1. $\int x \tan^{-1}x \, dx$

let $u = x\tan^{-1}x \Rightarrow du = \frac{1}{1+x^2}dx$

$dv = x\,dx$ and $v = \frac{x^2}{2}$

$\int u\,dv = uv - v\,du$

$\int x\tan^{-1}x\,dx = \frac{x^2}{2}.\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$

let $x = \tan\theta$, $1+x^2 = 1+\tan^2\theta = \sec^2\theta$, $dx = \sec^2\theta\,d\theta$

$$\int \frac{x^2}{1+x^2}dx = \int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\,d\theta = \int \tan^2\theta\,d\theta = \int(\sec^2\theta - 1)\,d\theta$$

$$= \int \sec^2\theta\,d\theta - \int d\theta$$

$$= \tan\theta - \theta + C$$

$$= x - \tan^{-1}x$$

$$\Rightarrow \int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + C$$

$$= \left(\frac{1+x^2}{2}\right)\tan^{-1}x - \frac{1}{2}x + C$$

2.* $\int \arcsin x\,dx$

let $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}$

$dv = dx$ and $v = x$

Using IBP,

$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$

let $1-x^2 = w \Rightarrow dw = -2x\,dx \Rightarrow x\,dx = \frac{1}{-2}dw$

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{dw}{\sqrt{w}} = -\frac{1}{2}\left(2w^{1/2}\right)$$

$$= -w^{1/2}$$

$$= -\sqrt{1-x^2}$$

$$\Rightarrow \int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + C$$

*3. $\int t^3 e^{t^2} dt$

Let $w = t^2 \Rightarrow dw = 2t\,dt \Rightarrow t\,dt = \frac{1}{2}dw$

$$\int t^3 e^{t^2} dt = \int t^2 e^{t^2}(t\,dt) = \frac{1}{2}\int w e^{w} dw$$

Let $u = w \Rightarrow du = dw$

$dv = e^{w}dw \quad , \quad v = e^{w}$

$$\int t^3 e^{t^2} dt = \frac{1}{2}\int w e^{w} dw \underset{\uparrow \text{IBP}}{=} \frac{1}{2}\left(we^{w} - \int e^{w} dw\right) = \frac{1}{2}(we^{w} - e^{w}) + C$$

$$= \frac{1}{2}(w-1)e^{w} + C$$

$$= \frac{1}{2}(t^2-1)e^{t^2} + C$$

*4. $\int \frac{\ln t}{t^2} dt$

Let $u = \ln t \Rightarrow du = \frac{1}{t}dt$

$dv = dt/t^2 \quad , \quad v = -1/t$

$$\int \frac{\ln t}{t^2} dt \underset{\uparrow \text{IBP}}{=} -\frac{\ln t}{t} - \int -\frac{1}{t}\cdot\frac{1}{t} dt$$

$$= -\frac{\ln t}{t} + \int \frac{1}{t^2} dt = -\frac{\ln t}{t} - \frac{1}{t} + C$$

$$= -\frac{1}{t}(1 + \ln t) + C$$

5. $\int e^{\sqrt{x}} dx$

Let $w = \sqrt{x} \quad , \quad dw = \frac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x}\,dw = 2w\,dw$

$$\Rightarrow \int e^{\sqrt{x}} dx = 2\int w e^{w} dw \underset{\uparrow \text{Problem 3}}{=} 2(w-1)e^{w} + C = 2(e^{\sqrt{x}} - 1)e^{\sqrt{x}} + C$$

6. $\displaystyle\int_0^{2\pi} e^{\cos t} \sin 2t \, dt$

I'll first compute the antiderivative of $e^{\cos t}\sin 2t$ and plug in the limits afterwards. I am choosing to do this so that I don't have to keep track of changes in limits as I do substitutions.

$$\int e^{\cos t} \sin 2t \, dt = \int e^{\cos t} \, 2 \sin t \cos t \, dt$$

$$= \int e^{w} \cdot 2w \, (-dw)$$

Let $w = \cos t$
$-dw = \sin t \, dt$

$$= -2 \int w e^{w} \, dw$$

$$= -2\left((w-1)e^{w} + C \right)$$

As seen in Problem 3

$$= -2\left((\cos t - 1) e^{\cos t} + C \right)$$

$$\Rightarrow \int_0^{2\pi} e^{\cos t} \sin 2t \, dt = -2\left((\cos t - 1) e^{\cos t} + C \right)\Big|_0^{2\pi}$$

$$= -2(\cos 2\pi - 1) e^{\cos 2\pi} + 2\left((\cos 0 - 1) e^{\cos 0}\right.$$

the constant C will cancel

$$= -2 \cdot 0 \cdot e + 2 \cdot 0 \cdot e = 0$$

7. $\displaystyle\int_{e^{1/2}}^{e} \frac{\arcsin(\ln x)}{x}\,dx$

Computing antiderivative first as before:

$$\int \frac{\arcsin(\ln x)}{x}\,dx \underset{\substack{\text{Let } w = \ln x \\ dw = \frac{1}{x}dx}}{=} \int (\arcsin w)\,dw$$

$$\underset{\substack{\text{Problem} \\ 2}}{=} w\arcsin w + \sqrt{1-w^2} + C$$

$$= (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C$$

$$\Rightarrow \int_{e^{1/2}}^{e} \frac{\arcsin(\ln x)}{x}\,dx = (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2}\,\Big]_{e^{1/2}}^{e}$$

$$= (\ln e)\arcsin(\ln e) + \sqrt{1-(\ln e)^2} - (\ln e^{1/2})\arcsin(\ln e^{1/2}) - \sqrt{1-(\ln e^{1/2})^2}$$

$$= \arcsin 1 + \sqrt{1-1} - \frac{1}{2}\arcsin\frac{1}{2} - \sqrt{1-\frac{1}{4}}$$

$$= \frac{\pi}{2} - \frac{1}{2}\frac{\pi}{6} - \frac{\sqrt{3}}{2} = \boxed{\frac{5\pi}{12} - \frac{\sqrt{3}}{2}}$$

