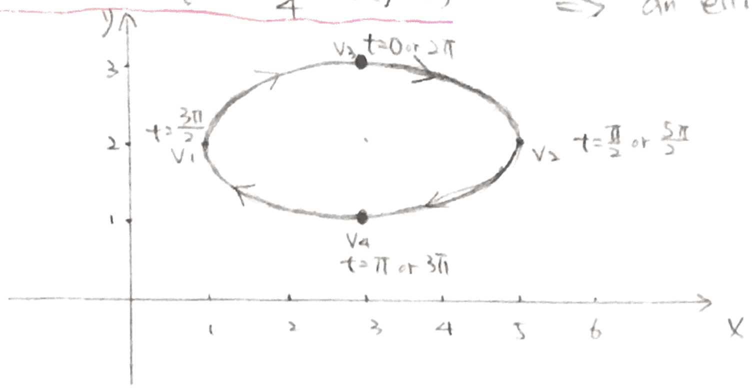
HW 6 Part 1

1. $x = 2\sin t + 3$ $y = \cos t + 2$ for $t \in [0, 3\pi]$. Sketch the parametric curve.

sol: $\sin t = \frac{1}{2}(x-3)$ $\cos t = y-2$

$1 = \sin^2 t + \cos^2 t = \frac{(x-3)^2}{4} + (y-2)^2$ $\Rightarrow$ an ellipse centered at $(3,2)$ ······ equation



If $x-3=0$, then $(y-2)^2=1$ $y=1$ or 3

so $V_3=(3,3)$ $V_4=(3,1)$

If $y-2=0$, then $\frac{(x-3)^2}{4}=1$ $x=1$ or 5

so $V_1=(1,2)$, $V_2=(5,2)$

When $t=0$, $x=2\sin 0+3=3$ $y=\cos 0+2=3$

When $t=\frac{\pi}{2}$, $x=2\sin\frac{\pi}{2}+3=5$ $y=\cos\frac{\pi}{2}+2=2$

When $t=3\pi$, $x=2\sin 3\pi+3=3$ $y=\cos 3\pi+2=1$

$(\cos t, \sin t)$ oriented counterclockwise, so $(\sin t, \cos t)$ oriented clockwise.

······ orientation 2"

So the curve is an ellipse starting at $(3,3)$, ending at $(3,1)$, oriented clockwise.

2. (a) Compute the slope of the parametric curve $x(t) = t\sin t$, $y(t) = \cos t$, for $t \in [0, \frac{\pi}{2}]$.

(b) Write down the tangent line equation at $t=0$.

(c) Compute the second derivative $\frac{d^2y}{dx^2}$ of the parametric curve $x(t)=\cos t$, $y(t)=\sin t$, for $t \in [0, \frac{\pi}{2}]$. Is the curve convex or concave?

Sol: (a) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{-\sin t}{t\cos t + \sin t}$ ······ 1"

(b) When $t=0$, $\frac{dy}{dt} = \frac{dx}{dt} = 0$, $\lim\limits_{t\to 0} \frac{dy}{dx} = \lim\limits_{t\to 0} \frac{-\sin t}{t\cos t + \sin t} \overset{\text{L'Hospital's Rule}}{=} \lim\limits_{t\to 0} \frac{-\cos t}{-t\sin t + \cos t + \cos t}$

$= \frac{-1}{-0+1+1} = -\frac{1}{2}$. ······ slope 1"

$x(0) = 0\sin 0 = 0$

$y(0) = \cos 0 = 1$

tangent line: $(y-1) = -\frac{1}{2}(x-0)$ $\Rightarrow$ $y = -\frac{1}{2}x + 1$ ······ equation 1"

(c) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\cos t}{-\sin t}\right)}{-\sin t} = \frac{\frac{d}{dt}(-\cot t)}{-\sin t} = \frac{\csc^2 t}{-\sin t} = -\frac{1}{\sin^3 t}$ ······ 1"

$t \in [0, \frac{\pi}{2}]$ $\Rightarrow$ $\sin t \geq 0$ so $\frac{d^2y}{dx^2} = -\frac{1}{\sin^3 t} < 0$. Therefore, the curve is concave

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