Yesterday

\textbf{Definition}

The plane

\[ z = f(a,b) + m(x-a) + n(y-b) \]

approximates to first order the graph

\[ \eta \not\equiv \theta (a,b) \quad \text{iff} \]

\[
\lim_{(x,y) \to (a,b)} \frac{f(x,y) - [f(a,b) + m(x-a) + n(y-b)]}{\| (x,y) - (a,b) \|} = 0
\]

\textbf{Observation 1}

If such a "well-approximating" plane exists, then

it is unique, and

\[
\begin{align*}
m &= \frac{\partial f}{\partial x} (a,b) \\
n &= \frac{\partial f}{\partial y} (a,b)
\end{align*}
\]
Example

\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{x^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \]

Note that \( f \) is continuous everywhere.

Check continuity at \((0, 0)\):

\[ 0 \leq |f(x, y)| \leq \left| \frac{x y^2}{y^2} \right| = |x| \]

Pinching lemma \(\Rightarrow\) \(\lim_{(x, y) \to (0, 0)} f(x, y) = 0 = f(0, 0)\)

\[ \lim_{(x, y) \to (0, 0)} |x| = 0 \]

Q: Is there a well-approximating plane to the graph of \( f \) at \((0, 0)\)?

A: If such a plane exists, its equation is given by

\[ z = f(0, 0) + m (x - 0) + n (y - 0), \]

where

\[ m = \frac{\partial f}{\partial x} (0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0 \]

\[ n = \frac{\partial f}{\partial y} (0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = 0 \]
Conclusion: If such a plane exists, it must be the plane 
\[ z = 0 \] (the xy-plane).

Verify the condition:
\[ \lim_{(x,y) \to (0,0)} \frac{f(x,y) - 0}{\| (x,y) \|} = 0, \]

in other words
\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \frac{xy^2}{(x^2 + y^2)^{3/2}} = 0. \]
**FALSE!**

\[ x = t \]

\[ y = t^2 \quad t \to 0 \]

\[ \lim_{t \to 0} \frac{t - t^2}{(t^2 + t^2)^{3/2}} = \frac{1}{2^{3/2}} \neq 0 \]

Therefore, if a well approximating plane existed, it would have been \( z = 0 \).

But this one does not do the job.

**Conclusion** \( f : \mathbb{R}^2 \to \mathbb{R} \) is differentiable at \((a, b) \in \mathbb{R}^2\) if the following two conditions are satisfied:

1) \( \frac{\partial f}{\partial x} (a, b), \frac{\partial f}{\partial y} (a, b) \) exist. \( \exists \psi \)

2) \( \lim_{(x, y) \to (a, b)} \frac{f(x, y) - f(a, b) - \frac{\partial f}{\partial x} (a, b) \cdot (x-a) - \frac{\partial f}{\partial y} (a, b) \cdot (y-b)}{|| (x, y) - (a, b) ||} = 0 \)
Compare to the situation of \( g: \mathbb{R} \to \mathbb{R} \).

If \( \frac{dg}{dt}(3) = \lim_{t \to 0} \frac{g(t+5) - g(t)}{t} \),

then

\[
\lim_{x \to 3} \frac{g(x) - g(3) + \frac{dg}{dt}(3) - (x-3)}{x-3}
\]

\[
= \lim_{x \to 3} \left( \frac{g(x) - g(3)}{x-3} - \frac{dg}{dt}(3) \right)
\]

\[
= \lim_{x \to 3} \frac{g(x) - g(3)}{x-3} - \frac{dg}{dt}(3)
\]

\[
= \frac{dg}{dt}(3) - \frac{dg}{dt}(3) = 0
\]

So, condition \( b \) is automatically satisfied!

Explanations
\[ f : \mathbb{R}^2 \rightarrow \mathbb{R} \]

\[ g : \mathbb{R} \rightarrow \mathbb{R} \]

\[ a) \quad g'(a) = \lim_{t \to a} \frac{g(t+a) - g(a)}{t} \]

\[ \exists \quad \begin{cases} \frac{\partial f}{\partial x} (a,b) = \lim_{t \to 0} \frac{f(a+t,b) - f(a,b)}{t} \\ \frac{\partial f}{\partial y} (a,b) = \lim_{t \to 0} \frac{f(a,b+t) - f(a,b)}{t} \end{cases} \]

More ways to approach \((a,b)\)!

