Math 109 HW5 Graded: 2, 3, 5, 7, 9.

1. Problem 3-6 on p598 of textbook

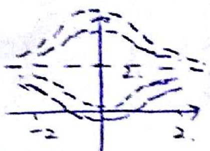
3. $y' = 2 - y$

4. $y' = x(2-y)$

5. $y' = x + y - 1$

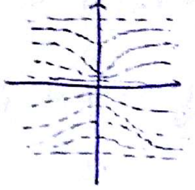
6. $y' = \sin x \sin y$

4 matches I:



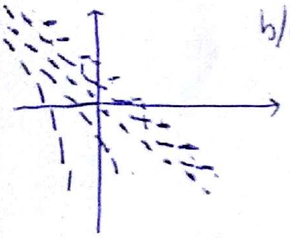
because $y' = 0$ when: $x = 0$ or $y = 2$

6 matches II.



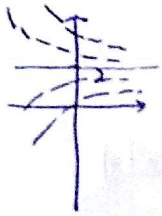
because $y' = 0$ when:
$x = 0$,
or $y = 0$
or $x = k\pi$
or $y = k\pi$.

5 matches IV



b/c $y' = 0$ only when $y = 1 - x$.

3 matches III.



because. $y' = 0$ ONLY when $y = 2$.

$\square$

$\boxed{2^*}$ Solve the differential equations:

(a) $\dfrac{dy}{dx} = x^3 y$.

Sol: $\dfrac{dy}{y} = x^3\,dx$ when $y \neq 0$.

Integrate $\Rightarrow \ln|y| = \dfrac{x^4}{4} + C$. C is a constant

$\Rightarrow y = \pm C' \cdot e^{\frac{x^4}{4}}$, where $C' > 0$. (because $C' = e^C$) ... 1'

. check $y = 0$; LHS $= \dfrac{d0}{dx} = 0$

RHS $= x^3 \cdot 0 = 0$

So $y = 0$ is also a solution (special) ... 1'

Thus the solutions are $y = \tilde{C} e^{\frac{x^4}{4}}$, $\tilde{C}$ is ANY (real) constant.

$\square$

(b). $\frac{x\,dy}{dx} = (x^2+1)y$, $y(1)=1$.

~~Proof~~. Sol: $\frac{dy}{y} = \frac{x^2+1}{x}\,dx = (x+\frac{1}{x})\,dx$

$$\Rightarrow \ln|y| = \frac{x^2}{2} + \ln|x| + C. \quad \cdots (2')$$

· Now. $y(1)=1$ means y is not identically 0, so we don't have to consider special solutions.

· Determin C: $y(1)=1$ means $\ln|1| = \frac{1^2}{2} + \ln|1| + C$

$$\Rightarrow C = -\frac{1}{2}$$

Thus the solution is

$$\ln|y| = \frac{x^2}{2} + \ln|x| - \frac{1}{2}.$$

$$\Rightarrow \ln\left|\frac{y}{x}\right| = \frac{x^2-1}{2}. \quad \Rightarrow \frac{y}{x} = \pm e^{\frac{x^2-1}{2}}$$

Since $y(1)=1$, we take the positive sign: $y = xe^{\frac{x^2-1}{2}}$. $\cdots (1')$.

□

$\boxed{3.^*}$ Solve the differential equation $\frac{dP}{dt} = P(1-\frac{P}{M})$. by separation of variables and graph the general solution.

Sol: $\int \frac{dP}{P(1-\frac{P}{M})} = dt$.

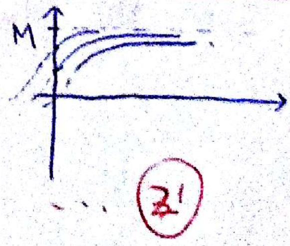
Note that $\frac{1}{P(1-\frac{P}{M})} = \frac{1}{P\cdot(\frac{M-P}{M})} = \frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$. $\cdots (2')$

Thus the equation becomes $\frac{dP}{P} + \frac{dP}{M-P} = dt$. Graph $\cdots (1')$.

Integrate: $\ln|P| - \ln|M-P| = t + C$.

$$\Rightarrow \frac{M-P}{P} = c'e^{-t} \quad \Rightarrow P(t) = \frac{M}{1+c'e^{-t}}.$$

for some $c'>0$. $\cdots (3')$



□

4. Find the orthogonal trajectories of the family of Curves $x^2+3y^2=k^2$.

Draw several members of each family on xy-plane.

