7.4: Integration of rational functions

Case 2. $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose $Q(x)$ contains a term $(a_1x + b_1)^{r_1}$. Instead of $\frac{A_1}{a_1x+b_1}$, we would have

$$\frac{A_1}{a_1x + b_1} + \cdots + \frac{A_{r_1}}{(a_1x + b_1)^{r_1}}.$$

For another repeated factor, say $(a_2x + b_2)^{r_2}$, we have

$$\frac{B_1}{a_2x + b_2} + \cdots + \frac{B_{r_2}}{(a_2x + b_2)^{r_2}}.$$

7.4: Integration of rational functions

Example 8. $\int \frac{x}{(x-1)^2(x+1)} dx$

$$\frac{x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Then

$$\begin{aligned} x =& A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ =& (C+A)x^2 + (B-2C)x + (-A+B+C). \end{aligned} \tag{46}$$

Thus $C = -\frac{1}{4}, A = \frac{1}{4}, B = \frac{1}{2}$.

$$\begin{aligned} &\int \frac{x}{(x-1)^2(x+1)} dx \\ =& \int \frac{x}{(x-1)^2(x+1)} = \int \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)} dx \\ =& \frac{1}{4} \ln|x-1| - \frac{1}{2(x-1)} - \frac{1}{4} \ln|x+1| + C. \end{aligned} \tag{47}$$

7.4: Integration of rational functions

Case 3. $Q(x)$ has irreducible quadratic factors $(a_1x^2+b_1x+c_1)\cdots(a_mx^2+b_mx+c_m)$, none of which is repeated. Then in addition to those terms of Case 1 and 2, $\frac{R(x)}{Q(x)}$ also contains $\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}+\cdots+\frac{A_mx+B_m}{a_mx^2+b_mx+c_m}$ in the decomposition.

- Example 9. $\int \frac{x^2}{(x+1)(x^2+1)}dx$

$$\frac{x^2}{(x+1)(x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}.$$

Thus

$$(A+C)x^2+(B+A)x+B+C=x^2.$$

7.4: Integration of rational functions

Then $A = \frac{1}{2}, B = -\frac{1}{2}, C = \frac{1}{2}$.

$$
\begin{aligned}
&\int \frac{x^2}{(x+1)(x^2+1)} dx \\
=&\int \frac{x-1}{2(x^2+1)} + \frac{1}{2(x+1)} dx \\
=&\int \frac{x}{2(x^2+1)} dx - \int \frac{1}{2(x^2+1)} dx + \frac{1}{2} \ln|2(x+1)|. \\
=&\frac{1}{4} \ln|x^2+1| - \frac{1}{2} \arctan x + \frac{1}{2} \ln|2(x+1)| + C.
\end{aligned}
\tag{48}
$$

7.4: Integration of rational functions

- Example 10. $\int \frac{x^2}{x^2+2x+3} dx$ By long division,

$$\frac{x^2}{x^2+2x+3} = 1 + \frac{-2x-3}{x^2+2x+3} = 1 + \frac{-2x-3}{(x+1)^2+2}.$$

Thus

$$\begin{aligned}
&\int \frac{x^2}{x^2+2x+3} dx \\
=&\int 1 + \frac{-2x-3}{(x+1)^2+2} dx \\
=&\int 1 - \frac{2(x+1)}{(x+1)^2+2} - \frac{1}{(x+1)^2+2} dx \\
=&x - \ln\left|(x+1)^2+2\right| - \frac{\sqrt{2}}{2} \arctan(\frac{x+1}{\sqrt{2}}) + C.
\end{aligned} \tag{49}$$

7.4: Integration of rational functions

Case 4. $Q(x)$ has a repeated irreducible quadratic factor $(ax^2+bx+c)^r$. Then in the decomposition, there are following terms in $\frac{R(x)}{Q(x)}$: (of course, there are other terms corresponding to other factors of Q.)

$$\frac{A_1 x + B_1}{ax^2+bx+c} + \cdots + \frac{A_r x + B_r}{(ax^2+bx+c)^r}. \tag{50}$$