Linear equations

A first order linear differential equation is

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We call it linear, because $\frac{dy}{dx} + P(x)y$ is linear in y.
Namely, if $y_1(x)$, $y_2(x)$ are both solutions to $\frac{dy}{dx} + P(x)y = 0$, then $ay_1(x) + by_2(x)$ is also a solution, for any constants a, b.

Linear equations

- Example 1. $xy' + y = 2x$. $\Rightarrow y' + \frac{y}{x} = 2$.
- Example 2. When P, Q are constants.

$$\frac{dy}{dx} + Py = Q$$

can be written as

$$\frac{dy}{Q - Py} = dx.$$

Linear equations

- But when P, Q are not constants, we need a different strategy to solve linear equations.

  Example 3.

  $$\frac{dy}{dx} + y = x$$

  We multiply both sides by e^x. Then

  $$e^x \frac{dy}{dx} + ye^x = xe^x.$$

  Then

  $$(e^x \cdot y(x))' = xe^x.$$

Linear equations

$$\Rightarrow e^{x} \cdot y(x) = \int xe^{x} dx.$$

$$\Rightarrow y(x) = e^{-x} \int xe^{x} dx.$$

Use integration by parts, we get the right hand side equals

$$y(x) = e^{-x}(xe^{x} - e^{x} + C) = x - 1 + Ce^{-x}.$$

Linear equations

This indicates a general method to deal with linear equations.

- Step 0. Before applying the following steps, we need to change the equation into this standard form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Step 1. Write down factor $I(x) := e^{\int P(x)dx}$ (we call it integrating factor), and multiply it on both sides so that

$$LHS = I(x)(\frac{dy}{dx} + P(x)y) = (I(x)y(x))'$$

This is because $I(x)P(x) = I'(x)$

- Step 2. We solve the equation

$$(I(x)y(x))' = I(x)Q(x)$$

by taking integration on both sides.

Linear equations

- Step 3. We get

$$I(x)y(x) = \int I(x)Q(x)dx.$$

Thus

$$y(x) = \frac{1}{I(x)} \int I(x)Q(x)dx$$

is the general solution.

Linear equations

- Example 4.

$$y' + 2xy = 1.$$

Solution: $P(x) = 2x, Q(x) = 1$. We take $I(x) = e^{\int P(x)dx} = e^{x^2}$. Thus

$$(e^{x^2}y)' = e^{x^2}.$$

Integrate both sides, we get

$$e^{x^2}y(x) = \int e^{x^2}dx + C.$$

$$\Rightarrow y(x) = e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}.$$

- Example 5. $2xy' + y = 2x$. (Integrating factor is $\frac{1}{\sqrt{x}}$.)