Math 109 HWK 10
solutions

1. $$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$$

Because $\frac{1}{n^2 \ln(n)} \le \frac{1}{n^2}$ for $n \ge 3$ (since $\ln(n) > 1$ for $n \ge 3$), and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we see that $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ $\boxed{\text{converges}}$

by the series comparison test.

2. $$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^4}$$

we note that

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2}$$ and using L-Hopital we see that

$$\lim_{n\to\infty} \frac{\ln(n)}{n^2} = \lim_{n\to\infty} \frac{d/dn\,(\ln(n))}{d/dn\,(n^2)} = \frac{\frac{1}{n}}{2n} = \frac{1}{2n^2} = 0$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we see that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^4}$ $\boxed{\text{converges}}$

as well by the limit comparison test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n+4}$

We note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+4}$ converges because

$\displaystyle \frac{1}{3^n+4} \leq \frac{1}{3^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, so

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+4}$ converges by series comparison test.

Because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+4}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n+4}$ $\boxed{\text{converges}}$ by

the absolute convergence test.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-3\ln(n)}$

We see that

$$(n+1)^2 - 3\ln(n+1) \geq n^2 - 3\ln(n) \quad \text{because}$$

$$\Updownarrow$$

$$n^2+2n+1 - n^2 \geq 3\ln(n+1) - \ln(n)$$

$$\Updownarrow$$

$$2n+1 \geq 3\ln\left(\frac{n+1}{n}\right)$$

$$\Updownarrow$$

$$2n+1 \geq 3\ln\left(1+\frac{1}{n}\right) \quad \text{which is true for all } n \geq 1.$$

Thus $\displaystyle \frac{1}{(n+1)^2-3\ln(n+1)} \leq \frac{1}{n^2-3\ln(n)}$ for all $n \geq 1$.

Also, $\displaystyle \lim_{n\to\infty} \frac{1}{n^2-3\ln(n)} = 0$, and thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-3\ln(n)}$

$\boxed{\text{converges}}$ by the alternating series test.

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln(n)}$

we note that

$$\sqrt{n+1}\ln(n+1) \geq \sqrt{n}\ln(n) \quad \text{as}$$

$$\Updownarrow$$

$$\frac{\sqrt{n+1}}{\sqrt{n}} \geq \frac{\ln(n)}{\ln(n+1)}$$

which is true as $\dfrac{\sqrt{n+1}}{\sqrt{n}} \geq 1$ while $\dfrac{\ln(n)}{\ln(n+1)} \leq 1$ for all $n \geq 2$

Thus $\dfrac{1}{\sqrt{n+1}\ln(n+1)} \leq \dfrac{1}{\sqrt{n}\ln(n)}$ for all $n \geq 2$

Since $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}\ln(n)} = 0$, we see that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln(n)}$ $\boxed{\text{converges}}$ by the alternating series test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{n^2-2\sqrt{n}}$

we note that $\{a_n\} \to 0$ if and only if $\{|a_n|\} \to 0$

Taking the contrapositive, we see that $\displaystyle\lim_{n\to\infty} |a_n| \neq 0 \Leftrightarrow \lim_{n\to\infty} a_n \neq 0$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^n(n^2+1)}{n^2-2\sqrt{n}}\right| = \lim_{n\to\infty} \frac{n^2+1}{n^2-2\sqrt{n}} = 1 \neq 0$, we see that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n(n^2+1)}{n^2-2\sqrt{n}} \neq 0$, and thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{n^2-2\sqrt{n}}$ $\boxed{\text{diverges}}$ due to divergence test.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^4-1}$

We see that $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ converges by the limit comparison test.

Indeed

$$\lim_{n\to\infty} \frac{\frac{n}{n^4-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^4-1} = 0 \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ converges and}$$

thus $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ converges.

Since $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ converges, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4-1}$ converges by the absolute convergence test.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{3\sqrt{n}}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{3\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n}}$ as $\cos(n\pi) = (-1)^n$ for all positive integers n.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n}}$ converges since $\dfrac{1}{3\sqrt{n+1}} \le \dfrac{1}{3\sqrt{n}}$ and $\displaystyle\lim_{n\to\infty} \frac{1}{3\sqrt{n}} = 0$

by the alternating series test

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}}$ diverges by the p-test for series.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{3\sqrt{n}}$ is $\boxed{\text{conditionally convergent}}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+n}$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+n}$ converges since $\dfrac{1}{n^2+n} \leq \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges.

Since $\left| \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+n} \right|$ converges, the series is $\boxed{\text{absolutely convergent}}$.