HW9 Solutions

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1 We prove by induction on n. First notice that since $a_1 = 1 \leq a_2 = 5/2 \leq 4$, the sequence is increasing and bounded by 4 at $n = 1$. This fulfills the base case.
Now assume $a_n \leq a_{n+1}$ and $a_n \leq 4$, then $a_{n+1} = (4 + a_n)/2 \leq (4 + a_{n+1})/2 = a_{n+2}$, and $a_{n+1} = (4 + a_n)/2 \leq (4 + 4)/2 = 4$. This proves the induction step.
Since the sequence is increasing and bounded from above, by monotone converge theorem, the sequence converges to a limit, say $\lim_{n\to\infty} a_n = L$. Then L satisfies $L = (4 + L)/2$. Solving for L gives $L = 4$, hence the sequence converges to 4.

2 This is a geometric series with $a = 4/3$ and $r = 1/3$. Since $|r| < 1$, the series converges to $a/(1 - r) = 2$.

3 Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$, the series diverges by the divergence test.

4 This is geometric with $a = 3/x$ and $r = 3/x$. This converges when and only when $|3/x| < 1$, i.e., $3 < |x|$. If so, the limit is $a/(1 - r) = 3/(x - 3)$.

6 By the integral test, it suffices to determine if $\int_1^\infty \frac{1}{x \ln x} dx$ converges. But using $u = \ln x$, we have

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln u = \ln \ln x$$

Thus the indefinite integral evaluates as $\lim_{b\to\infty} \ln \ln b - 0 = \infty$, hence the series diverges.

7 Since $0 < n^4/2 \leq n^4 - 1$ for $n \geq 2$, we have $\frac{2n}{n^4} = \frac{2}{n^3} \geq \frac{n}{n^4-1} \geq 0$. By the integral and p-test, $\sum_{n=2}^\infty \frac{2}{n^3}$ converges, and therefore by comparison theorem for series, the original series converges as well.

8 Since $0 \leq \frac{3^{-n}}{4+3^{-n}} \leq \frac{3^{-n}}{4}$, by comparison theorem, it suffices to prove $\sum_{n=1}^\infty \frac{3^{-n}}{4}$ converges. But this is a geometric series with $r = 1/3$, hence it converges, and therefore so does the original series.