7.4: Integration of rational functions

A rational function is a function of the form:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials in x.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

$$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0.$$

How to express a rational function as a sum of partial fractions?

7.4: Integration of rational functions

First of all, we only want to discuss integration of proper rational functions.

Definition

$f(x) = \frac{P(x)}{Q(x)}$ is called proper if $deg(P(x)) < deg(Q(x))$.
Otherwise, it is called improper.

Using long division, a rational function can always be written as:

$$f(x) = \text{a polynomial function} + \text{a proper rational function.}$$

7.4: Integration of rational functions

Why?

If $f(x)$ is improper, then we perform long division to divide $P(x)$ by $Q(x)$, and get

$$P(x) = S(x)Q(x) + R(x)$$

where $S(x)$ and $R(x)$ are polynomials, satisfying $deg(R(x)) < deg(Q(x))$. Then

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}.$$

with $\frac{R(x)}{Q(x)}$ being proper.

7.4: Integration of rational functions

- Example 2. How to write $\frac{x^3}{x-2}$ as a sum of a polynomial and a proper rational function?

  By long division, (Review long division if you are not sure how to do it!!!)

$$x^3 = (x-2)(x^2+2x+4)+8. \tag{38}$$

$$\frac{x^3}{x-2} = (x^2+2x+4)+\frac{8}{x-2}. \tag{39}$$

7.4: Integration of rational functions

- Example 3. How to write $\frac{x^3+1}{x^2+x+2}$ as a sum of a polynomial and a proper rational function?

  By long division,

$$x^3 + 1 = (x^2 + x + 2)(x - 1) + (-x + 3). \tag{40}$$

$$\frac{x^3 + 1}{x^2 + x + 2} = (x - 1) + \frac{-x + 3}{x^2 + x + 2}. \tag{41}$$

  Thus it suffices to only discuss integration of proper rational functions.

7.4: Integration of rational functions

Some more theory about polynomial.

Definition

A polynomial $P(x)$ is irreducible if it cannot be written into a product of two polynomials of lower degrees. Otherwise, it is reducible.

7.4: Integration of rational functions

- All degree 1 polynomials are irreducible. Example. $x + 3$, $2x + 4$, etc.
- Some degree 2 polynomials are irreducible, but not all of them.
  Example 4. $x^2 - 4 = (x - 2)(x + 2)$, so it is reducible.
  Example 5. $x^2 + 4$ is irreducible.

7.4: Integration of rational functions

How to determine if a degree 2 polynomial is irreducible or reducible?

For

$$p(x) = ax^2 + bx + c,$$

- if the discriminant $\Delta := b^2 - 4ac \geq 0$, then reducible;
- if the discriminant $\Delta := b^2 - 4ac < 0$, then irreducible.

7.4: Integration of rational functions

If $P(x)$ has degree≥ 3, it can always be factored into a product:

$$P(x) = P_1(x) \cdots P_k(x),$$

where each $P_i(x)$ is a irreducible polynomial with degree ≤ 2.

P_i is either of degree 1, or of degree 2 and $\Delta < 0$.

7.4: Integration of rational functions

To integrate a rational function, we first need to factorize the denominator $Q(x)$.

$$Q(x) = Q_1(x) \cdots Q_k(x).$$

Case 1. $Q(x)$ is a product of distinct linear factors.
$\frac{R(x)}{Q(x)}$ takes the form

$$\frac{A_1}{a_1 x + b_1} + \cdots + \frac{A_k}{a_k x + b_k}$$

.

7.4: Integration of rational functions

Example 6.

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx.$$

Solution: Factorize the denominator

$$2x^3+3x^2-2x = x(2x-1)(x+2).$$

Claim: We can express

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

for some real numbers A, B, C.

7.4: Integration of rational functions

$$\begin{aligned} RHS =& \frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{x(2x-1)(x+2)} \\ =& \frac{(2A+B+2C)x^2+(3A+2B-C)x-2A}{x(2x-1)(x+2)}. \end{aligned} \tag{42}$$

Thus

$$\begin{cases} 2A+B+2C =& 1 \\ 3A+2B-C =& 2 \\ -2A =& -1 \end{cases} \tag{43}$$

$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

7.4: Integration of rational functions

Thus

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx \\
=&\int \frac{1}{2x}+\frac{1}{5(2x-1)}-\frac{1}{10(x+2)}dx \\
=&\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+C.
\end{aligned} \tag{44}$$

7.4: Integration of rational functions

- Example 7. $\int \frac{1}{x^2-3} dx$

$$\frac{1}{x^2-3} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}}.$$

Then $1 = A(x+\sqrt{3}) + B(x-\sqrt{3})$ Thus $A+B=0$ and $\sqrt{3}A - \sqrt{3}B = 1$ We solve the equations to get $A = \frac{1}{2\sqrt{3}}$ and $B = -\frac{1}{2\sqrt{3}}$.

$$\begin{aligned} &\int \frac{1}{x^2-3} dx \\ =&\int \frac{1}{2\sqrt{3}}\left(\frac{1}{x-\sqrt{3}} - \frac{1}{x+\sqrt{3}}\right) dx \qquad (45) \\ =&\frac{1}{2\sqrt{3}}(\ln|x-\sqrt{3}| - \ln|x+\sqrt{3}|) + C. \end{aligned}$$

7.4: Integration of rational functions

Case 2. $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose $Q(x)$ contains a term $(a_1x+b_1)^{r_1}$. Instead of $\frac{A_1}{a_1x+b_1}$, we would have

$$\frac{A_1}{a_1x+b_1}+\cdots+\frac{A_{r_1}}{(a_1x+b_1)^{r_1}}.$$

For another repeated factor, say $(a_2x+b_2)^{r_2}$, we have

$$\frac{B_1}{a_2x+b_2}+\cdots+\frac{B_{r_2}}{(a_2x+b_2)^{r_2}}.$$