* 1) $\int x \tan x \sec x \, dx$

Let $u = x$, $du = dx$

$dv = \tan x \sec x \, dx$, $v = \sec x$

Using IBP,

$$\int x \tan x \sec x \, dx = x \sec x - \int \sec x \, dx$$

$$= x \sec x - \ln|\sec x + \tan x| + C$$

* 2) $\int \sin\frac{1}{2}x \cos x \, dx$

Recall: $\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$$\Rightarrow \sin \underbrace{\tfrac{1}{2}x}_{A} \cos \underbrace{x}_{B} = \frac{1}{2}\left(\sin \underbrace{\tfrac{3}{2}x}_{A+B} + \sin(\underbrace{-\tfrac{1}{2}x}_{A-B})\right)$$

$$= \frac{1}{2}\left(\sin \frac{3}{2}x - \sin \frac{1}{2}x\right)$$

$$\int \sin\frac{1}{2}x \cos x \, dx = \frac{1}{2}\left(\int \sin \frac{3}{2}x \, dx - \int \sin \frac{1}{2}x \, dx\right)$$

$$= \frac{1}{2}\left(-\frac{2}{3} \cdot \cos \frac{3}{2}x + 2\cos\frac{1}{2}x\right) + C$$

$$= -\frac{1}{3} \cdot \cos \frac{3}{2}x + \cos\frac{1}{2}x + C$$

3) $\int \tan^2 x \sec x \, dx$

let $u = \tan x$ $\quad du = \sec^2 x$

$dv = \tan x \sec x$ $\quad v = \sec x$

$$\int \tan^2 x \sec x \, dx = \tan x \sec x - \int \sec^3 x \, dx \quad \text{——(1)}$$

$$\boxed{\begin{aligned} &\text{Notice, also that } \int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \\ &\Rightarrow \int \sec^3 x \, dx = \int \tan^2 x \sec x \, dx + \int \sec x \, dx \end{aligned}}$$

Plugging this into (1),

$$\int \tan^2 x \sec x \, dx = \tan x \sec x - \int \tan^2 x \sec x \, dx - \int \sec x \, dx$$

$$2 \int \tan^2 x \sec x \, dx = \tan x \sec x - \int \sec x \, dx$$

$$\int \tan^2 x \sec x \, dx = \frac{1}{2}\left(\tan x \sec x - \ln|\sec x + \tan x|\right) + C$$

* 4) $\int \frac{x}{\sqrt{1+x^2}} \, dx$

let $x = \tan\theta \Rightarrow \sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

& $dx = \sec^2\theta \, d\theta$

$$\int \frac{x}{\sqrt{1+x^2}} \, dx = \int \frac{(\tan\theta)(\sec^2\theta \, d\theta)}{(\sec\theta)} = \int \tan\theta \sec\theta \, d\theta$$

$$= \sec\theta + C$$

$$= \sqrt{1+x^2} + C$$

5) $\int \frac{\sqrt{t^2-3}}{t^4}\, dt$

Let $t = \sqrt{3}\sec\theta$

$\sqrt{t^2-3} = \sqrt{3\sec^2\theta - 3} = \sqrt{3\tan^2\theta} = \sqrt{3}\tan\theta$

$dt = \sqrt{3}\tan\theta\sec\theta\, d\theta$

$$\int \frac{\sqrt{t^2-3}}{t^4}\, dt = \int \frac{\sqrt{3}\tan\theta}{(\sqrt{3})^4\sec^4\theta} \cdot \sqrt{3}\tan\theta\sec\theta\, d\theta$$

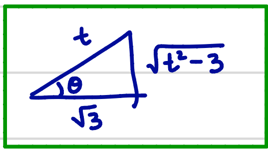
$$= \int \frac{3\tan^2\theta\sec\theta\, d\theta}{9\sec^4\theta} = \frac{1}{3}\int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^4\theta \cdot \frac{1}{\cos\theta}\, d\theta$$

$$= \frac{1}{3}\int \sin^2\theta\cos\theta\, d\theta$$

$$= \frac{1}{3}\int u^2 du = \frac{1}{3}\left(\frac{u^3}{3}\right) + C$$

Let $u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= \frac{1}{9}\sin^3\theta + C$$



