HW8 Solutions

October 29, 2018

1 Observe that $0 \leq \sin^2 x \leq 1$ on $[0, \pi/2]$, and therefore $0 \leq \frac{\sin^2 x}{x^{1/4}} \leq \frac{1}{x^{1/4}}$ on $(0, \pi/2]$. Because

$$\int_0^{\pi/2} x^{-\frac{1}{4}} dx = \lim_{c \to 0} \int_c^{\pi/2} x^{-\frac{1}{4}} dx = \lim_{c \to 0} \frac{4}{3}(\frac{\pi}{2})^{\frac{3}{4}} - \frac{4}{3}c^{\frac{3}{4}} < \infty$$

by the comparison theorem, the original integral converges as well.

2

1. There is a factorization $x^2 - x - 2 = (x-2)(x+1)$. Because on $(2, 3]$, $x + 1 \leq 4$, we have also $\frac{1}{x+1} \geq \frac{1}{4}$, and therefore $0 \leq \frac{1}{4(x-2)} \leq \frac{1}{x^2-x-2}$. But

   $$\int_2^3 \frac{1}{4(x-2)} dx = \frac{1}{4} \lim_{c \to 2^+} (\ln(1) - \ln(c-2)) \to \infty$$

   by the comparison theorem, the original integral diverges.

2. Here we have $x^2 - 4x + 4 = (x-2)^2$. We can therefore shift the integral to a more familiar form using a substitution $u = x - 2$. Now the integral becomes

   $$\int_{-1}^{\infty} \frac{1}{u^2} du = \int_{-1}^{0} \frac{1}{u^2} du + \int_0^1 \frac{1}{u^2} du + \int_1^{\infty} \frac{1}{u^2} du$$

   We know the first two terms do not converge, therefore the integral diverges.

3. Again, using a substitution $u = x - 1$ we get the integral to a more familiar form:

   $$\int_{-1}^{3} u^{-1/3} du = \int_{-1}^{0} u^{-1/3} du + \int_0^3 u^{-1/3} du$$

   Both of which we know to converge, thus the integral converges.

3 $a_n = (-1)^{n+1} \cdot 2n$

4 $a_n = (-1)^{n+1} \cdot 4^{2-n}$

5 $a_1 = 1, a_2 = 3$ are the initial terms. Inductively, for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

6

1. Repeat L'Hopital's rule a number of times to see that this converges.

$$\lim_{n\to\infty} \frac{n^3}{2n^3 - n} = \lim_{n\to\infty} \frac{3n^2}{6n^2 - 1} = \lim_{n\to\infty} \frac{6n}{12n} = \lim_{n\to\infty} \frac{6}{12} = \frac{1}{2}$$

2. The sequence diverges. To see why, observe $a_n = 1 + 3n$, which clearly goes to infinity as n approaches infinity.

3. This limit can be computed directly using some algebra.

$$\lim_{n\to\infty} (e^{3n+4})^{1/n} = \lim_{n\to\infty} e^{4/n} \cdot e^3 = e^0 \cdot e^3 = e^3$$

   In particular, the sequence converges.

7

1. Observe that $-\pi \leq \arctan n \leq \pi$, hence $-\frac{\pi}{n} \leq \frac{\arctan n}{n} \leq \frac{\pi}{n}$. Since $\{-\frac{\pi}{n}\}$ and $\{\frac{\pi}{n}\}$ both converge to 0, by the squeeze theorem, so does $\{a_n\}$.

2. Since $-1 \leq \sin n/2 \leq 1$, apply squeeze theorem again with bounds $\{-1/n\}$ and $\{1/n\}$ to see that $\{a_n\}$ also converges to 0.

3. Simplifying notation we have $a_n = (\frac{1}{(2n)(2n+1)})^{1/3}$. But because $f(n) = a_n$ for $f(x) = (\frac{1}{2x(2x+1)})^{1/3}$, and $f(x) \to 0$ as $x \to \infty$, a_n converges to 0 as well.