9.3: Separable Equations

An equation is separable if one can move terms so that each side of the equation only contains 1 variable. Consider the 1st order equation

$$\frac{dy}{dx} = F(x, y).$$

When $F(x, y) = f(x)g(y)$, this differential equation is separable. We have a strategy called separation of variables to solve this type of equations.

- Example 1. $y' = x(y-1)$

Solution: Rewrite the equation as

$$\frac{dy}{dx} = x(y-1).$$

$y = 1$ is a solution. Suppose $y \neq 1$, then we separate the variables:

$$\frac{dy}{y-1} = xdx.$$

Example 1

Integrate both sides,

$$\int \frac{dy}{y-1} = \int x dx.$$

$$\Rightarrow \ln|y-1| + C_1 = \frac{x^2}{2} + C_2.$$

$$\Rightarrow \ln|y-1| = \frac{x^2}{2} + C_3.$$

Thus $|y-1| = e^{C_3} \cdot e^{\frac{x^2}{2}}$.

$$\Rightarrow y = 1 \pm e^{C_3} e^{\frac{x^2}{2}}, \quad \text{for all constants} \quad C_3.$$

$$\Rightarrow y = 1 + C_4 e^{\frac{x^2}{2}}, \quad \text{for all constants} \quad C_4.$$

Example 2

Example 2. Find all the solutions to

$$y' = 2x(1-y)^2.$$

Solution: First note that there is a constant solution $y \equiv 1$.

Next we use separation method as above

$$\Rightarrow \frac{dy}{(1-y)^2} = 2xdx.$$

$$\Rightarrow \frac{1}{1-y} = x^2 + C.$$

$$\Rightarrow y = 1 - \frac{1}{x^2 + C}.$$

Example 2

Note that the special solution $y \equiv 1$ is "lost" from the general one. The problem comes in the separation step, as $\frac{dy}{(1-y)^2}$ is valid only if $y \neq 1$. In general,

$$y' = f(x)g(y).$$

We apply separation of variables to get

$$\frac{dy}{g(y)} = f(x)dx.$$

All the values of y s.t. $g(y) = 0$ give rise to a "lost" solutions.

Population growth/decay

Example 3. $y' = ky$ (for a constant k) can be solved by separation of variables method.

$$\frac{dy}{y} = kdt.$$

Integrate both sides,

$$\ln|y| = kt + C_1.$$

$\Rightarrow y = \pm e^{C_1} e^{kt}$ for all constant C_1.

$\Rightarrow y = C_2 e^{kt}$ for all constant $C_2 \neq 0$.

Population growth/decay

There is a lost solution in the separation of variable step, which is $y \equiv 0$. When $C_2 = 0$, we recover the lost solution. So the general solutions are

$$y = C_2 e^{kt}$$

for all constant C_2.

Example 4

- Example 4. Heat diffusion. A body at temperature T sits in an environment of temperature T_E. Newton's law of cooling models the rate of change in temperature by

$$T' = -k(T - T_E)$$

where $k > 0$ is a constant.

Example 4

- $$T' = -k(T - T_E)$$

  Solution:

  $$\begin{aligned}
  \frac{dT}{dt} &= -k(T - T_E) \\
  \Rightarrow \frac{dT}{T - T_E} &= -kdt \\
  \Rightarrow \ln|T - T_E| &= -kt + C_1 \\
  \Rightarrow T - T_E &= \pm e^{C_1} \cdot e^{-kt} \\
  \Rightarrow T &= T_E + C_2 e^{-kt}.
  \end{aligned}$$

Example 5

Newton's law of motion, constant gravity.

$$\frac{d^2y}{dt^2} = -g.$$

where y is the height of the body and g is the acceleration rate due to gravity, $9.8m/sec^2$

$$y = -\frac{1}{2}g \cdot t^2 + C_1 t + C_2.$$

Example 6 Orthogonal trajectories.

Take a family of curves $x = ky^2$, where k is any constant. Find another family of curves such that any member of this family intersects any given one at a right angle.

$$x = ky^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{2ky} = \frac{y}{2x}.$$

The orthogonal trajectories satisfy the differential equation:

$$\frac{dy}{dx} = -\frac{2x}{y}.$$
$$\Rightarrow \int y dy = -\int 2x dx.$$
$$\Rightarrow x^2 + \frac{y^2}{2} = C, \quad C > 0.$$

Population growth/decay

We have talked about the population model: $P'(t) = kP$ with $P(0) = 10^5$. can be solved by separation of variables method.

Solution: Case 1: $P(t) = 0$ is a solution.

Case 2: Suppose $P(t) \neq 0$, we can divide both sides of the equation by $P(t)$.

$$\Rightarrow \frac{dP}{P} = kdt.$$

Integrate both sides,

$$\ln|P| = kt + C_1.$$

Population growth/decay

$\Rightarrow P(t) = \pm e^{C_1} e^{kt}$ for all constant C_1.

$\Rightarrow P(t) = C_2 e^{kt}$ for all constant C_2.

Since $P(0) = 10^5$, $C_2 = 10^5$.

Thus $P(t) = 10^5 e^{kt}$.

Logistic model of population

Modified model:

If P is small, $\frac{dP}{dt} = kP$.

If $P > M$, $\frac{dP}{dt} < 0$.

A simple modification would be

$$\frac{dP}{dt} = kP(t)(1 - \frac{P(t)}{M}).$$

Logistic model of population

Separation of variable method:

$$\int \frac{dP}{P(1-P/M)} = \int k dt.$$

Since

$$\frac{1}{P(1-P/M)} = \frac{1}{P} + \frac{1}{M-P}.$$

Thus

$$\ln|P| - \ln|M-P| = kt + C.$$

Logistic model of population

$$\frac{M-P}{P} = Ae^{-kt}.$$

Thus

$$P(t) = \frac{M}{1+Ae^{-kt}}$$

where $A = \frac{M-P_0}{P_0}$.

Logistic model of population

- $\lim_{t \to \infty} P(t) = M$.
- Graph of $P(t)$.
- Comparison with the natural growth model.