Sol: $x^2+3y^2=k^2$.

① Differentiate the equation

$$2x\,dx+6y\,dy=0.$$

$\Rightarrow \frac{dy}{dx}=-\frac{x}{3y}$. , This is the slope of the tangent line of $x^2+3y^2=k^2$ at the point (x,y)

② The slope of the tangent line of orthogonal trajectory at (x,y) is

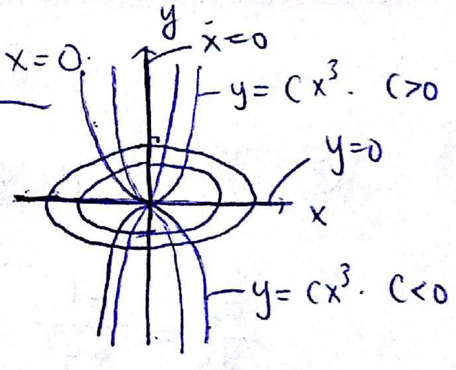
$\frac{dy}{dx}=\frac{-1}{\left(-\frac{x}{3y}\right)}=\frac{3y}{x}$ $\Rightarrow \frac{dy}{3y}=\frac{dx}{x}$. $\Rightarrow \cancel{\tfrac{1}{3}}\ln|y|=3\ln|x|+C$

$\Rightarrow |y|=C'|x|^3$. $C'>0$.

$\Rightarrow y=C'x^3$. for $C'>0$.

③ Also $y=0$, $x=0$ are of the orthogonal trajectories.

So the answer is $y=Cx^3$ for any C, and $x=0$.



□.

5.* Find the orthogonal trajectories of $y = \frac{1}{k+x}$.
Draw several members of each family in xy-plane.

Sol.

① Compute the slope of tangent line of $y = \frac{1}{k+x}$ at (x, y).

$$\frac{dy}{dx} = -\frac{1}{(k+x)^2} \quad \cdots (1')$$

② Now we have

$$\begin{cases} y = \frac{1}{k+x} & \cdots \langle 1 \rangle \\ \frac{dy}{dx} = -\frac{1}{(k+x)^2} & \cdots \langle 2 \rangle \end{cases}$$

We eliminate k: ~~From the first~~

$$\langle 1 \rangle^2 + \langle 2 \rangle \Rightarrow \frac{dy}{dx} + y^2 = 0 \Rightarrow \frac{dy}{dx} = -y^2 \quad \cdots (2')$$

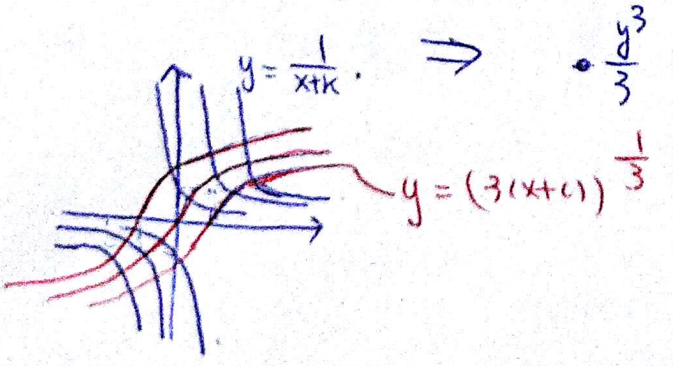
$$\cancel{\Rightarrow \int \frac{dy}{y^2} = \int dx \text{ when } y \neq 0}$$

$$\cancel{\Rightarrow \frac{1}{y} = x + C \Rightarrow y = \frac{1}{x+C}}$$

③ The slope of the tangent line of the orthogonal trajectory at (x, y) is: $\cdots (1')$

$$\frac{dy}{dx} = \frac{-1}{-y^2} \Longrightarrow y^2\, dy = dx.$$

$$\Rightarrow \frac{y^3}{3} = x + C. \Rightarrow y = (3(x+C))^{\frac{1}{3}} \text{ for any constant } C. \quad \cdots (1')$$



□

6. Solve ~~the~~ $\frac{dP}{dt} = P(1-\frac{1}{1000}P)$.

with (a) $P(0)=300$

(b) $P(0)=1500$. Is the solution increasing / decreasing?

Sol: From exercise 3 we know, the solution is (Let $M=1000$).

$$P(t) = \frac{1000}{1+C'e^{-t}} \quad \text{for some } c'>0.$$

(a) $P(0)=300 \Rightarrow 300 = \frac{1000}{1+c'} \Rightarrow c' = \frac{7}{3}$.

$$\Rightarrow P(t) = \frac{1000}{1+\frac{7}{3}e^{-t}}.$$ increasing.

