HW 7

1. Evaluate the improper integral $\int_1^\infty \frac{1}{x^2+x}\,dx$

Sol: $\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+x}\,dx$

$= \lim_{b\to\infty}\int_1^b \frac{1}{x} - \frac{1}{x+1}\,dx$

$= \lim_{b\to\infty}\left(\ln|x| - \ln|x+1|\right)\Big|_1^b$

$= \lim_{b\to\infty} \ln\left|\frac{x}{x+1}\right|\Big|_1^b$ $\qquad b \geq 1 > 0 \Rightarrow \frac{b}{b+1} > 0$

$= \lim_{b\to\infty} \ln\frac{b}{b+1} - \ln\frac{1}{1+1}$

$= \cancel{\ln 1}^{\,0} - \ln\frac{1}{2} = \boxed{-\ln\frac{1}{2}}$

* 2. Evaluate the improper integral $\int_{-\infty}^{\sqrt{2}-1} \frac{1}{x^2+2x+3}\,dx$

Sol: $\int_{-\infty}^{\sqrt{2}-1} \frac{1}{x^2+2x+3}\,dx = \lim_{a\to-\infty}\int_a^{\sqrt{2}-1} \frac{1}{x^2+2x+3}\,dx$ …… 1"

$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$ $\quad$ [Let $u = \frac{1}{\sqrt{2}}(x+1)$ $\quad du = \frac{1}{\sqrt{2}}\,dx$]

$= \int \frac{\sqrt{2}}{2u^2+2}\,du$

$= \frac{\sqrt{2}}{2}\int \frac{1}{u^2+1}\,du$

$= \frac{\sqrt{2}}{2}\arctan(u) + C$

$= \frac{\sqrt{2}}{2}\arctan\left(\frac{1}{\sqrt{2}}(x+1)\right) + C$ …… 2"

So $\int_{-\infty}^{\sqrt{2}-1} \frac{1}{x^2+2x+3}\,dx = \lim_{a\to-\infty} \frac{\sqrt{2}}{2}\arctan\left(\frac{1}{\sqrt{2}}(x+1)\right)\Big|_a^{\sqrt{2}-1}$

$= \frac{\sqrt{2}}{2}\arctan(1) - \frac{\sqrt{2}}{2}\arctan(-\infty)$ …… 1"

$= \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} - \frac{\sqrt{2}}{2}\cdot\left(-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{8}\pi + \frac{\sqrt{2}}{4}\pi = \boxed{\frac{3\sqrt{2}}{8}\pi}$. …… 1"

Determine whether the following improper integrals are convergent or divergent.

Evaluate it if it is convergent.

* 3. $\int_3^\infty \frac{1}{x^2+4x+4}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x+2)^2}\,dx$ ...... 2"

$= \lim_{b\to\infty} -\frac{1}{x+2}\Big|_3^b$ ...... 1"

$= \lim_{b\to\infty} -\frac{1}{b+2} + \frac{1}{3+2}$ ...... 1"

$= -0 + \frac{1}{5} = \boxed{\frac{1}{5}} < \infty$ convergent. 1

4. $\int_0^\infty x\ln|x|\,dx$ $\left[\begin{array}{ll} u=\ln|x| & v=\frac{1}{2}x^2 \\ du=\frac{1}{x}dx & dv=x\,dx \end{array}\right]$

$\int x\ln|x|\,dx = \frac{1}{2}x^2\ln|x| - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln|x| - \frac{1}{4}x^2 + C$

$\int_0^\infty x\ln|x|\,dx = \lim_{b\to\infty}\int_0^b x\ln|x|\,dx = \lim_{b\to\infty}\left(\frac{1}{2}x^2\ln|x| - \frac{1}{4}x^2\right)\Big|_0^b$

$= \lim_{b\to\infty}\left(\frac{1}{2}b^2\ln b - \frac{1}{4}b^2\right) - \lim_{x\to0}\left(\frac{1}{2}x^2\ln x\right)$

$= \frac{1}{2}\lim_{b\to\infty} b^2\left(\ln b - \frac{1}{2}\right) - \frac{1}{2}\lim_{x\to0}\frac{\ln x}{\frac{1}{x^2}}$

