TWO VECTOR FIELDS

1. Setup

1.1. In \( \mathbb{R}^3 \):  

Functs. \( \phi \xrightarrow{\nabla} \) V. fields \( \vec{F} \xrightarrow{\text{curl}} \) V. fields \( \vec{G} \xrightarrow{\text{div}} \) Functs. \( f \)

Pts. \( \xrightarrow{\partial} \) Curves \( \gamma \xrightarrow{\partial} \) Surfaces \( \Sigma \xrightarrow{\partial} \) Solid domains \( W \)

Note that \( \partial(\partial W) = 0 \), \( \partial(\partial \Sigma) = 0 \)

and the analogue of this is \( \text{curl}(\nabla \phi) = 0 \), \( \text{div}(\text{curl} \vec{F}) = 0 \)

The three theorems of "integrals on the boundary" are:

\[
\int_{\gamma} \nabla \phi = \phi(Q) - \phi(P) \quad P, Q = \partial \gamma = \text{endpoints of } \gamma \\
\int_{\partial \Sigma} \vec{F} = \iint_{\Sigma} \text{curl} \vec{F} \cdot dS \quad \text{[Stokes]} \\
\int_{\partial W} \vec{G} \cdot dS = \iiint_{W} (\text{div} \vec{G}) dV \quad \text{[Gauss]}
\]

2. Two vector fields

2.1. Consider the vector field \( \vec{G} = \frac{r}{r^3} \) (the negative of the gravitational vector field).

a) Prove that \( \text{div} \vec{G} = 0 \) (\( \vec{G} \) is incompressible).

b) Prove that

\[
\int_{\Sigma_{R}} \vec{G} \cdot dS = 4\pi
\]

where \( \Sigma_{R} \) is the sphere of radius \( R \) centered at the origin.

c) Prove (using Stokes theorem) that \( \vec{G} \) is not the curl of another vector field \( \vec{F} \) (well-defined on \( \mathbb{R}^3 - (0,0,0) \)) such that \( \text{curl} \vec{F} = \vec{G} \).

d) What is so special about the geometry of \( \mathbb{R}^3 - (0,0,0) \) that a vector field with the properties of \( \vec{G} \) exists?

e) (Gauss’s law) Assume \( \Sigma \) is a closed (simple, oriented) surface in \( \mathbb{R}^3 \), not passing through the origin \( (0,0,0) \). Use Gauss’ divergence theorem to prove that

\[
\int_{\Sigma} \vec{G} \cdot dS = \begin{cases} 4\pi, & \text{if } (0,0,0) \text{ is in the interior of } \Sigma \\ 0, & \text{otherwise} \end{cases}
\]

2.2. (Side computation) Let \( \vec{F} = P(x,y)i + Q(x,y)j \) a vector field in \( \mathbb{R}^2 \). Prove that \( \text{curl} \vec{F} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)k \).
2.2.1. Consider the following vector field in $\mathbb{R}^2$:

$$\vec{F} = \frac{-yi + xj}{x^2 + y^2}$$

a) Prove that curl $\vec{F} = 0$ (i.e. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$)

b) Compute the work of $\vec{F}$ along $C_R$, the circle of radius $R$ oriented counterclockwise.

c) Prove that $\vec{F}$ is not a conservative vector field. In other words, prove that there does not exist a (potential) function $\phi$ (well defined on $\mathbb{R}^2 - (0, 0)$) such that $\vec{F} = \nabla \phi$.

d) What is so special about the geometry of $\mathbb{R}^2 - (0, 0)$ that a vector field with the properties of $\vec{F}$ exists?

e) Let $\gamma$ an arbitrary simple (without self-intersections) closed curve in $\mathbb{R}^2$, not passing through the origin (oriented counter-clockwise). Use Green’s formula to prove that

$$\frac{1}{2\pi} \int_{\gamma} \vec{F} = \begin{cases} 
1, & \text{\gamma circles around the origin (0, 0)} \\
0, & \text{otherwise}
\end{cases}$$