$$= \frac{1}{9}\left(\frac{\sqrt{t^2-3}}{t}\right)^3 + C$$

$$= \frac{1}{9}\frac{(t^2-3)^{3/2}}{t^3} + C$$

6) $\int \frac{1}{\sqrt{x^2+4x+7}}\, dx$

Now, $x^2+4x+7 = x^2+4x+2^2+3 = (x+2)^2+3$

$$\int \frac{1}{\sqrt{x^2+4x+7}}\, dx = \int \frac{1}{\sqrt{(x+2)^2+3}}\, dx$$

$$\text{Let} \quad x+2 = \sqrt{3}\tan\theta$$

$$dx = \sqrt{3}\sec^2\theta \, d\theta$$

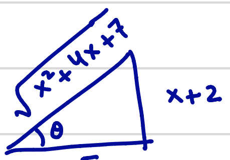
$$\sqrt{(x+2)^2+3} = \sqrt{3\tan^2\theta+3} = \sqrt{3\sec^2\theta} = \sqrt{3}\sec\theta$$

$$\int \frac{1}{\sqrt{(x+2)^2+3}}\,dx = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{\sqrt{3}\sec\theta}$$

$$= \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4x+7}}{\sqrt{3}} + \frac{x+2}{\sqrt{3}}\right| + C$$



(using $\tan\theta = \frac{x+2}{\sqrt{3}}$)

7) $\displaystyle\int \frac{\sqrt{x^2-3}}{x^3}\,dx$

Let $x = \sqrt{3}\sec\theta \Rightarrow dx = \sqrt{3}\tan\theta\sec\theta\,d\theta$

(i.e., $\cos\theta = \frac{\sqrt{3}}{x}$)

and $\sqrt{x^2-3} = \sqrt{3\sec^2\theta-3} = \sqrt{3}\tan\theta$

$$\int \frac{\sqrt{x^2-3}}{x^3}\,dx = \int \frac{\sqrt{3}\tan\theta}{(\sqrt{3}\sec\theta)^3}\,\sqrt{3}\tan\theta\sec\theta\,d\theta$$

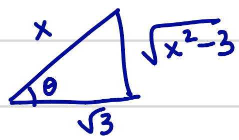
$$= \frac{1}{\sqrt{3}}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{\sqrt{3}}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{2\sqrt{3}}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2\sqrt{3}}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{\theta}{2\sqrt{3}} - \frac{2\sin\theta\cos\theta}{2\cdot 2\sqrt{3}} + C$$

$$= \frac{\theta}{2\sqrt{3}} - \frac{\sin\theta\cos\theta}{2\sqrt{3}} + C$$



$$= \frac{\arccos(\sqrt{3}/x)}{2\sqrt{3}} - \frac{\sqrt{x^2-3}\cdot\sqrt{3}}{x^2\cdot 2\sqrt{3}} + C$$

$$= \frac{\arccos(\sqrt{3}/x)}{2\sqrt{3}} - \frac{\sqrt{x^2-3}}{2x^2} + C$$

8) $\int (x-1)^2\sqrt{1+4x-2x^2}\,dx = \int (x-1)^2\sqrt{2\left(\frac{1}{2}+2x-x^2\right)}\,dx$

$$\frac{1}{2}+2x-x^2 = -\left(x^2-2x-\frac{1}{2}\right) = -\left(x^2-2x+1-\frac{3}{2}\right)$$

$$= -\left((x-1)^2-\frac{3}{2}\right) = \frac{3}{2}-(x-1)^2$$

$$\int (x-1)^2\sqrt{2\left(\frac{1}{2}+2x-x^2\right)}\,dx = \int (x-1)^2\sqrt{2\left(\frac{3}{2}-(x-1)^2\right)}\,dx$$

$$= \int (x-1)^2\sqrt{3-2(x-1)^2}\,dx$$

Let $\sqrt{2}(x-1) = \sqrt{3}\sin\theta \iff x = \sqrt{\frac{3}{2}}\sin\theta + 1$

(i.e. $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}(x-1)$)

$$dx = \sqrt{\frac{3}{2}}\cos\theta\,d\theta$$

and $\sqrt{3-2(x-1)^2} = \sqrt{3-\left(\sqrt{2}(x-1)\right)^2}$

$$= \sqrt{3-3\sin^2\theta} = \sqrt{3}\cos\theta$$

$$\int (x-1)^2\sqrt{3-2(x-1)^2}\,dx = \int \left(\sqrt{\frac{3}{2}}\sin\theta\right)^2\left(\sqrt{3}\cos\theta\right)\sqrt{\frac{3}{2}}\cos\theta\,d\theta$$

$$= \int \frac{3}{2}\cdot\frac{3}{\sqrt{2}}\sin^2\theta\cos^2\theta\,d\theta = \frac{9}{2\sqrt{2}}\int (\sin\theta\cos\theta)^2\,d\theta$$

