PRACTICE PROBLEMS

1. Vector geometry

1.1. Find the equation of the plane that passes through \( A(1,2,0), B(0,1,-2), \) and \( C(4,0,1). \)

1.2. Find the distance from the point \( (2,1,-2) \) to the plane \( x - 2y + 2z + 5 = 0 \)

1.3. Given two vectors \( \vec{a} \) and \( \vec{b} \), do the equations
\[ \vec{v} \times \vec{a} = \vec{b} \quad \text{and} \quad \vec{v} \cdot \vec{a} = \|a\| \]
determine the vector \( \vec{v} \) uniquely? If so, find an explicit formula of \( \vec{v} \) in terms of \( \vec{a} \) and \( \vec{b} \).

2. Tangent planes & lines

2.1. Find the points on the surface \( z = x^2y^2 + y + 1 \) where the tangent plane (to the surface) is parallel to the plane \(-2x - 3y + z = 1\).

2.2. Find the tangent plane to the graph of the function \( f(x,y) = \frac{x}{x+y} \) at the point \((x_0, y_0) = (1,0)\).

2.3. Find a unit vector normal to the graph of \( f(x,y) = e^{xy} \) at the point \((-1,1)\).

2.4. Show that the graphs of \( f(x,y) = x^2 + y^2 \) and \( g(x,y) = -x^2 - y^2 + xy^3 \) are tangent at \((0,0)\).

2.5. Consider the surface \( z^2 = x^2 + y^2 \). Is the tangent plane at the point \((0,0,0)\) well defined?

3. Curves in \( \mathbb{R}^3 \)

3.1. The curve \( c(t) = (t, t^2, t^3) \) crosses the plane \( 4x + 2y + z = 24 \) at a single point. Find that point and calculate the cosine of the angle between the tangent vector at \( c \) at that point and the normal vector to the plane.

3.2. A particle travels on the surface of a fixed sphere of radius \( R \) centered at the origin, i.e.
\[ \|\gamma(t)\| = R, \quad \forall t \]
where \( \gamma(t) \in \mathbb{R}^3 \) is the position of the particle at time \( t \). Prove that the velocity is always perpendicular on the position vector, i.e.
\[ \gamma'(t) \cdot \gamma(t) = 0, \quad \forall t \]
3.3. a) Let \( V : \mathbb{R}^3 \to \mathbb{R} \) given by \( V(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \), or in other words \( V(\vec{r}) = \frac{1}{r} \). Compute \( \nabla V \).

b) The equation of motion of a planet that orbits the Sun satisfies the equation of motion
\[
c''(t) = -\frac{GM}{\|c(t)\|^3} \quad c(t) = (x(t), y(t), z(t))
\]
where \( c(t) = (x(t), y(t), z(t)) \) is the position of the planet at time \( t \). Prove that the vector (angular momentum)
\[
\vec{L} = c(t) \times c'(t)
\]
is independent of time.

c) Prove that the quantity (energy)
\[
E = \frac{1}{2} m \|c'(t)\|^2 - \frac{GMm}{\|c(t)\|}
\]
is independent of time.

3.4. The position of a particle in time is given by
\[
c(t) = (\cos(\pi t), \sin(\pi t), \pi t)
\]
At time \( t_0 = \frac{5\pi}{2} \) the particle is freed of any constraints and starts travelling along the tangent at the constant speed \( v = v(t_0) \). Determine how long does it take (starting from \( t_0 \)) the particle to hit the wall given by the equation \( x = 0 \).

4. LIMITS

4.1. Determine whether the following limit exists:
\[
\lim_{(x, y) \to (0, 0)} \frac{\sin(2x) - 2x + y}{x^3 + y}
\]

5. DIFFERENTIABILITY

5.1. Compute \( f_x, f_y, f_z \) and evaluate them at the indicated points

a) \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2}; \quad (3, 0, 4) \)

b) \( f(x, y, z) = x\sqrt{x^2 + y^2 + z^2}; \quad (2, 1, 3) \)

c) \( f(x, y, z) = xy + e^x \cos y; \quad (1, \frac{\pi}{2}, 0) \)

5.2. Compute the matrix of partial derivatives of the function \( f(x, y) = (x + y, x - y, xy) \).

5.3. a) True or false: If \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and \( \frac{\partial f}{\partial x}(1, 0) \) and \( \frac{\partial f}{\partial y}(1, 0) \) exist, then \( f \) is differentiable at \( (1, 0) \).

