Hwk 10 solutions

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n n^2}$$

we see that $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}/2^{n+1}(n+1)^2}{3^n/2^n n^2}\right| = \left|\frac{3}{2}\frac{n^2}{(n+1)^2}\right|$

so $\lim_{n\to\infty} \left|\frac{3}{2}\frac{n^2}{(n+1)^2}\right| = \frac{3}{2} > 1$, so the series diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

we see that $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!/10^{n+1}}{n!/10^n}\right| = \left|\frac{n+1}{10}\right|$,

so $\lim_{n\to\infty} \left|\frac{n+1}{10}\right| = \infty$, so the series diverges.

2. $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n(n+1)}\right)^n$: we have $|a_n|^{\frac{1}{n}} = \frac{n^2}{2^n(n+1)}$, so

$\lim_{n\to\infty} \frac{n^2}{2^n(n+1)} = 0$ and thus the series converges

$\sum_{n=1}^{\infty} \frac{(-2)^n+5^n}{n^n}$: we have $|a_n|^{\frac{1}{n}} = \frac{((-2)^n+5^n)^{\frac{1}{n}}}{n}$ and noticing that $\lim_{n\to\infty} \frac{(2\cdot 5^n)^{\frac{1}{n}}}{n} = \frac{2^{\frac{1}{n}}5}{n} = 0$ we see that $\lim |a_n|^{\frac{1}{n}} = 0$, so the series converges

$\sum_{n=1}^{\infty} (1-\frac{1}{n})^{n^2}$: we have $|a_n|^{\frac{1}{n}} = (1-\frac{1}{n})^n = \frac{1}{e}$, so

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 1/e$, so the series converges

3. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

we see that $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \frac{n^3}{(n+1)^3} = 1$, so

ratio test fails

$\sum_{n=1}^{\infty} \frac{n}{2^n}$

we see that $\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{n} \cdot \frac{1}{2} = \frac{1}{2}$,

So ratio test works in this case (the series converges)

$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

we see that $\lim_{n\to\infty} \frac{\left|\frac{(-3)^n}{\sqrt{n+1}}\right|}{\left|\frac{(-3)^{n-1}}{\sqrt{n}}\right|} = \frac{\frac{3^n}{\sqrt{n+1}}}{\frac{3^{n-1}}{\sqrt{n}}} = \frac{3^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^{n-1}} = \frac{3\sqrt{n}}{\sqrt{n+1}} = 3$,

so ratio test works (the series diverges)

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

we see that $\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{(n+1)^2+1}}{\frac{\sqrt{n}}{n^2+1}} = \frac{\sqrt{n+1}}{(n+1)^2+1} \frac{n^2+1}{\sqrt{n}} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{n^2+1}{n^2+2n+2} = 1 \cdot 1 = 1$,

so the ratio test fails (the series is indeterminate from the test alone)

4. $\displaystyle\sum_{n=1}^{\infty} n!\,(5x-4)^n$

we use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}, \text{ so } \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!\,(5x-4)^{n+1}}{n!\,(5x-4)^n} = n+1\,(5x-4)$$

So since $\lim\limits_{n\to\infty} (n+1)(5x-4) = \infty$ unless $x = 4/5$, we see that radius of convergence is 0 and interval of convergence is $\{4/5\}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$$

we use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}, \text{ so } \frac{|a_{n+1}|}{|a_n|} = \frac{(x-2)^{n+1}/3^{n+1}(n+1)^2}{(x-2)^n/3^n n^2} = \frac{n^2}{(n+1)^2} \cdot \frac{(x-2)}{3},$$

so $\lim\limits_{n\to\infty} \left|\frac{n^2}{(n+1)^2} \cdot \frac{(x-2)}{3}\right| = \left|\frac{x-2}{3}\right|$. Thus the series converges when $|x-2| < 3$. At $|x-2| = 3$ (when $x = 5, -1$),

we see that at $x=5$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

which converges

$x = -1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n (-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

which converges.

Thus the interval of convergence is $[-1, 5]$ and radius of convergence is 3.

5.

a) we see that for $f(x) = \frac{x}{x^2+2}$,

$$f(x) = \frac{x}{x^2+2} = \frac{\frac{x}{2}}{1+\frac{x^2}{2}} = \frac{\frac{x}{2}}{1-\left(-\frac{x^2}{2}\right)}$$, so the

power series representation is just

$$f(x) = \frac{x}{2} - \frac{x^3}{4} + \frac{x^5}{8} - \frac{x^7}{16} \cdots$$

For $f(x) = \frac{x+a}{x^2+a^2}$, we see that

$$f(x) = \frac{x+a}{x^2+a^2} = \frac{\frac{x+a}{a^2}}{\frac{x^2+a^2}{a^2}} = \frac{\frac{x+a}{a^2}}{1+\frac{x^2}{a^2}} = \frac{\frac{x+a}{a^2}}{1-\left(-\frac{x^2}{a^2}\right)}$$, so

the power series representation is just

$$\sum_{n=0}^{\infty} \frac{x+a}{a^2}\left(\frac{-x^2}{a^2}\right)^n$$

6.

$$f(x) = \ln(1-x^2)$$

we see that $f'(x) = \frac{-2x}{1-x^2}$, so

$$f'(x) = \cancel{\sum_{n=0}^{\infty}(-2x)(1-x^2)^n} \quad \sum_{n=0}^{\infty}(-2x)(x^2)^n$$

and thus

$$f(x) = \int -\sum_{n=0}^{\infty} 2x^{2n+1}\, dx = -\sum_{n=0}^{\infty}\frac{1}{n+1}x^{2n+2}$$

as desired. The ~~[illegible]~~ radius of convergence is

1 as $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$

$f(x) = x^2 \tan^{-1}(x^3)$

we note that since $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$, the power series expansion of $\arctan(x)$ is just

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Thus

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

and thus

$x^2 \tan^{-1}(x^3)$ has power series expansion

$$x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+5}}{2n+1}$$

Using the ratio test we see that

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(-1)^{n+1} x^{6n+11}}{2n+3}}{\frac{(-1)^n x^{6n+5}}{2n+1}} \right| = \left| x^6 \left(\frac{2n+1}{2n+3} \right) \right|$$

Since we want $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, we see that

$\lim_{n \to \infty} \left| x^6 \left(\frac{2n+1}{2n+3} \right) \right| = \lim_{n \to \infty} |x^6| \leq 1$, so $|x| < 1$ and

radius of convergence is 1

7. $\int \frac{t}{-t^4+1}\,dt$

we see that $\frac{t}{1-t^4}$ has power series $\sum_{n=0}^{\infty} t\,(t^4)^n$

and thus $\int \sum_{n=0}^{\infty} t^{4n+1}\,dt = \underline{\sum_{n=0}^{\infty} \frac{1}{4n+2} t^{4n+2}}$

The radius of convergence of $\sum_{n=0}^{\infty} t^{4n+1}$ is 1, so

the radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{4n+2} t^{4n+2}$ is 1

$\int \frac{\tan^{-1} t}{t}\,dt$

we see that arctan(t)'s power series expansion is

$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{2n+1}$, so the power series expansion of

$\arctan(t)/t$ is just $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n+1}$, and thus

$\int \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n+1}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)^2}$ as desired

The radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n+1}$ is just 1, so

the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)^2}$ is just 1