Several possible directions

\([\text{Infinitely many!}]\)

Not necessarily "compatible"

Only one direction

Compatible with

Computing \( \frac{df}{dt} (a) \)

not necessarily "compatible"

with the directions

along which you compute

\( \frac{df}{dx} \) and \( \frac{df}{dy} \).
Similarly for \( f : \mathbb{R}^3 \to \mathbb{R} \)

**Definition**

\( f \) is differentiable at \( \mathbf{a} = (x_0, y_0, z_0) \in \mathbb{R}^3 \) if the following two conditions are satisfied:

1) \( \frac{\partial f}{\partial x} (\mathbf{a}), \frac{\partial f}{\partial y} (\mathbf{a}), \frac{\partial f}{\partial z} (\mathbf{a}) \) exist

2) \( \lim_{(x,y,z) \to \mathbf{a}} \frac{f(x,y,z) - f(x_0,y_0,z_0) - \frac{\partial f}{\partial x} (\mathbf{a})(x-x_0) - \frac{\partial f}{\partial y} (\mathbf{a})(y-y_0) - \frac{\partial f}{\partial z} (\mathbf{a})(z-z_0)}{|| (x,y,z) - \mathbf{a} ||} = 0. \)
**GENERAL CASE**: \( f : \mathbb{R}^m \to \mathbb{R}^n \).

1. **Matrices and Linear Maps**.

Let \( A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \) be an \( n \times m \) matrix.

Then \( A \) determines a **map**.

\[ [A] : \mathbb{R}^m \to \mathbb{R}^n \quad \text{or} \quad T_A : \mathbb{R}^m \to \mathbb{R}^n \]

given by \( T_A (x_1, \ldots, x_m) = (y_1, \ldots, y_n) \)

where \( A \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \)

**Example 1**. \( A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix} \) \( 2 \times 3 \) matrix

\[ T_A : \mathbb{R}^3 \to \mathbb{R}^2 \]

\[ T_A (x, y, z) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ 2x + y + z \end{pmatrix} = (x - z, 2x + y + z) \]

2) \( A = (3) \). \( T_A : \mathbb{R} \to \mathbb{R} \), \( T_A (x) = 3x \)

i.e. \( [(3)] : \mathbb{R} \to \mathbb{R} \), \( [(3)] (x) = 3x \).
Proposition

A = \text{n x m matrix}, \quad B = \text{k x n matrix}

\[ \mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n \xrightarrow{T_B} \mathbb{R}^k \]

Then \( T_B \circ T_A = T_{BA} \)

where \( BA \) = matrix multiplication of \( B \) and \( A \)

**Main Feature (check!)**

**LINEARITY**

\[
\begin{align*}
T_A (\xi + \eta) &= T_A (\xi) + T_A (\eta), \quad \forall \xi, \eta \in \mathbb{R}^m \\
T_A (c \xi) &= c T_A (\xi), \quad \forall c \in \mathbb{R}, \xi \in \mathbb{R}^m 
\end{align*}
\]

Proposition [Homework]

Prove that any map \( T: \mathbb{R}^m \to \mathbb{R}^n \) which satisfies the linearity conditions is of the type \( T = T_A \), with \( A = \text{some n x m matrix} \).
2. **Definition of Differentiability**

**Definition 1** \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is differentiable at \( \mathbf{a} \in \mathbb{R}^m \) iff there exists a linear map \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) such that

\[
\lim_{x \to \mathbf{a}} \frac{f(x) - f(\mathbf{a}) - T(x - \mathbf{a})}{\|x - \mathbf{a}\|} = 0.
\]

We set \( f'(\mathbf{a}) = T \) if such a \( T \) exists.

