# Population of an animal

A mathematical model is an equation or a set of equations that mimic the behavior of some phenomenon under certain assumptions in nature. If the phenomena involve rates/ changes, they can be likely modeled with differential equations.

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Assumption: The population grows at a rate proportional to the size of the population. (We assume to have unlimited environment, adequate nutrition, absence of predators, immunity from disease.)

- $t =$ time
- $P(t) =$ the population at time $t$
- $\frac{dP}{dt} =$ rate of growth

The population model is given by

$$\frac{dP}{dt} = kP.$$

This is a separable equation.

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Separation of variable method: Suppose $P(t) \neq 0$, (we will discuss the case $P(t) = 0$ later) then

$$\frac{dP}{P} = kdt$$

Integrate both sides, we get

$$\ln|P(t)| = C + kt.$$

Thus

$$P(t) = \pm e^C e^{kt}$$

Since $C$ is an arbitrary constant, $\tilde{C} := \pm e^C$ is an arbitrary nonzero constant. On the other hand, we still need to know what happens to the case $P(t) = 0$. Notice $P(t) = 0$ is a solution, as we can plug it into the differential equation to see it satisfies the equation.)

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- In conclusion, the general solution is $P(t) = \tilde{C}e^{kt}$ for an arbitrary constant $\tilde{C}$.
  For example, $k = 2$, then $P(t) = \tilde{C}e^{2t}$.
- If in addition we have initial value condition, say $P(2008) = 10^10$, then we can determine the value of $\tilde{C}$.

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In nature, making assumptions that are the most suitable for the reality is the key to understand the problem.

If $P$ is small, $\frac{dP}{dt} = kP$.

If $P > M$, $\frac{dP}{dt} < 0$.

A simple modification would be

$$\frac{dP}{dt} = kP(t)(1 - \frac{P(t)}{M}).$$

Solve this equation using separation of variable method after the class. (Do not lose any solution!)

# 9.2: Direction fields

For $y' = F(x, y)$, there is a direction vector with slope $F(x_0, y_0)$ at each point $(x_0, y_0)$. Without solving the equation, we can draw a rough picture of the integral curves (graph of solutions).

- Example $F(x, y) = x^2 + y^2 - 1$. Take $x_0 = 1, y_0 = 2$, then $F(x_0, y_0) = 4$. This tells us that if an integral curve for the equation $y' = F(x, y)$ exists, passing through $(1, 2)$, then the curve has slope 4 at $(1, 2)$.

# 9.2: Direction fields

■ Example $y' = x^2 + y^2 - 1$.

▶ Draw a vector (line segment) with slope $F(x, y)$ at many points $(x, y)$. Then connecting these line segments to get connecting curves. They are approximation of the actual integral curves (graph of solutions).

▶ The denser the chosen points are distributed on the plane, the closer the connecting curves are to the actual integral curves.

# 9.2: Direction fields

- This method gives numerical approximation of the integral curves (without knowing how to solve the equation!). It allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.
- It is easy to draw integral curves by connecting direction vectors. Actual solutions may sometimes be too hard to find.