Math 109 Solutions to Hw4

*1. Evaluate $I = \int \frac{dx}{x^3\sqrt{x^2-4}}$

Solution: Do substitution $x = 2\sec\theta$, then $dx = 2\sec\theta\tan\theta\, d\theta$. ... 1'

then $I = \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^3\theta \cdot 2\tan\theta}$

$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$.

$= \frac{\theta}{16} + \frac{1}{32}\sin 2\theta + C$. ... 2'

Now: $\frac{2}{\cos\theta} = x \Rightarrow \theta = \arccos\left(\frac{2}{x}\right)$. (i.e, $\cos\theta = \frac{2}{x}$).

$\sin 2\theta = 2\sin\theta\cos\theta = 2\sqrt{1-\cos^2\theta}\cos\theta = 2\sqrt{1-\frac{4}{x^2}} \cdot \frac{2}{x} = \frac{4\sqrt{x^2-4}}{x^2}$.

So, $I = \frac{1}{16}\arccos\left(\frac{2}{x}\right) + \frac{1}{8x^2}\sqrt{x^2-4} + C$. ... 2'

□

2. Evaluate $I = \int x^2\sqrt{1-x^6}\, dx$

Solution: Let $t = x^3$, then $dt = 3x^2\, dx$

$\Rightarrow I = \frac{1}{3}\int \sqrt{1-t^2}\, dt = \frac{1}{3}\int 1\cdot\sqrt{1-t^2}\, dt$.

Let $u'(t) = 1$, $v(t) = \sqrt{1-t^2}$

then $u(t) = t$, $v'(t) = \frac{-t}{\sqrt{1-t^2}}$

~~Let $t = \sin\theta$, then $dt = \cos\theta\, d\theta$~~

~~$\Rightarrow I = \frac{1}{3}\int\sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \frac{1}{3}\int\cos^2\theta$~~

~~Similarly as in exercise 1, we know $I = \frac{\theta}{6} + \frac{1}{12}\sin 2\theta + C$.~~

Integrate by part: $I = u(t)v(t) - \int u(t)v'(t)$.

$I = \frac{1}{3}\left(t\sqrt{1-t^2}\right) + \frac{1}{3}\int \frac{t^2}{\sqrt{1-t^2}}\, dt$

Observe that, this term is exactly I

$= \frac{1}{3}\left(t\sqrt{1-t^2}\right) + \frac{1}{3}\int \frac{t^2-1+1}{\sqrt{1-t^2}}\, dt = \frac{1}{3}\left(t\sqrt{1-t^2}\right) - \frac{1}{3}\int\sqrt{1-t^2}\, dt + \frac{1}{3}\int\frac{1}{\sqrt{1-t^2}}\, dt$

Thus, we have $I = \frac{1}{3}(t\sqrt{1-t^2}) - \cancel{\phi} I + \frac{1}{3}\arcsin(t) + c$.

$$\Rightarrow \quad 2I = \frac{1}{3}\left(t\sqrt{1-t^2} + \arcsin t\right) + \text{constant}.$$

$$\Rightarrow \quad I = \frac{1}{6}\left(t\sqrt{1-t^2} + \arcsin t\right) + \text{constant}$$

use $t = x^3$ $\rightarrow = \frac{1}{6}\left(x^3\sqrt{1-x^6} + \arcsin(x^3)\right) + \text{constant}.$

$\square$

3.* For (a). (b). (c). write out the form of the partial fraction decomposition of the function (as in Example 7 in page 499). Do not determine the numerical values of the coefficients.

(a). $$\int \frac{3+x}{x(x^2+2x+1)}\,dx = \int \frac{3+x}{x(x+1)^2}\,dx$$

$$\frac{3+x}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{(x+1)^2} \quad \cdots (1')$$

Errata: The third term should be C/(x+1)^2

(b). $$\int \frac{2x+1}{(x^3+x^2+x)}\,dx = \int \frac{2x+1}{x(x^2+x+1)}\,dx$$

$$\frac{2x+1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}. \quad \cdots (2')$$

(c). $$\int \frac{x^5+1}{x^2+x^4}\,dx$$

$$\frac{x^5+1}{x^2+x^4} = \frac{x^5+x^3+1-x^3}{x^2+x^4} = \frac{x(x^2+x^4)+(1-x^3)}{x^2+x^4} = x + \frac{1-x^3}{x^2+x^4}$$

$$= x + \frac{1-x^3}{x^2(x^2+1)} = x + \frac{A}{x} + \frac{Bx+C}{x^2} + \frac{Dx+E}{x^2+1} \quad \cdots (2')$$

Errata: The third term should be B/x^2

$\square$

*4. Evalute the integral $I = \int \frac{x^2+1}{x^2-2x-3}\,dx$.

Sol: $\frac{x^2+1}{x^2-2x-3} = \frac{(x^2-2x-3)+(2x+4)}{x^2-2x-3} = 1 + \frac{2x+4}{x^2-2x-3} = 1 + \frac{2x+4}{(x-3)(x+1)}$. ... 2'

Suppose $\frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{(A+B)x+(A-3B)}{(x-3)(x+1)}$.

this requires $A+B=2$

$A-3B=4 \quad \Rightarrow \quad B = -\frac{1}{2}. \quad A = \frac{5}{2}.$ ... 2'

$\Rightarrow I = \int 1 + \frac{5}{2}\frac{1}{x-3} - \frac{1}{2}\frac{1}{x+1}\,dx = x + \frac{5}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C.$ ... 1'

□

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5. Evaluate $I = \int_1^3 \frac{x^3+2x^2+x-1}{x^3+x}\,dx$.