**Observation 2** If such a linear map exists,

Then \( T = \left[ Df(\mathbf{a}) \right] \),

\[
Df(\mathbf{a}) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\mathbf{a}), & \cdots, & \frac{\partial f_1}{\partial x_m}(\mathbf{a}) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1}(\mathbf{a}), & \cdots, & \frac{\partial f_n}{\partial x_m}(\mathbf{a})
\end{pmatrix}
\]

is the matrix of partial derivatives.

Based on this observation →
Definition 2 \( f : \mathbb{R}^m \to \mathbb{R}^n \) is differentiable at \( a \in \mathbb{R}^m \) if:

\( \alpha \) \( \frac{\partial f_j}{\partial x_i} (a) \) exist, \( 1 \leq i \leq m, \ 1 \leq j \leq n \).

\( \beta \) \( \lim_{x \to a} \frac{f(x) - f(a) - [Df(a)](x-a)}{||x-a||} = 0 \)

We say that \( f'(a) \) is the linear map

\( [Df(a)] : \mathbb{R}^m \to \mathbb{R}^n \).

Matrix of partial derivatives.
A function \( f : \mathbb{R}^m \to \mathbb{R}^n \)
is **differentiable** at \( x_0 \in \mathbb{R}^m \) **iff**

\( f \) can be approximated (to first order) by a linear map \( T : \mathbb{R}^m \to \mathbb{R}^n \)

near \( x_0 \).

If \( f \) is differentiable, such a \( T \) exists, it is given by the matrix of partial derivatives at \( x_0 \):

\[
T = \left[ \text{D} f (x_0) \right].
\]
Example

Let \( M \) a \( n \times m \) matrix, and

\[
\varphi = T_M : \mathbb{R}^m \to \mathbb{R}^n, \quad \varphi(x) = Mx, \quad x \in \mathbb{R}^m.
\]

Prove that \( \varphi \) is differentiable at \( a \in \mathbb{R}^m \).

\[\rightarrow \text{need to show that there exists a linear map} \quad T_A : \mathbb{R}^m \to \mathbb{R}^n, \quad A = n \times m \text{ matrix such that}
\]

\[
\lim_{x \to a} \frac{T_M(x) - T_M(a) - T_A(x-a)}{\|x-a\|} = 0
\]

\[\Rightarrow \text{simply take} \quad A = M !
\]

Therefore \( \varphi'(a) = [M] = \varphi \)

Answer \( \forall \ a \in \mathbb{R}^m, \ \varphi'(a) = \varphi \), when \( \varphi \) is a linear mapping.
Main Tool:

\[ \text{Theorem } \quad C^1 \Rightarrow \text{Diff } \quad f = (f_1, \ldots, f_n) \]

A) Let \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \), \( a \in \mathbb{R}^n \).

\[ \text{IF } \quad \frac{\partial f_i}{\partial x_j} \quad \text{exist, } \quad (1 \leq i \leq n, 1 \leq j \leq m) \]

and are continuous in a ball \( B_r(a) \), for some \( r > 0 \)

[i.e. "near" \( a \)]

then \( f \) is called differentiable at \( a \).

B) If \( \frac{\partial f_i}{\partial x_j} \) exist and are continuous everywhere, \( \frac{\partial f_i}{\partial x_j} : \mathbb{R}^m \rightarrow \mathbb{R} \)

then \( f \) is differentiable everywhere.
Example

\[ f : \mathbb{R}^2 \to \mathbb{R} , \quad f(x,y) = x \sin y \]

\[ \frac{\partial f}{\partial x} (x,y) = \sin y \]

\[ \frac{\partial f}{\partial y} (x,y) = x \cos y \]

Clearly (WHY?)

\[ \frac{\partial f}{\partial x} , \quad \frac{\partial f}{\partial y} : \mathbb{R}^2 \to \mathbb{R} \quad \text{are continuous everywhere}. \]

\[ \Rightarrow \quad f \text{ is continuous everywhere}. \]
II. Main Application of Differentiability:

1) Computing Directional Derivatives.

Let \( f: \mathbb{R}^3 \to \mathbb{R} \), \( a \in \mathbb{R}^3 \).

