CONTINUITY OF MULTIVARIABLE FUNCTIONS. EXAMPLES

1. Definitions

1.1. Limit. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) some function, \( x_0 = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( y_0 = (y_1, \ldots, y_m) \in \mathbb{R}^m \). Then

\[
\lim_{x \to x_0} f(x) = y_0
\]

if and only for \( x \) "close to" \( x_0 \), \( f(x) \) is "close to" \( y_0 \). In other words, given \( \epsilon > 0 \), there exists \( \delta \) depending on \( \epsilon \) and \( x_0 \) (so we write \( \delta = \delta(\epsilon, x_0) \)) such that

\[
d_{\mathbb{R}^n}(x, x_0) < \delta \Rightarrow d_{\mathbb{R}^m}(f(x), y_0) < \epsilon
\]

where the distance on the left is taken in \( \mathbb{R}^n \), whereas the one on the right-hand side is taken in the target space \( \mathbb{R}^m \). We will drop the subscript \( \mathbb{R}^n \) from \( d \), and we will often write \( \|x - x_0\| \) instead of \( d(x, x_0) \). We will also drop the bold-face notation.

Note: a function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is clearly given by a row vector \( f = (f_1, \ldots, f_m) \) where \( f_i \)'s are the components of \( f \). In fact, \( f_i = p_i \circ f \) (see below). Then

\[
\lim_{x \to x_0} f(x) = y_0 \text{ if and only if } \lim_{x \to x_0} f_i(x) = y_i, \ i = 1, \ldots, m.
\]

1.2. Continuity. A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous at \( x_0 \in \mathbb{R}^n \) iff

\[
\lim_{x \to x_0} f(x) = f(x_0)
\]

We say that \( f \) is continuous (everywhere) if it is continuous at every point of the domain \( \mathbb{R}^n \).

Note: if \( f = (f_1, \ldots, f_m) \), where \( f_1, \ldots, f_m \) are the components of \( f \), then \( f \) is continuous iff \( f_1, \ldots, f_m \) are continuous, as functions : \( \mathbb{R}^n \to \mathbb{R} \).

2. Tools

Operations with limits: addition, subtraction, multiplication, division (when possible), etc.

Components. If \( f : \mathbb{R}^n \to \mathbb{R}^m \), \( f = (f_1, \ldots, f_m) \), then \( \lim_{x \to x_0} f(x) = y_0 = (y_1, \ldots, y_m) \) iff \( \lim_{x \to x_0} f_i(x) = y_i \), for every \( i = 1, \ldots, m \).

Projections. The functions \( p_i : \mathbb{R}^n \to \mathbb{R}, \ p_i(x_1, \ldots, x_n) = x_i \) are continuous, and this can be easily checked with the \((\epsilon, \delta)\)-definition of continuity.

Composition of functions. If \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuous and \( g : \mathbb{R}^m \to \mathbb{R}^N \) is continuous, then \( g \circ f : \mathbb{R}^n \to \mathbb{R}^N \) is continuous, where \( g \circ f(x) := g(f(x)) \).
3. Examples

1. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x \sin(z)$.
   Then $f = p_1 \circ (\sin \circ p_3)$ is continuous.

2. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = (xy, x + y + z^2)$.
   - $f_1(x, y) = xy = p_1 \circ p_2$ is cont.
   - $f_2(x, y) = x + y + z^2 = p_1 + p_2 + (p_3)^2$ is cont.
   Therefore $f = (f_1, f_2)$ is continuous.

3. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{x^2}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$
   Proof that $f$ is continuous.
   On $\mathbb{R}^2 - (0, 0)$, $f$ is continuous since it is a ratio of continuous functions (with non-vanishing denominator), namely $f = \frac{p_1^3}{\sqrt{p_1^2 + p_2^2}}$ is continuous.

   NB: To make sure we understand why is the denominator continuous, we can write the denominator as the composition of functions $\mathbb{R}^2 - \{0, 0\} \to [0, \infty) \to \mathbb{R}$

   We are thus left to show that $f$ is continuous at $(0, 0)$, in other words to prove that
   $$\lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = 0$$
   Squeeze:
   $$0 \leq \frac{x^2}{\sqrt{x^2 + y^2}} \leq \frac{x^2}{\sqrt{x^2}} = |x|$$
   However, as $(x, y) \to (0, 0)$ necessarily $x \to 0$ and hence $|x| \to 0$, therefore by the Pinching Lemma $\frac{x^2}{\sqrt{x^2 + y^2}} \to 0$ as $(x, y) \to 0$.

   Different way of saying it ($(\varepsilon, \delta)$-proof). Note that
   $$d((x, y), (0, 0)) \leq \varepsilon \iff \sqrt{x^2 + y^2} \leq \varepsilon \Rightarrow |x| \leq \varepsilon \Rightarrow \frac{x^2}{\sqrt{x^2 + y^2}} \leq |x| \leq \varepsilon$$

4. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$
   Proof that $f$ is not continuous at $(0, 0)$. Consider the followin sequences of points in $\mathbb{R}^2$:
   $$x_k = (\frac{1}{k}, 0), \quad k \geq 1$$
   $$y_k = (\frac{1}{k}, \frac{1}{k}), \quad k \geq 1$$
   Then $x_k \to (0, 0)$ and $y_k \to (0, 0)$ as $k \to \infty$, and yet
   $$f(x_k) = 0 \to 0, \quad k \to \infty$$
   $$f(y_k) = \frac{1}{2} \to \frac{1}{2}, \quad k \to \infty$$
   which shows that the limit
   $$\lim_{x \to (0,0)} f(x)$$
   does not exist. In particular, the function $f$ is not continuous at $(0, 0)$. 
5. Let
\[ g(x, y) = \begin{cases} 
  x \sin(1/y), & y \neq 0 \\
  0, & y = 0 
\end{cases} \]
Determine the points \((x, y) \in \mathbb{R}^2\) where \(g\) is continuous.

6. Tricky example. Let \(f: \mathbb{R}^2 \to \mathbb{R}\) defined by
\[ f(x, y) = \begin{cases} 
  \frac{xy^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\
  0, & (x, y) = (0, 0) 
\end{cases} \]
Show that \(\lim_{k \to \infty} f(x_k) = 0\), for any sequence \(x_k \to (0, 0)\) such that \(x_k, k \geq 1\) is on a fixed line passing through \((0, 0)\). Is \(f\) continuous at \((0, 0)\)?