b) True or false: if \( \partial_{\vec{v}} f \) is the directional derivative of \( f \) along the vector \( \vec{v} \), then \( \partial_{\vec{v}} f(1, 0, 1) = f_y(1, 0, 1) \).

c) True or false: if \( f : \mathbb{R}^2 \to \mathbb{R} \) is a function such that \( \frac{\partial f}{\partial x}(1, 0) \) and \( \frac{\partial f}{\partial y}(0, 1) \) exist, then \( f \) is continuous at \( (1, 0) \).

d) Give the definition of the directional derivative \( \partial_{\vec{v}} f(x_0, y_0) \) where \( \vec{v} \) is some unit vector in \( \mathbb{R}^2 \) and \( f : \mathbb{R}^2 \to \mathbb{R} \) a function.
e) Given a function \( f : \mathbb{R}^2 \to \mathbb{R} \), give the definition of the tangent plane at \((1, 0)\) to the graph of \( f \) (provided it exists).

f) Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is differentiable and \( \frac{\partial f}{\partial x}(1, 0) \) and \( \frac{\partial f}{\partial y}(1, 0) \) exist.

True or false: if \( \vec{v} = p \hat{i} + q \hat{j} \) is a unit vector, then \( \partial_{\vec{v}} f(1, 0) = p \).

g) Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous and \( \frac{\partial f}{\partial x}(1, 0) \) and \( \frac{\partial f}{\partial y}(1, 0) \) exist.

True or false: if \( \vec{v} = p \hat{i} + q \hat{j} \) is a unit vector, then \( \partial_{\vec{v}} f(1, 0) = \nabla f(1, 0) \cdot \vec{v} \).

5.4. a) Argue that the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
f(x, y, z) = (x + e^z + y, yx^2)
\]
is differentiable.

b) Compute an approximate value for \( f(1.1, 0, -0.9) \).

5.5. \( f : \mathbb{R}^2 \to \mathbb{R} \),
\[
f(x, y) = \begin{cases} 
\frac{x^2}{x^2+y^2}, & (x, y) \neq 0 \\
0, & (x, y) = (0, 0)
\end{cases}
\]
a) Compute \( \frac{\partial f}{\partial x}(0, 0) \), \( \frac{\partial f}{\partial y}(0, 0) \), and \( \partial_{\vec{v}} f(0, 0) \) for any unit vector \( \vec{v} \).

b) Is \( f \) differentiable at \((0, 0)\)?

6. Chain rule

6.1. Use the chain rule to find \( u_x, u_y, u_z \) for \( u = e^x \cos(yz^2) \).

6.2. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) a differentiable map, such that \( g_z(0, 1) = 1 \), \( g_y(0, 1) = 2 \). Determine the rate of change of \( g \) (at \((1,0)\)) along the circle centered at the origin and radius 1. In other words, compute \( \frac{\partial g}{\partial \theta} \) at \((1,0)\) where \( x = r \cos \theta \) and \( y = r \sin \theta \).

6.3. Let \((x(t), y(t))\) a path in the plane \( 0 \leq t \leq 1 \), and let \( f(x, y) \) a \( C^1 \) function of two variables. Assume that
\[
x'(t) f_x(x(t), y(t)) + y'(t) f_y(x(t), y(t)) \leq 0
\]
Prove that \( f(x(1), y(1)) \leq f(x(0), y(0)) \).

6.4. Let \( f : \mathbb{R} \to \mathbb{R} \) a differentiable function and \( h = f(x^2+y) \). Prove that \( h(z, y) \) satisfies the equation
\[
x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = 0
\]

6.5. Let \( f : \mathbb{R}^2 \to \mathbb{R} \), \( f(x, y) = x^2 + y \) and \( h : \mathbb{R}^2 \to \mathbb{R} \), \( h(u) = (\sin(3u), \cos(8u)) \).

Let \( g(u) = f(h(u)) \), for \( u \in \mathbb{R} \). Compute \( g'(0) \) both directly and by using the chain rule.

7. Gradients

7.1. Let \( f(x, y, z) = (\sin(xy))e^{-z^2} \). In what direction from \((1, \pi, 0)\) should one proceed to increase \( f \) most rapidly? Express your answer as a unit vector.