**Definition**

\[
\frac{D_\vec{v} f(a)}{D\vec{v}}(a) \triangleq \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t}
\]

For \( 0 \neq \vec{v} \in \mathbb{R}^3 \), provided the limit exist.

**Geometric Interpretation:** Rate of change of \( f \) in the \( \vec{v} \) direction.

NB: preferably \( \vec{v} \) is taken to be a unit vector.
**Theorem**

Assume \( f : \mathbb{R}^3 \to \mathbb{R} \) is differentiable at \( a \in \mathbb{R}^3 \).

Let \( a + \vec{v} \in \mathbb{R}^3 \) a non-zero vector.

Thus \( \nabla f(a) \) exists, and is given by the formula:

\[
\nabla f(a) = \frac{\partial f}{\partial x}(a) \cdot \frac{\vec{v}}{2x} + \frac{\partial f}{\partial y}(a) \cdot \frac{\vec{v}}{2y} + \frac{\partial f}{\partial z}(a) \cdot \frac{\vec{v}}{2z}
\]

**Proof.** \( f \) differentiable at \( a \), means

\[
\lim_{x \to a} \frac{f(x) - f(a) - [Df(a)] \cdot (x-a)}{\|x-a\|} = 0.
\]

Let \( x = a + tv, t \to 0 \), get:

\[
\lim_{t \to 0} \frac{f(a + tv) - f(a) - [Df(a)] \cdot tv}{\|tv\|} = 0.
\]

\[
\Leftrightarrow \lim_{t \to 0} \left( \frac{f(a + tv) - f(a)}{t} - [Df(a)](v) \right) = 0
\]

\[
\Leftrightarrow \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = [Df(a)](v)
\]

\[
\nabla f(a) = \left( \frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a), \frac{\partial f}{\partial z}(a) \right) \cdot (\frac{v_1}{v_2}, \frac{v_2}{v_3})
\]

\[
\nabla f(a) \cdot \vec{v}
\]
Note: If \( f \) is differentiable at \( a \in \mathbb{R}^3 \),

\[
\nabla f(a) = \frac{\partial f}{\partial x}(a) \cdot \mathbf{v}_1 + \frac{\partial f}{\partial y}(a) \cdot \mathbf{v}_2 + \frac{\partial f}{\partial z}(a) \cdot \mathbf{v}_3
\]

Directions know about each other!
27) Directions of greatest maximal increase.

\[ \vec{V}_+ = \text{unit vector such that} \]

\[ \nabla f(a) \cdot \hat{v} = \| \nabla f(a) \| \cdot \| \hat{v} \| \cdot \cos \theta \]

\[ = \| \nabla f(a) \| \cdot \cos \theta \]

maximum if \( \theta = 0 \)

\[ \text{i.e.} \quad \vec{V}_+ = \frac{\nabla f(a)}{\| \nabla f(a) \|} \]

Direction of greatest increase

Similarly, direction of greatest decrease is

\[ \vec{V}_- = -\frac{\nabla f(a)}{\| \nabla f(a) \|} \]

Corollary: The greatest possible change in rate of change along a unit vector \( \vec{u} \) is

\[ \nabla f(a) \cdot \vec{u} = \nabla f(a) \cdot \vec{V}_+ = \nabla f(a) \cdot \frac{\nabla f(a)}{\| \nabla f(a) \|} = \frac{\| \nabla f(a) \|^2}{\| \nabla f(a) \|} = \| \nabla f(a) \| \]
Corollary

The greatest possible increase along a unit vector is,

\[ \frac{\partial f}{\partial \mathbf{v}} f(a) = \nabla f(a) \cdot \mathbf{v} \]

\[ = \nabla f(a) \cdot \frac{\nabla f(a)}{\|\nabla f(a)\|} \]

\[ = \frac{\|\nabla f(a)\|^2}{\|\nabla f(a)\|} \]

\[ = \|\nabla f(a)\| \]

\[ = \sqrt{(\frac{\partial f}{\partial x}(a))^2 + (\frac{\partial f}{\partial y}(a))^2 + (\frac{\partial f}{\partial z}(a))^2} \]