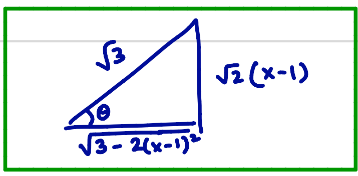
as $(\sin 2\theta)^2 = (2\sin\theta\cos\theta)^2$

$$= \frac{9}{2\sqrt{2}}\int \frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \frac{9}{8\sqrt{2}}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{9}{16\sqrt{2}}\left(\theta - \frac{\sin 4\theta}{4}\right) + C$$

Now, $\sin 4\theta = 2(\sin 2\theta)(\cos 2\theta) = 2\left(2\sin\theta\cos\theta\right)\left(\cos^2\theta - \sin^2\theta\right)$



$$= 4\left(\frac{\sqrt{2}(x-1)\cdot\sqrt{3-2(x-1)^2}}{3}\right)\left(\frac{3-2(x-1)^2}{3} - \frac{2(x-1)^2}{3}\right)$$

$$= \frac{4}{9}\left(\sqrt{2}(x-1)\cdot\sqrt{3-2(x-1)^2}\right)\left(3-4(x-1)^2\right)$$

$$\int \quad = \frac{9}{16\sqrt{2}}\left(\theta - \frac{\sin 4\theta}{4}\right) + C$$

$$= \frac{9}{16\sqrt{2}}\left(\arcsin \frac{\sqrt{2}(x-1)}{\sqrt{3}} - \frac{1}{9}\left(\sqrt{2}(x-1)\sqrt{3-2(x-1)^2}\right)\left(3-4(x-1)^2\right)\right) + C$$

$$= \frac{9}{16\sqrt{2}} \arcsin \frac{\sqrt{2}(x-1)}{\sqrt{3}} - \frac{1}{16}(x-1)\left(\sqrt{3-2(x-1)^2}\right)\left(3-4(x-1)^2\right) + C$$

$$= \frac{9}{16\sqrt{2}} \arcsin \frac{\sqrt{2}(x-1)}{\sqrt{3}} + \sqrt{3-2(x-1)^2}\left(\frac{(x-1)^3}{4} - \frac{3(x-1)}{16}\right) + C$$

*9)

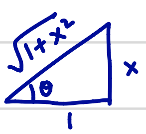
$$\int_0^{\sqrt{3}} \frac{1}{(x^2+1)^2}\,dx$$

Computing the indefinite integral first:

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta = \int \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$



$$= \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta = \int \frac{(1+\cos 2\theta)}{2}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

$$= \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C$$

Plugging in limits :

$$\int_0^{\sqrt{3}} = \frac{1}{2}\left(\arctan\sqrt{3} + \frac{\sqrt{3}}{1+3}\right) - \frac{1}{2}\left(\arctan 0 + 0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$$

10)* $$\int_0^{\pi/3} \frac{\tan t \sec t}{\sqrt{1+\sec^2 t}}\, dt$$

computing the indefinite integral first :

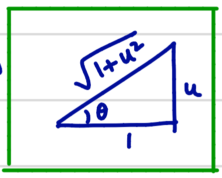
let $u = \sec t$, $du = \tan t \sec t\, dt$ —(1)

$$\int \frac{\tan t \sec t}{\sqrt{1+\sec^2 t}}\, dt = \int \frac{du}{\sqrt{1+u^2}}$$

let $u = \tan\theta$ $du = \sec^2\theta\, d\theta$

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$

$$= \ln\left|\sqrt{1+\sec^2 t} + \sec t\right| + C$$

Plugging in limits :

$$\int_0^{\pi/3} = \ln\left|\sqrt{1+\sec^2(\pi/3)} + \sec(\pi/3)\right| - \ln\left|\sqrt{1+\sec^2 0} + \sec 0\right|$$

$$= \ln\left|\sqrt{1+4} + 2\right| - \ln\left|\sqrt{1+1} + 1\right|$$

$$= \ln(\sqrt{5}+2) - \ln(\sqrt{2}+1)$$

$$= \ln\left(\frac{\sqrt{5}+2}{\sqrt{2}+1}\right)$$