L'Hospital's Rule

$= \frac{1}{2}\infty - \frac{1}{2}\lim_{x\to0}\frac{\frac{1}{x}}{-2\frac{1}{x^3}} = \frac{1}{2}\infty - \frac{1}{2}0 = \infty$ $\boxed{\text{divergent}}$

* 5. $\int_{-\infty}^\infty x^3-3x^2\,dx = \lim_{a\to-\infty}\int_a^0 x^3-3x^2\,dx + \lim_{b\to\infty}\int_0^b x^3-3x^2\,dx$ ...... 2"

$= \lim_{a\to-\infty}\left(\frac{x^4}{4}-x^3\right)\Big|_a^0 + \lim_{b\to\infty}\left(\frac{x^4}{4}-x^3\right)\Big|_0^b$ ...... 1"

$= \lim_{a\to-\infty}\left(-\frac{a^4}{4}+a^3\right) + \lim_{b\to\infty}\left(\frac{b^4}{4}-b^3\right)$ ...... 1"

$\quad\quad\quad = -\infty \quad\quad\quad\quad\quad = \infty$

Since both ~~lim~~ $\int_{-\infty}^0 x^3-3x^2\,dx$ & $\int_0^\infty x^3-3x^2\,dx$ diverge,

$\int_{-\infty}^\infty x^3-3x^2\,dx$ $\boxed{\text{diverges}}$. ...... 1"

6. $\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta = \lim_{b\to\infty}\int_0^b \sin\theta\, e^{\cos\theta}\,d\theta$ ...... 2" $= \lim_{b\to\infty} -e^{\cos\theta}\Big|_0^b$ ...... 1"

$= \lim_{b\to\infty} -e^{\cos\theta} + e$ limit does not exist. $\Rightarrow$ $\boxed{\text{divergent}}$ ...... 2"

$\ast$ 7. $\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{x}{x^4+4}\,dx$ ...... 1"

$u = \frac{x^2}{2}$ $\int \frac{x}{x^4+4}\,dx = \int \frac{1}{4u^2+4}\,du = \frac{1}{4}\arctan(u) + C = \frac{1}{4}\arctan\left(\frac{x^2}{2}\right) + C$ ...... 1"

$du = x\,dx$

...... 1"

$\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx = \lim_{a\to-\infty} \frac{1}{4}\arctan\left(\frac{x^2}{2}\right)\Big|_a^0 = \lim_{a\to-\infty} -\frac{1}{4}\arctan\left(\frac{a^2}{2}\right) = -\frac{1}{4}\cdot\frac{\pi}{2} = \boxed{-\frac{\pi}{8}}$

convergent ...... 2"

$\ast$ 8. $\int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\sqrt{x}}\,dx$ ...... 1"

$u = \sqrt{x}$ $\int e^{-\sqrt{x}}\,dx = \int e^{-u}\cdot 2u\,du$ $\begin{bmatrix} v = 2u & w = -e^{-u} \\ dv = 2du & dw = e^{-u}du \end{bmatrix}$ derivative by part ...... 1"

$du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\,du = 2u\,du$ $= -2u\cdot e^{-u} + 2\int e^{-u}\,du$

Substitution ...... 1" $= -2ue^{-u} - 2e^{-u} + C$

$= -2e^{-\sqrt{x}}(\sqrt{x}+1) + C$

So $\int_0^\infty e^{-\sqrt{x}}\,dx = \lim_{b\to\infty} -2e^{-\sqrt{x}}(\sqrt{x}+1)\Big|_0^b$

...... 1"

$= -2\lim_{b\to\infty} \frac{\sqrt{b}+1}{e^{\sqrt{b}}} + 2 = -2\cdot 0 + 2 = \boxed{2}$ convergent.

$\lim_{b\to\infty} \frac{\sqrt{b}+1}{e^{\sqrt{b}}} = \frac{\text{"}\infty\text{"}}{\text{"}\infty\text{"}} \overset{\text{L'Hospital's Rule}}{=} \lim_{b\to\infty} \frac{\frac{1}{2\sqrt{b}}}{e^{\sqrt{b}}\cdot\frac{1}{2\sqrt{b}}} = 0$ ...... L'Hospital's Rule 1"