}{(1+z\cdot\bar{z})^N} = O(e^{-\lambda N})$$

for all $z \in \mathbb{C}^m \setminus \mathbb{R}^m$. Statements (3) through (6) could be proved similarly, noting that f_q is holomorphic so that $\frac{\partial f_q}{\partial z_p} = \frac{\partial f_q}{\partial x_p}$. ∎

Lemma 1.8. *We have for all q, q', p, p':*

(1) $E_{\gamma_{cx}}(f_q \bar{f}_{q'}) = E_{\gamma_{real}}(f_q \bar{f}_{q'})$

(2) $E_{\gamma_{cx}}\left(f_q \overline{\dfrac{\partial f_{q'}}{\partial z_{p'}}}\right) = E_{\gamma_{real}}\left(f_q \overline{\dfrac{\partial f_{q'}}{\partial z_{p'}}}\right)$

(3) $E_{\gamma_{cx}}\left(\dfrac{\partial f_q}{\partial z_p}\overline{\dfrac{\partial f_{q'}}{\partial z_{p'}}}\right) = E_{\gamma_{real}}\left(\dfrac{\partial f_q}{\partial z_p}\overline{\dfrac{\partial f_{q'}}{\partial z_{p'}}}\right)$

Proof. We prove (1), and (2) and (3) are proved similarly. We have

$$\begin{aligned}
&E_{\gamma_{cx}}(f_q \bar{f}_{q'}) \\
&= E_{\gamma_{cx}}\left[\left(\sum_{|J|=0}^{N} c_J \binom{N}{J}^{1/2} \frac{\partial}{\partial z_q} z^J\right)\left(\overline{\sum_{|K|=0}^{N} c_K \binom{N}{K}^{1/2} \frac{\partial}{\partial z_{q'}} z^K}\right)\right] \\
&= \sum_{|J|=0}^{N}\sum_{|K|=0}^{N} E_{\gamma_{cx}}(c_J \overline{c_K}) \binom{N}{J}^{1/2}\binom{N}{K}^{1/2} \frac{\partial}{\partial z_q} z^J \overline{\frac{\partial}{\partial z_{q'}} z^K} \\
&= \sum_{|J|=0}^{N} \binom{N}{J} \frac{\partial}{\partial z_q} z^J \overline{\frac{\partial}{\partial z_{q'}} z^J}
\end{aligned}$$

since $E_{\gamma_{cx}}(c_J\overline{c_K}) = 1$, when $J = K$, and is zero otherwise. Likewise, since $E_{\gamma_{real}}(c_J\overline{c_K}) = E_{\gamma_{real}}(a_J\overline{a_K}) = E_{\gamma_{real}}(a_Ja_K)$, we have

$$\begin{aligned} E_{\gamma_{real}}(f_q\bar{f}_{q'}) &= \sum_{|J|=0}^{N}\sum_{|K|=0}^{N} E_{\gamma_{real}}(c_J\overline{c_K})\binom{N}{J}^{1/2}\binom{N}{K}^{1/2}\frac{\partial}{\partial z_q}z^J\overline{\frac{\partial}{\partial z_{q'}}z^K} \\ &= \sum_{|J|=0}^{N}\binom{N}{J}\frac{\partial}{\partial z_q}z^J\overline{\frac{\partial}{\partial z_{q'}}z^J} = E_{\gamma_{cx}}(f_q\bar{f}_{q'}) \end{aligned}$$

$\square$

Using these 3 lemmas, we can write A, B, C and therefore Λ in more detail. We have

$$\frac{A_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\dfrac{E_{\gamma_{cx}}(f_q^r f_{q'}^r)}{(1+z\cdot\bar{z})^N}\right)_{q,q'} & 0 \\ 0 & \left(\dfrac{E_{\gamma_{cx}}(f_q^i f_{q'}^i)}{(1+z\cdot\bar{z})^N}\right)_{q,q'} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

$$\frac{A_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\dfrac{E_{\gamma_{real}}(f_q^r f_{q'}^r)}{(1+z\cdot\bar{z})^N}\right)_{q,q'} & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & \left(\dfrac{E_{\gamma_{real}}(f_q^i f_{q'}^i)}{(1+z\cdot\bar{z})^N}\right)_{q,q'} \end{pmatrix} = \begin{pmatrix} A & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & -A \end{pmatrix}$$

so that we get

$$\frac{A_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \frac{A_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}),\ \text{uniformly on } K.$$

Likewise, we have

$$\frac{B_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\frac{E_{\gamma_{cx}}\left(f_q^r \frac{\partial f_{q'}^r}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,q',p'} & 0 \\ 0 & \left(\frac{E_{\gamma_{cx}}\left(f_q^i \frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,q',p'} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$$

$$\frac{B_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\frac{E_{\gamma_{real}}\left(f_q^r \frac{\partial f_{q'}^r}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,q',p'} & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & \left(\frac{E_{\gamma_{real}}\left(f_q^i \frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,q',p'} \end{pmatrix} = \begin{pmatrix} B & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & -B \end{pmatrix}$$

so that we get

$$\frac{B_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \frac{B_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}),\ \text{uniformly on } K.$$

