Initial value problem

- If besides the differential equation, the value of y at a specific point, say x_0 is given: $y(x_0) = y_0$ is given, we can determine the value of C. For example, if $y(0) = 3$, then

$$y(0) = C + 1 = 3$$

implies that $C = 2$.

Initial value problem

We call the condition $y(x_0) = y_0$ an initial value condition.

And we call a differential equation with a condition $y(x_0) = y_0$ an initial value problem.

Geometric meaning of general solution

Each solution is a graph (curve) on the xy-plane.

The set of general solution is a family of curves on the xy-plane.

See the picture.

Geometric meaning of general solution

Geometrically, the initial condition has the effect of isolating the integral curve that passes through the point (x_0, y_0).

9.3: Separable Equations

An equation is separable if one can move terms so that each side of the equation only contains 1 variable. Consider the 1st order equation

$$\frac{dy}{dx} = F(x, y).$$

When $F(x, y) = f(x)g(y)$, this differential equation is separable. We have a strategy called separation of variables to solve this type of equations.

- Example 1. $y' = x(y - 1)$

  Solution: Rewrite the equation as

$$\frac{dy}{dx} = x(y - 1).$$

$y = 1$ is a solution. Suppose $y \neq 1$, then we separate the variables:

$$\frac{dy}{y - 1} = xdx.$$

Example 1

Integrate both sides,

$$\int \frac{dy}{y-1} = \int x dx.$$

$$\Rightarrow \ln|y-1| + C_1 = \frac{x^2}{2} + C_2.$$

$$\Rightarrow \ln|y-1| = \frac{x^2}{2} + C_3.$$

Thus $|y-1| = e^{C_3} \cdot e^{\frac{x^2}{2}}$.

$$\Rightarrow y = 1 \pm e^{C_3} e^{\frac{x^2}{2}}, \quad \text{for all constants} \quad C_3.$$

$$\Rightarrow y = 1 + C_4 e^{\frac{x^2}{2}}, \quad \text{for all constants} \quad C_4.$$

Example 2

Example 2. Find all the solutions to

$$y' = 2x(1-y)^2.$$

Solution: First note that there is a constant solution $y \equiv 1$.

Next we use separation method as above

$$\Rightarrow \frac{dy}{(1-y)^2} = 2xdx.$$

$$\Rightarrow \frac{1}{1-y} = x^2 + C.$$

$$\Rightarrow y = 1 - \frac{1}{x^2 + C}.$$

Example 2

Note that the special solution $y \equiv 1$ is "lost" from the general one. The problem comes in the separation step, as $\frac{dy}{(1-y)^2}$ is valid only if $y \neq 1$. In general,

$$y' = f(x)g(y).$$

We apply separation of variables to get

$$\frac{dy}{g(y)} = f(x)dx.$$

All the values of y s.t. $g(y) = 0$ give rise to a "lost" solutions.

Population growth/decay

Example 3. $y' = ky$ (for a constant k) can be solved by separation of variables method.

$$\frac{dy}{y} = kdt.$$

Integrate both sides,

$$\ln|y| = kt + C_1.$$

$\Rightarrow y = \pm e^{C_1} e^{kt}$ for all constant C_1.

$\Rightarrow y = C_2 e^{kt}$ for all constant $C_2 \neq 0$.

Population growth/decay

There is a lost solution in the separation of variable step, which is $y \equiv 0$. When $C_2 = 0$, we recover the lost solution. So the general solutions are

$$y = C_2 e^{kt}$$

for all constant C_2.

Example 4

- Example 4. Heat diffusion. A body at temperature T sits in an environment of temperature T_E. Newton's law of cooling models the rate of change in temperature by

$$T' = -k(T - T_E)$$

where $k > 0$ is a constant.

Example 4

- $$T' = -k(T - T_E)$$

  Solution:

  $$\frac{dT}{dt} = -k(T - T_E)$$

  $$\Rightarrow \frac{dT}{T - T_E} = -kdt$$

  $$\Rightarrow \ln|T - T_E| = -kt + C_1$$

  $$\Rightarrow T - T_E = \pm e^{C_1} \cdot e^{-kt}$$

  $$\Rightarrow T = T_E + C_2 e^{-kt}.$$