7.4: Integration of rational functions

Integrate each term of Case 1 and 2:

$$\int \frac{A}{ax+b}dx = \frac{A}{a}\ln|ax+b| + C.$$

$$\begin{aligned}&\int \frac{A_1}{ax+b} + \cdots + \frac{A_r}{(ax+b)^r}dx \\ =&\frac{A_1}{a}\ln|ax+b| + \frac{A_2}{a}\frac{(ax+b)^{-1}}{-1} + \cdots + \frac{A_r}{a}\frac{(ax+b)^{-r+1}}{-r+1} + C.\end{aligned} \tag{51}$$

7.4: Integration of rational functions

Integrate each term in Case 3: The terms in Case 3 take the form

$$\int \frac{Ax+B}{ax^2+bx+c}dx$$

We need to complete the square of denominator first, and the break the numerator into two pieces:

- Example 11. $\int \frac{3x+1}{x^2+2x+3}dx$

$$\frac{3x+6}{x^2+2x+3} = \frac{3x+6}{(x+1)^2+2} = \frac{3(x+1)-3+6}{(x+1)^2+2}.$$

We break the integral into two pieces: $\int \frac{3(x+1)}{(x+1)^2+2}dx$ and $\int \frac{3}{(x+1)^2+2}dx$.

7.4: Integration of rational functions

By substitution $u = (x+1)^2$, $du = 2(x+1)dx$,

$$\begin{aligned}\int \frac{3(x+1)}{(x+1)^2+2}dx =& \frac{3}{2}\int \frac{du}{u+2} \\ =& \frac{3}{2}\ln|u+2| + C = \frac{3}{2}\ln|(x+1)^2+2| + C.\end{aligned} \tag{52}$$

$$\int \frac{3}{(x+1)^2+2}dx = \int \frac{3}{2[(\frac{x+1}{\sqrt{2}})^2+1]}dx \tag{53}$$

Substitute $u = \frac{x+1}{\sqrt{2}}$, and then $du = \frac{1}{\sqrt{2}}dx$,

$$\begin{aligned}\int \frac{3}{2[(\frac{x+1}{\sqrt{2}})^2+1]}dx =& \frac{3}{2}\sqrt{2}\int \frac{1}{u^2+1}du \\ =& \frac{3}{2}\sqrt{2}\arctan u + C = \frac{3}{2}\sqrt{2}\arctan(\frac{x+1}{\sqrt{2}}) + C.\end{aligned} \tag{54}$$

7.4: Integration of rational functions

Integrate each term in Case 4: The terms in Case 4 take the form

$$\int \frac{Ax + B}{(ax^2 + bx + c)^r} dx$$

The method is similar. We first complete the square in the denominator, and break the numerator into two pieces according to that. Then the first integral will be an integration of a power function (with a negative power), which is easy to integrate. The second integral needs to use the trig substitution: $x = \tan\theta$.

For example, we have showed how to do $\int \frac{2x+3}{[(x+1)^2+1]^2}$ in the class.

AS.110.109: Calculus II (Eng)

Chapter 9: Differential Equations

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Differential equations

- The equation involves derivatives.
- Solution is a function $y = y(x)$.
- If the highest order of derivative appears in the equation is the n-th order, we say such a differential equation an n-th order differential equation.
  - Example 1(a). $\frac{dy}{dx} = y + e^{2x}$ is a 1st order differential equation.
  - Example 1(b). $\frac{d^2y}{dx^2} = \sqrt{x^2+1}$ is a 2nd order differential equation.

Example

What are solutions in Example 1(a) $\frac{dy}{dx} = y + e^{2x}$?

- $y(x) = e^{2x}$ is a solution on the interval $(-\infty, \infty)$.
- There are many more solutions: $y(x) = Ce^{x} + e^{2x}$ for all

constant C are solutions.

Example

How to check $y(x) = Ce^x + e^{2x}$ is a solution?

We compute the derivative of y.

$$\frac{dy}{dx} = Ce^x + 2e^{2x} = (Ce^x + e^{2x}) + e^{2x} = y(x) + e^{2x}. \tag{1}$$

Yes, for all constants C, $y(x) = Ce^x + e^{2x}$ is a solution. The first solution $y(x) = e^{2x}$ is when taking $C = 0$.

Example

- Example 2. Show that $y = x + 1 + Ce^x$ is a solution to

$y' = y - x$.

Plug $y = x + 1 + Ce^x$ into the left hand side (LHS) of the equation $y'(x) = 1 + 0 + Ce^x$. And the right hand side (RHS) of the equation is $y - x = x + 1 + Ce^x - x$. Thus they are equal.

General Solution

Definition

The set of all solutions is called general solution. One single solution is called a (special) solution.

In the previous example, $y(x) = e^{2x}$ is a (special) solution, while $y(x) = Ce^{x} + e^{2x}$ is the general solution to this differential equation.