and

$$\frac{C_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\frac{E_{\gamma_{cx}}\left(\frac{\partial f_q^r}{\partial x_p}\frac{\partial f_{q'}^r}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,p,q',p'} & 0 \\ 0 & \left(\frac{E_{\gamma_{cx}}\left(\frac{\partial f_q^i}{\partial x_p}\frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,p,q',p'} \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

$$\frac{C_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \left(\frac{E_{\gamma_{real}}\left(\frac{\partial f_q^r}{\partial x_p}\frac{\partial f_{q'}^r}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,p,q',p'} & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & \left(\frac{E_{\gamma_{real}}\left(\frac{\partial f_q^i}{\partial x_p}\frac{\partial f_{q'}^i}{\partial x_{p'}}\right)}{(1+z\cdot\bar{z})^N}\right)_{q,p,q',p'} \end{pmatrix} = \begin{pmatrix} C & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & -C \end{pmatrix}$$

so that we get

$$\frac{C_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \frac{C_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}), \text{ uniformly on } K.$$

Finally, using $\Lambda_\gamma = C_\gamma - B_\gamma^T A_\gamma^{-1} B_\gamma$, we have

$$\frac{\Lambda_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} = \begin{pmatrix} \frac{C_{\gamma_{cx}} - B_{\gamma_{cx}}^T A_{\gamma_{cx}}^{-1} B_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} & 0 \\ 0 & -\frac{(C_{\gamma_{cx}} - B_{\gamma_{cx}}^T A_{\gamma_{cx}}^{-1} B_{\gamma_{cx}})}{(1+z\cdot\bar{z})^N} \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$$

$$\begin{aligned} \frac{\Lambda_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} &= \begin{pmatrix} \frac{C_{\gamma_{real}} - B_{\gamma_{cx}}^T A_{\gamma_{real}}^{-1} B_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}) & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & -\frac{(C_{\gamma_{cx}} - B_{\gamma_{cx}}^T A_{\gamma_{real}}^{-1} B_{\gamma_{cx}})}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}) \end{pmatrix} \\ &= \begin{pmatrix} \Lambda + O(e^{-\lambda N}) & O(e^{-\lambda N}) \\ O(e^{-\lambda N}) & -\Lambda + O(e^{-\lambda N}) \end{pmatrix} \end{aligned}$$

so that we get

$$\frac{\Lambda_{\gamma_{real}}}{(1+z\cdot\bar{z})^N} = \frac{\Lambda_{\gamma_{cx}}}{(1+z\cdot\bar{z})^N} + O(e^{-\lambda N}), \text{ uniformly on } K.$$

Note that each term in $E_{\Lambda_{\gamma_{cx}}}(\det \xi)$ has m factors each of which is an element of $\Lambda_{\gamma_{cx}}$, and likewise for $E_{\Lambda_{\gamma_{real}}}(\det \xi)$. This gives

$$\frac{E_{\Lambda_{\gamma_{real}}}(\det \xi)}{(1+z\cdot\bar{z})^{Nm}} = \frac{E_{\Lambda_{\gamma_{cx}}}(\det \xi)}{(1+z\cdot\bar{z})^{Nm}} + O(e^{-\lambda N}), \text{ uniformly on } K.$$

Also note that we have

$$\frac{\det A_{\gamma_{real}}}{(1+z\cdot\bar{z})^{2Nm}} = \frac{\det A_{\gamma_{cx}}}{(1+z\cdot\bar{z})^{2Nm}} + O(e^{-\lambda N}), \text{ uniformly on } K.$$

This means that we have

$$\begin{aligned}
E_{\gamma_{real}}(Z_{f_1\cdots f_m}) &= \frac{1}{\pi^m}\frac{E_{\Lambda_{\gamma_{real}}}(\det \xi)}{\sqrt{\det A_{\gamma_{real}}}} = \frac{1}{\pi^m}\frac{1}{\sqrt{\frac{\det A_{\gamma_{real}}}{(1+z\cdot\bar{z})^{2Nm}}}}\frac{E_{\Lambda_{\gamma_{real}}}(\det \xi)}{(1+z\cdot\bar{z})^{Nm}} \\
&= \frac{1}{\pi^m}\frac{1}{\sqrt{\frac{\det A_{\gamma_{cx}}}{(1+z\cdot\bar{z})^{2Nm}} + O(e^{-\lambda N})}}\left(\frac{E_{\Lambda_{\gamma_{cx}}}(\det \xi)}{(1+z\cdot\bar{z})^{Nm}} + O(e^{-\lambda N})\right) \\
&= \frac{1}{\pi^m}\frac{1}{\sqrt{\frac{\det A_{\gamma_{cx}}}{(1+z\cdot\bar{z})^{2Nm}}}}\frac{E_{\Lambda_{\gamma_{cx}}}(\det \xi)}{(1+z\cdot\bar{z})^{Nm}} + O(e^{-\lambda N}) \\
&= \frac{1}{\pi^m}\frac{E_{\Lambda_{\gamma_{cx}}}(\det \xi)}{\sqrt{\det A_{\gamma_{cx}}}} + O(e^{-\lambda N}) \\
&= E_{\gamma_{cx}}(Z_{f_1\cdots f_m}) + O(e^{-\lambda N}), \text{ uniformly on } K.
\end{aligned}$$

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