8. $\displaystyle\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int (2\sin x\cos x)^2\,dx = \frac{1}{4}\int (\sin 2x)^2\,dx$

$$\underset{\text{double angle formula}}{=} \frac{1}{4}\int \frac{1-\cos(2(2x))}{2}\,dx$$

Recall:
$\cos 2y = 1 - 2\sin^2 y$
$= 2\cos^2 y - 1$
$\therefore \sin^2 y = \frac{1-\cos 2y}{2}$

$$= \frac{1}{8}\left(\int dx - \int \cos 4x\,dx\right)$$

Let $u = 4x$. $du = 4dx$, $dx = \frac{1}{4}du$

$\displaystyle\int \cos 4x\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u = \frac{1}{4}\sin 4x$

$$\therefore \int \sin^2 x\cos^2 x\,dx = \frac{1}{8}\left(\int dx - \int \cos 4x\,dx\right) = \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

9.* $\int \sin x \cos^2 x \, dx$

Let $u = \cos x \Rightarrow -du = \sin x \, dx$

$$\int \sin x \cos^2 x \, dx = \int u^2 (-du) = -\int u^2 du$$

$$= \frac{-u^3}{3} + C$$

$$= -\frac{\cos^3 x}{3} + C$$

10.* $\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \cdot \sec^2 x \, dx$

$$= \int \tan^2 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$

$$\int \tan^2 x \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx = \int u^2 du + \int u^4 du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

11. Show: $\int_0^\pi \sin^n x \, dx = \frac{n-1}{n} \int_0^\pi \sin^{n-2} x \, dx, \quad n \geq 2$

Let $u = \sin^n x \Rightarrow du = n(\sin^{n-1} x) \cos x \, dx$

$dv = dx \quad , \quad v = x$

$$\int_0^\pi \sin^n x \, dx = x \sin^n x \Big|_0^\pi - n \int_0^\pi x \cos x \cdot \sin^{n-1} x \, dx$$

$$= \pi(\sin \pi)^n - 0(\sin 0)^n - n \int_0^\pi x \cos x \cdot \sin^{n-1} x \, dx$$

$$= -n \int_0^\pi x \cos x \sin^{n-1} x \, dx$$

Now, $\int_0^{\pi} x\cos x \cdot \sin^{n-1}x\, dx$

Let $u = \sin^{n-1}x \qquad du = (n-1)\sin^{n-2}x\, \cos x\, dx$

$dv = x\cos x\, dx$;

$v \underset{\text{IBP}}{=} x\sin x - \int \sin x\, dx = x\sin x + \cos x$

IBP
$f = x \qquad f' = 1$
$g' = \cos x\, dx \qquad g = \sin x$

$$\int_0^{\pi} x\cos x \cdot \sin^{n-1}x\, dx = (x\sin x + \cos x)\sin^{n-1}x \Big|_0^{\pi} - \int_0^{\pi} (x\sin x + \cos x)(n-1)\sin^{n-2}x\, \cos x\, dx$$

$$\int_0^{\pi} x\cos x \cdot \sin^{n-1}x\, dx = 0 - (n-1)\int_0^{\pi} (x\sin x + \cos x)\sin^{n-2}x\, \cos x\, dx$$

$$= -(n-1)\int_0^{\pi} x\sin^{n-1}x\, \cos x\, dx \quad -(n-1)\int_0^{\pi} \sin^{n-2}x\, \cos^2 x\, dx$$

$$\Rightarrow \quad -n\int_0^{\pi} x\cos x \cdot \sin^{n-1}x\, dx = (n-1)\int_0^{\pi} \sin^{n-2}x\, \cos^2 x\, dx$$

$$= (n-1)\int_0^{\pi} \sin^{n-2}x\,(1-\sin^2 x)\, dx$$

$$= (n-1)\int_0^{\pi} \sin^{n-2}x\, dx - (n-1)\int_0^{\pi} \sin^n x\, dx$$

$$\Rightarrow \quad \int_0^{\pi} \sin^n x\, dx = (n-1)\int_0^{\pi} \sin^{n-2}x\, dx - (n-1)\int_0^{\pi} \sin^n x\, dx$$

$$\Rightarrow \quad n\int_0^{\pi} \sin^n x\, dx = (n-1)\int_0^{\pi} \sin^{n-2}x\, dx$$

$$\Rightarrow \quad \int_0^{\pi} \sin^n x\, dx = \frac{(n-1)}{n}\int_0^{\pi} \sin^{n-2}x\, dx \qquad ✓$$