(b) $P(0) = 1500 \Rightarrow 1500 = \frac{1000}{1+c'} \Rightarrow c' = -\frac{1}{3}$.

$$\Rightarrow P(t) = \frac{1000}{1-\frac{1}{3}e^{-t}}$$ decreasing. □

$\boxed{7.}^{*}$ Solve the differential eqn: $(x\ln x)\cdot y' + y = xe^x$.

Sol: Divide ~~x on~~ both side. by x

$\Rightarrow (\ln x)\cdot y' + \frac{1}{x}y = e^x$

$\Rightarrow (\ln x)\cdot y' + (\ln x)'\cdot y = e^x. \overset{\text{product rule}}{\Rightarrow} (y\cdot \ln x)' = e^x$ ... 3'

$(\ln x)' = \frac{1}{x}$

Integrate $\Rightarrow$. $y\ln x = \int e^x = e^x + C$. is the solution ... 2'

□

8. Solve the diff. eq: $xy' = x^2 + y$.

Sol: $xy' = x^2 + y$

$\Rightarrow xy' - y = x^2$.

$x' = 1$ $\Rightarrow xy' - (x)' \cdot y = x^2$

$\Rightarrow \dfrac{y' \cdot x - y \cdot (x')}{x^2} = 1$.

$\Rightarrow \left(\dfrac{y}{x}\right)' = 1$. $\xrightarrow{\text{Integrate.}}$ $\dfrac{y}{x} = x + C$.

$\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$.

here $f(x) = y(x)$

$g(x) = x$

$\Rightarrow y = x(C + x)$.

□.

$\boxed{9^*}$ Solve the initial value problem:

$$\begin{cases} xy' + 4xy = e^{-4x} \ln x. & \cdots \langle 1 \rangle \\ y(1) = 1. & \cdots \langle 2 \rangle \end{cases}$$

Sol: Multiply e^{4x} on both sides of $\langle 1 \rangle$.

we have $xe^{4x}y' + 4xe^{4x}y = \ln x$. $\cdots \langle 3 \rangle$

Then, divide $\langle 3 \rangle$ by x:

$$e^{4x}y' + 4e^{4x}y = \frac{\ln x}{x}.$$

Note that $(e^{4x})' = 4e^{4x}$ $\Rightarrow e^{4x} \cdot y' + (e^{4x})' \cdot y = \dfrac{\ln x}{x}$

product rule $\Rightarrow (e^{4x} \cdot y)' = \dfrac{\ln x}{x}$ $\cdots$ 3'.

Integrate $\Rightarrow e^{4x} \cdot y = \displaystyle\int \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 + C$ $\cdots$ 1'

Finally, we use $\langle 2 \rangle$ to determine C:

$y(1) = 1$

$\Rightarrow e^{4 \times 1} \cdot 1 = \frac{1}{2}(\ln 1)^2 + C$

$\Rightarrow C = e^4$ $\cdots$ 1'

Thus

$y = e^{-4x}\left(\frac{1}{2}(\ln x)^2 + e^4\right)$ $\cdots$ 1'

□

10. Solve $\begin{cases} t^3 \frac{du}{dt} + 4t^2 u = 1 & \cdots \langle 1 \rangle \\ u(1) = 1. & \cdots \langle 2 \rangle \end{cases}$

Sol: Mutiply t on both sides of $\langle 1 \rangle$

$$\langle 1 \rangle \Rightarrow t^4 \frac{du}{dt} + 4t^3 u = t$$

$$\Rightarrow t^4 \frac{du}{dt} + (t^4)' u = t.$$

$$\Rightarrow \frac{d(t^4 u)}{dt} = t. \quad \cdots \langle 3 \rangle$$

Integrate $\langle 3 \rangle$ with respect to t:

$$t^4 u = \int t = \frac{1}{2} t^2 + C$$

$$\Rightarrow u = t^{-4}\left(\frac{1}{2}t^2 + C\right).$$

Finally we determine C via $\langle 2 \rangle$.

$$u(1) = 1 \Rightarrow 1 = 1^{-4}\left(\frac{1}{2}1^2 + C\right) \Rightarrow C = \frac{1}{2}.$$

Thus: $u = \frac{t^{-4}(t^2+1)}{2} = \frac{t^2+1}{2t^4}$

$\square$