Sol: $\frac{x^3+2x^2+x-1}{x^3+x} = \frac{(x^3+x)+(2x^2-1)}{x^3+x} = 1 + \frac{2x^2-1}{x^3+x}$.

$= 1 + \frac{2x^2-1}{x(x^2+1)}$

Let $\frac{2x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$.

$= \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$.

$\Rightarrow$ This requires: $A+B=2$

$C=0 \quad \Rightarrow \quad A=-1. \quad B=3 \quad C=0.$

$A=-1$

$\Rightarrow \frac{2x^2-1}{x(x^2+1)} = -\frac{1}{x} + \frac{3x}{x^2+1}$

For this integral, do substitution $t = x^2$

$\Rightarrow I = \int_1^3 1 - \frac{1}{x} + \frac{3x}{x^2+1}\,dx. = \int_1^3 1\,dx - \int_1^3 \frac{1}{x}\,dx + \int_1^3 \frac{3x}{x^2+1}\,dx.$

$= 2 - \ln x\Big|_{x=1}^{x=3} + \frac{3}{2}\ln(x^2+1)\Big|_{x=1}^{x=3}$

$= 2 - \ln 3 + \frac{3}{2}\ln 10 - \frac{3}{2}\ln 2.$

$\cancel{= 2 + \ln(10^{\frac{3}{2}}}$

□

6. Show that $\frac{1}{2}x\sin x$ is a solution to $y''+y=\cos x$.

Proof: Let $\overset{y(x)}{y}=\frac{1}{2}x\sin x$

then $y'(x)=\frac{1}{2}(\sin x+x\cos x)$

$y''(x)=\frac{1}{2}\cos x+\frac{1}{2}\cos x-\frac{1}{2}x\sin x$

$=\cos x-y(x)$ $\Rightarrow y''(x)+y(x)=\cos x$.

$\square$

* 7. Show that every member of the family of functions $y(x)=\frac{\ln x+C}{x}$.

is a solution to $x^2y'+xy=1$.

Proof: $y(x)=\frac{\ln x+C}{x}$ $\Rightarrow y'(x)=\frac{(\ln x+C)'\cdot x-(\ln x+C)(x)'}{x^2}$

$=\frac{\frac{1}{x}\cdot x-(\ln x+C)\cdot 1}{x^2}$

$=\frac{1-\ln x-C}{x^2}$ ... (3')

$x^2y'+xy=\cancel{x^2}\frac{1-\ln x-C}{\cancel{x^2}}+\cancel{x}\frac{\ln x+C}{\cancel{x}}$

$=1-\ln x-C+\ln x+C=1$ ... (2') $\square$

* 8. $v(t)$ is a solution of $v'=-v(v+1)(v-1)$, For what value of v, is v unchanging? Explain your answer. We call the unchanging value of v is an equillibrum status.

Sol: This is the find the value v, such that $v'(t)=0$. ... (2')

$\Rightarrow -v(v+1)(v-1)=0$

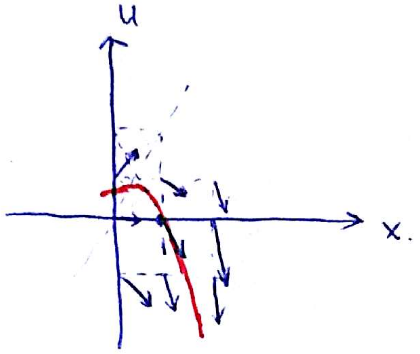
$\Rightarrow v=0$ or -1 or 1. ... (3')

$\square$.

9. (a) $u(x)$ satisfies $\frac{du}{dx} = u - 2x$. Use the direction field method to sketch a solution curve that passes the point $(1,0)$.

(b) $u(x)$ satisfies $\frac{du}{dx} = xu + u$. Use the direction field method to sketch a solution curve that passes the point $(0,1)$.

Sol: (a).



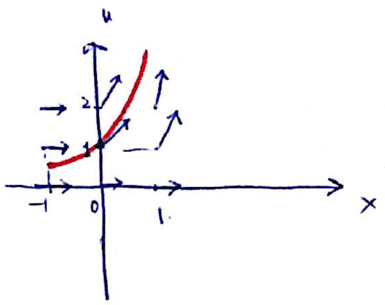
at $(1,0)$, $u - 2x = 0 - 2\times 1 = -2$

So the slope of the tangent line of the curve passing $(1,0)$ is -2.

x	1	0	2	1	0	2	1	0	2	...
u	0	0	0	1	1	1	-1	-1	-1	...
$\frac{du}{dx}$	-2	0	-4	-1	1	-3	-3	-1	-5	...

Rmk: In fact, the solution $u(x) = 2x-2-4e^{x-1}$

(b).



x	0	-1	1	0	-1	1	0	-1	1
u	1	1	1	0	0	0	2	2	2
$\frac{du}{dx}$	1	0	2	0	0	0	2	0	4

$$\frac{du}{dx} = xu + u = u(x+1).$$

$$\Rightarrow \frac{du}{u} = (x+1)\,dx.$$

$$\Rightarrow u = ce^{\frac{x(x+2)}{2}}.$$

$u(0) = 1 \Rightarrow c = 1. \Rightarrow u = e^{\frac{x(x+2)}{2}}$. ← solution

(You don't have to compute this)

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